Selected Topics in Mathematical Physics

Prof. V. Balakrishnan

Department Of Physics

Indian Institute Of Technology, Madras

Module - 03

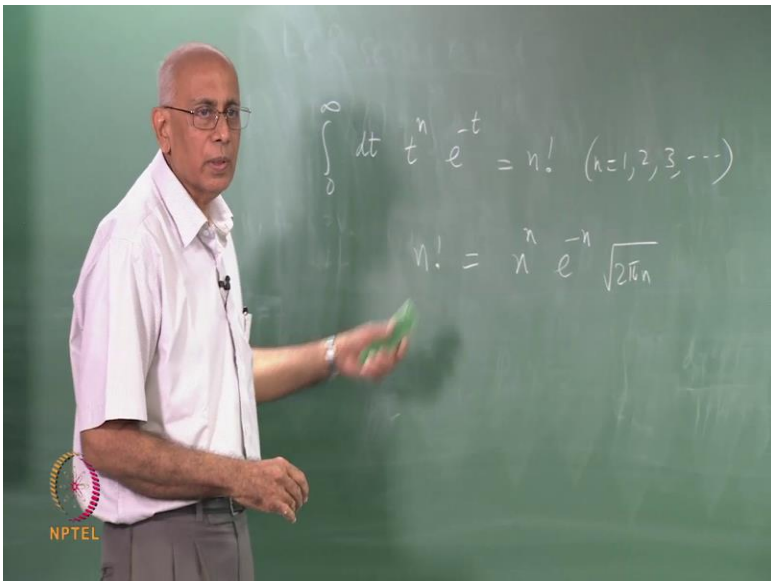
Lecture - 09

Analytic Continuation and the Gamma Function (Part -I)

Page 1, 6

Most instructive example to look at for analytic continuation not by power series, but by other techniques like integration by parts would be the gamma function. I presume all of you familiar with the grammar function, but just in case, we would like to see the together shor of complete picture of it. Let me repeat information, which probably already have.

(Refer Slide Time: 00:39)



So, let start with where the gamma function originally got define by Euler. He notice the following he notice the integral from zero to infinity of the d t, t to the power n e to the minus t was equal to n factorial for n equal to one two three etcetera. This is fact to the integrated do integrated by parts integrated this answer here. Now of course, we (( )) any could to zero then you again get unity. So, you could in principle use this integral to define zero factorial n. So, you will could say integral zero to infinity d t, t to the zero is t equal to one is equal to zero factorial. So, you can use this integral to define what is meant by zero factorial in which as you know is taken to be unity, and reason it is in this is fashion.

Now the question is what is this by the way what as one a do for computing large factorials. If you have little pocket calculator, you can compute the factorial pressing this button, but then at some point it shows error message, you know the number becomes greater than ten to the ninety nine typically. When does it happen know as the calculator think that become much more where does it happen?

Student: (( ))

Professor: What factorial is greater than ten to the ninety nine permit.

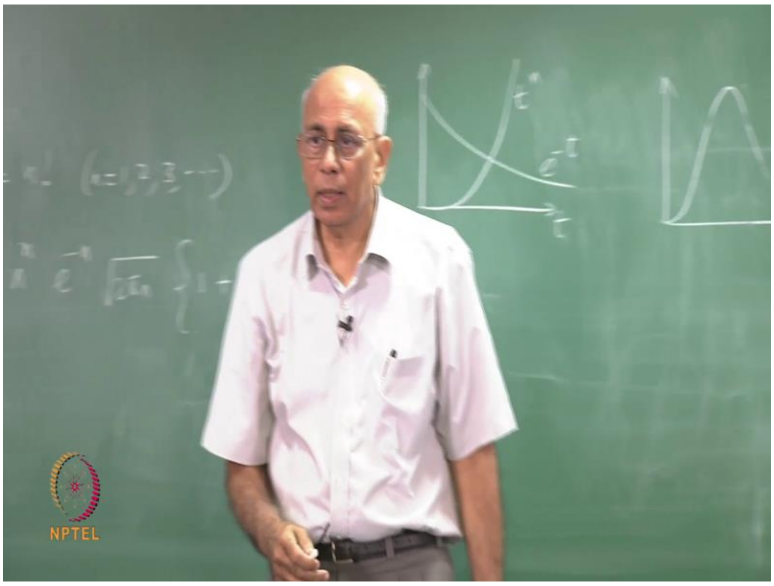
Student: Seventy around.

Professor: Around seventy, seventy I believe is get it will shown error messages so that means, at beyond seventy you cannot compute factorial using this bigger than ten to the ninety nine. So, we need an a approximation or we need a formula for computing the factorial, we need a factorial very often, you have commentarial problems involving more than a hundred objects in then of course immediately n factorial. Typically you have n to Avogadro number ten to the twenty three. So, you certainly need in the factorial and there is famous approximation called Stirling's approximation, I should call it is Stirling's formula because it is much more than a approximation as will see. It has a seed of something very general unit and that gives you even expression for the factors it what is stirling's approximation by the way or formula.

It says n factorial is equal to the first term is n to the power n because it says n times n minus one n minus two etcetera set the all equal to n took a power n, but that is a overestimate. So, we got a compensate for it and the compensate in factor is e to the minus n and then there is a square root of two by n that comes out is a very small correction to when you take the lot of n factorial it. These things will separate out in the leading term is n log n in in separate ten and then there is plus half long two pi n. So, logarithm correction, but actually this terms out to be part of the series there is a one and then a one over twelve n plus order n square this infinite series on this side.

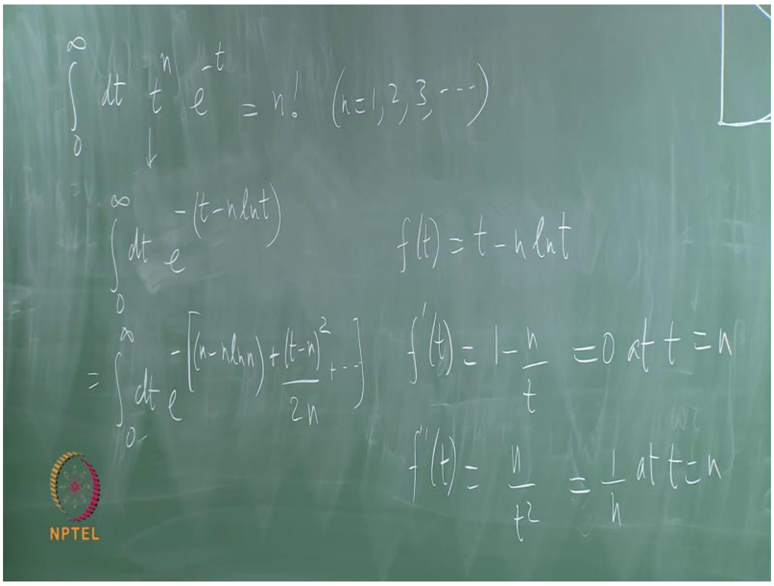
The next term is of the order one over three hundred time three hundred and twelve times and n square are something like that. So, these are powers of one over n keeps going. So, it is actually telling a something quite remarkable will come back to this this is sterling's formula the will come back to this in the minute. But I would like ask you how you get this formula how would you get this formula what would you do? I will do what is calls the Gaussian integration use the fact that you have then you know what the integral of e to the minus x square does. So, what we do is the following you realize that as a function of t this factor t to the power n increases like this t square t cube that etcetera again plot other plot in increase more and more sharply on the other hand e to the minus t

(Refer Slide Time: 04:47)



Decrease like this the starts at one finite value go sharply to zero. This guy starts a zero very plat and increases go infinity not quite as rapidly as this drops, but still goes up anything and clearly in you have something which is going to be zero multiplying some large, but finite number you going get zero is the answer. So, the product of the two is clearly go to look like sub cannot this (( )), and idea behind Gaussian integration is to say that the contribution from all this things is go to be negligible in the limitable end becomes very large you just go to get a contribution from this let me portion that. So, you might as well replace this by curves in inertial for minus infinity to infinity and say that the first approximation to it that is the idea behind Gaussian integration in way is implemented is to say that this integral

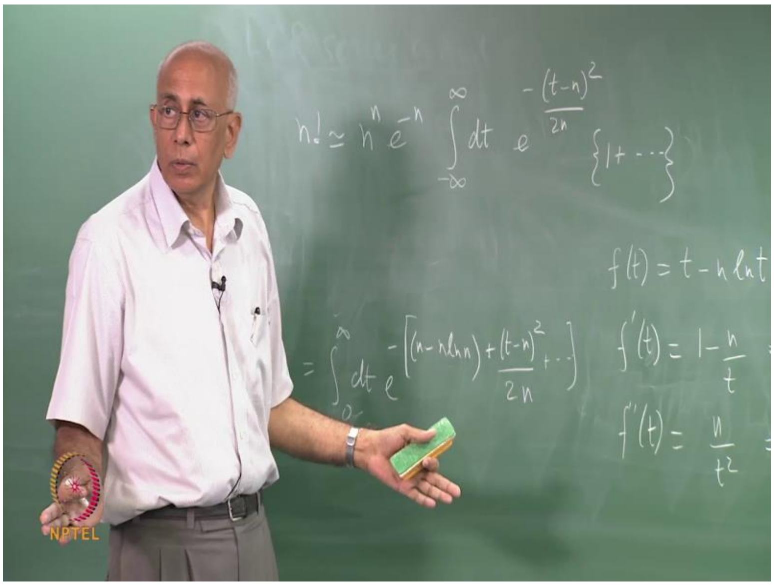
(Refer Slide Time: 05:49)



Here I first write it as zero infinity the d t e to the part minus t minus n log t. So, i write t to the n is it is a n lot t and then it is the guy here the integrated now this seeing here this function when this function e as near zero is near zero or near and (( )) that is when is equal to contribute most this integral that times its going to die down. So, let see where this function has an (( )). You have a f of t equal to t minus n log t. So, f prime of t equal to one minus n over t equal to zero at t equal to n because the make sure and ask whether these a minimum or a maximum. So, f double prime of t equal to n over t squared which is equal to one over n at t equal n.

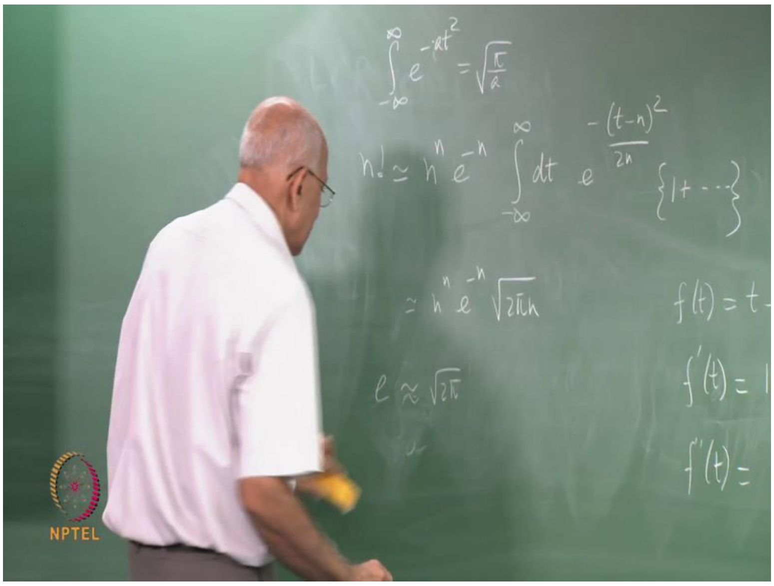
So, let us expand this, this is equal to integral zero to infinity d t t to the power now the first term is n minus n log n that f of n the next term is t minus n f prime of n, but prime of n is zero. The term after that is plus t minus n the whole square the two factorial which is the same as two times f double prime that is a one over n plus corrections in this fashion. Now look at this integrant by the way this correction terms are sitting on the exponents, but I could bring them down and expand the mount in powers of p minus n or something, but the leaning behavior is going to be given choice by what we looking down already show this whole things is constant.

(Refer Slide Time: 07:56)



So, in the right the size there for n factorial or n is n to the power n that is e to the n log n that is the n to the power n and thus e to the minus n and then n integral zero to infinity d t e to the minus t minus n the whole square over twice n. And then one plus dot dot of you interested and find the other terms you need to go those kinds here now it can be shown fine regress quite regressively that if I extend this integral for minus infinity to infinity then the correction is exponentially damned the exponentially small. And what is this tell you this is just a Gaussian integral here, I shift to t equal to n in the integration region is till minus infinity to infinity and the timidity tells you since

(Refer Slide Time: 08:50)

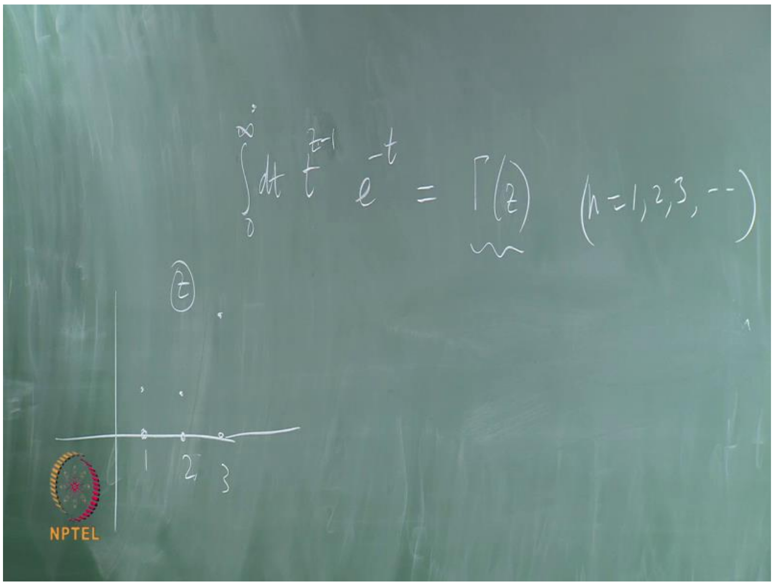


We know that integral minus infinity to infinity e would a minus a t square equal to square root of psi over a for a sets the real part of the real positive. It immediately tells you one this is a approximately n to the n e to the minus n square root of two psi n that is stirling's approximation. You can find the correction systematically by using this you have well at more careful, but you can right the corrections now i already mention that the first correction is one over twelve n diameter.

So, we see immediately that even for n as big as one as small as one lets first integer you have you still have a correction which is you have a answer here which is correct to something like to ninety two percent because a correction is a one part into a twelve. You have one plus one over twelve and if n is a of the order of ten then the correction is a what a one percent even is thousand the next point one percent. And also correction becomes smaller and smaller you know after all you put this equal to that.

For any could one the question you are asking is how could you is it to say that e is approximately equal to root two pi and statement we did go to ninety percent we two different number or together. So, this is not such a bad approximation its actually quite an a acquired an accurate formula by the time you hate time equal to hundred or two hundred its already quite accurate there will accurate this is the power of the settle point this is a Gaussian an a integration that (( )). In actually exponentially down the leave leave out the the extend this region of integration. So, these very familiar trick and might use it again. So, anyway what this tells you is a formula for in factorial way, but coming back to the original definition the question if could ask us.

(Refer Slide Time: 10:59)



I had integral zero to infinity d t treat to the power n e to the minus t equal to n factorial and Euler introduce can integral that for convention region for historical region it got shifted. So, let me call this n minus one and we call this the gamma function of n that n runs to one two three four etcetera. The Euler gamma function its n minus one factorial and n is one two three etcetera etcetera, but now they observation is that I do not need to restrictions to values of n which are positive in teachers of troll, I can use this formula now look at it instead of n.

Let us call it a complex variable z and you defined the for n equal to one two three etcetera there are just the factorials of n minus one. So, gamma one is one that zero factorial gamma of two is also one that is one factorial gamma is three is two and one gamma of four is three factorial which is six out care etcetera. And this formula actually provides you with an interpretation it actually makes sense if instead of n you re place it by real positive variable x. Let us go the whole log and re place it by complex variable z and call this gamma or z.

If you actually draw this graph in do the integral numerically then you discover that that a coverage looks like this and increase is extremely rapidly. In fact, we know how fast is count to increase its call increase likes x to the power x because the leading term also into behave then there is a need to the minus x. So, so it increases faster than any power faster than exponential and. So, on the other hand if you put this n equal to zero or z equal to zero the integral is d t over t which is bad news at the origin and go some. So, in did if you do this integral numerically you discovered that as a diversions here at zero and at looks like this. So, what a float it is x here versus gamma of x which makes sense as long as the integral exists and when does it exists it exists has known as this number this exponent here is greater than minus one or x greater than zero.

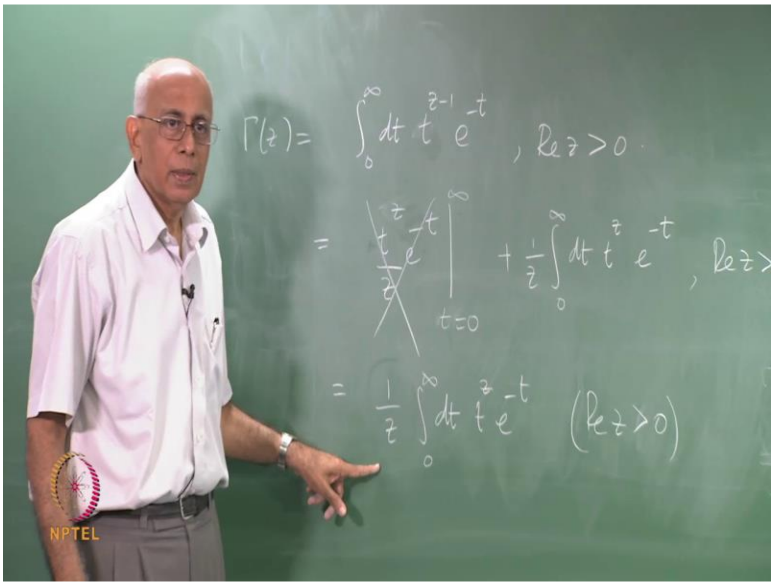
In this region, but I could not so make moving into the complex plane instead of x or call at z now and give it an imaginary part will that change, the converges of this integrate when does this converges. In the variable z where does the integral conversion where it converges has long as z minus one is greater than minus one or z greater than zero. But z could be complex could have in a imaginary part, now it obvious that if in the z plane and move in to and look at the imaginary and I give it an imaginary part. The integral will continued to exists as long as a real part is greater than zero because that is the only thing that control is conversions here if i write z as x plus i y there is a term.

So, i have t to the x plus i y minus one which is t to the call x minus one t to the part i y, but this is e to i y log t and its magnitude is unity is bounded is just ocelot he does not affect the conversion of the integral at all what effects the conversion. So, integral is the value of x and we required x to be greater than minus greater than zero, we want x minus one to be greater than minus one or x to be greater than zero. Now in the complex z plane this is x and this is y where is its region we want to x to be positive. So, you want this region real z greater than zero that is the region of conversion of this integral. Is that clear yes everybody agrees.

So, this integral defines an an analytic function explicitly and a analytic function of side define in the region real z greater than zero, namely the write half plane in the complex plane and when z hits positive meant is a value is like one two three etcetera. Then this function reduces when z hits the value n this functions value reduces to n minus one factorial otherwise its well defined by the integral explicitly that is in a explicit representation for this function in that region. We suspect there is going to be some trouble with this function on the boundary of the this conversions a region of conversion. So, I expect that gamma of z which is an look it in the write half plane these region has one or more singularity is on this line just as when we define the function by conversion power series. I said that this function is bound to have at least one singularity on the circle of conversions.

In the same way when i define that by definitely integral of this kind and this region is bounded by this line here, I expect that somewhere on this line at one or more points this representation based down because there is a singularity of some kind. And like to find out what this similarity is it now what should one do well why did this thing break down it broke down because if the behavior act of this factor t to the z minus one.

(Refer Slide Time: 17:46)



So, let us write to now I have gamma of z i defined to be zero to infinity d t t to the power z minus one t to the minus t and in the same i write real z greater than zero that is the region and which the integral makes sense. And gives me a finite value for any z in this region and it is this region at like to go to the left of this region. And see that the gamma can be analytically continued what was the factor that give me trouble this factor here, and that a rows because of this power here it we power had been larger when you would have no difficult at all. So, what should I do to make this power to larger how do i increase is power integrate by parts and keep it an integrate by parts. So, let us write basic as equal to t to the point z minus one a t to the power z over z t to the minus t t equal to zero to infinity and then minus an integral one over z comes out zero.

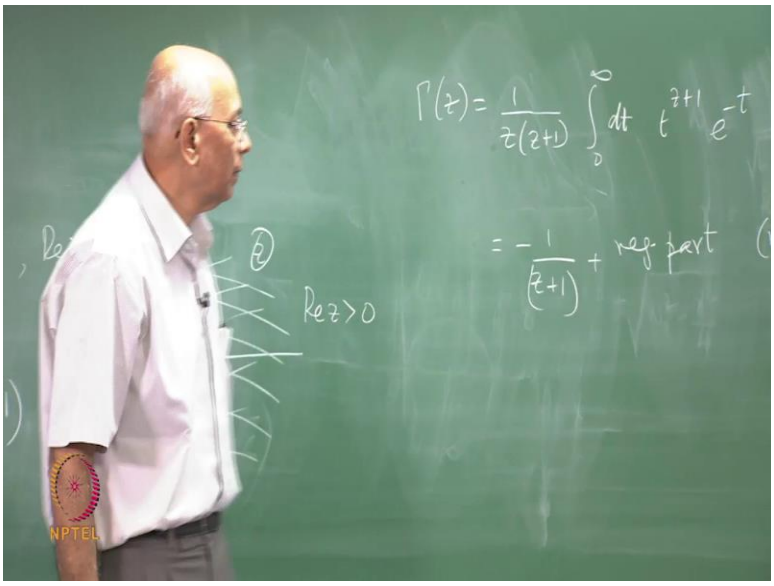
To infinity d t t to the power z there derivative of e to the minus t is a minus science were gives me plus i still have real z greater than zero. So, let us keep writing at like that I just integrated by a part. So, real z has to be kept greater than zero or otherwise original integrate in make sense, but what is this equal to were at infinity this factor going kill everything is want to go to zero what happens at equal to zero said give you one what happens this factor t t zero. So, what happens use factor it varnishes provided this is true provided this is true.

If you said z equal to zero that is not true in provided that is true it varnishes right, but we have kept z in in that region. So, this is definitely zero at all points in the z plane to the right of z in the imaginary access is zero. So, I can also right gamma or z as one over z integral zero to infinity d t t to the z e to the minus t real z greater than zero I have anything. So, I got an alternative representation for the same function which is point by point equal to the original representation no definition around, but there is a pole explicit setting out here at z equal to zero and where does this integral convert to the write of real z greater than minus one.

So, what we discovered is that this function has a pole here and view got an integral which convergence here apart from that pole these representation is now actually valid for real z greater than minus one and there is a explicit pole in this point. So, we found then alternative representation which is valid in a bigger region and it also exposes the singularity at that is equal to zero. It is a simple pole because it will put z equal to zero in this you just get unity and there for the coefficient of one over z is one which means it is a simple pole with residue equal to plus one off course this is got a regular part at z equal to zero.

So, this whole function has a singular part plus a hole regular part we near z equal to zero these are the form one over z plus regular part near z equal to zero near z equal to zero that is definitely true. So, we conclude that is the gamma function has a simple pole at z equal to zero big residue plus one, but now I can say the same trick I want to go for that to the left to this i do the same thing once again and integrate by parts once again. And then what is going to happen is that if want to get a same gamma of z equal to one over z and now a integrate these by part

(Refer Slide Time: 22:39)

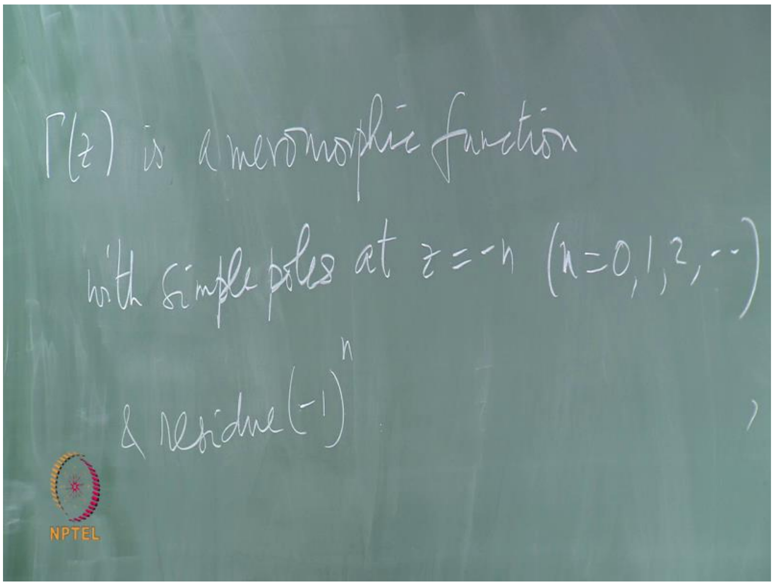


So, z plus one and then an a integral zero to infinity d t t to the part z plus one e to the minus t pole system where is this going to be convergent this integral.

This is greater from real z plus one greater than minus one or real z greater than minus two. So, and what about the behavior at z equal to minus one its singular what is sort of singularity, it is a simple pole which is explicit seeing here. And what is the recipe that point you got a multiply by z plus one and then take the limit now we multiplied by z plus one is goes away that gives you unity and these gives you minus sign minus. So, these is a equal to minus one over z plus one plus regular part near z equal to minus one. So, what is a pole at z equal to zero is quite nice at z equal to minus one.

There is no problem there is a value is a minus one, but it contributes the residue in that point and. So, want. So, what to do guess is the general structure of this function there continue here this is zero this is minus one i continue next time its minus two and. So, what. So, what would you conclude is about this structure of this function it as simple poles at all the non positive enticers zero minus one minus two etcetera. So, in general its called a simple pole at z equal to minus n and what is the residue at that point minus one to the power n because these things go to alternate inside. So, gamma or z is a meromorphic function.

(Refer Slide Time: 24:57)

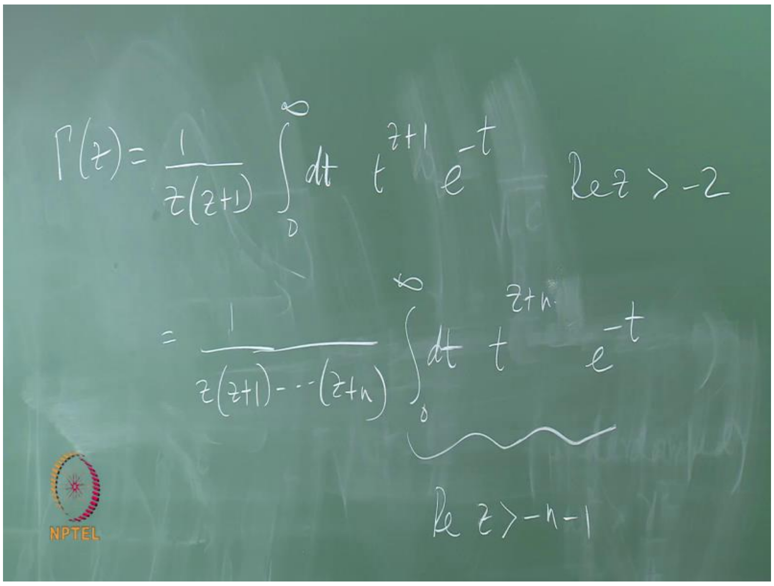


With simple poles and residue minus one to the power.

Student: (( ))

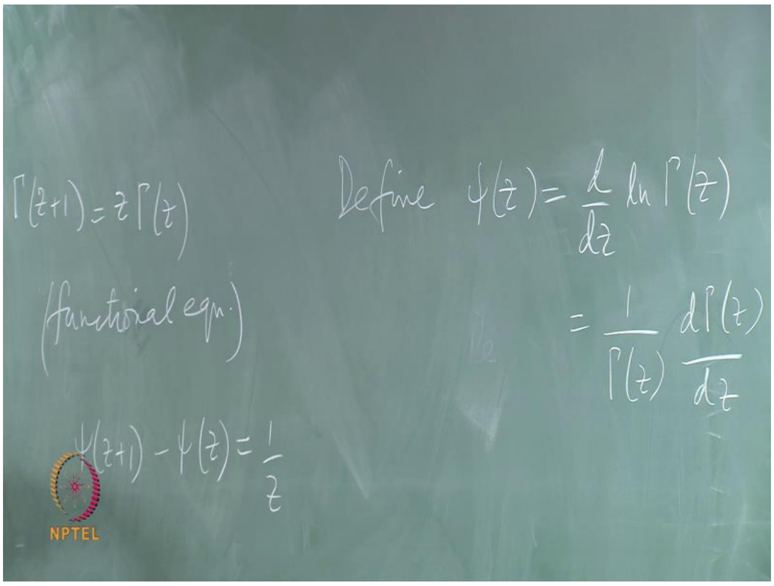
Professor: Yes, it is a minus, yes, because this is goes to all residue z plus two z plus three etcetera minus one n factorial. So, what we done is analytically continue this gamma of side by the strip of integration by parts strip by strip by strip you could ask. Can I write something which is one short it is valid everywhere shows the poles explicitly everywhere I do this en times and end of it something which is still valid only enough finite region.

(Refer Slide Time: 26:11)



So, this is equal to one over z and z plus one dot dot dot up to z plus n integral zero infinity d t t to the part z plus n plus one e to the minus t. Now whatever z plus n, but this integral still is convergent only for z greater than minus n minus one real part. So, at some finite point its stops and i can keep doing this, but the question asked is can I write single formula down valid everywhere in the complex z plane. We will do that when we talk about branch points and show you there is a master representation you have write this down, but this is as good method step by step by step, we can analytically continue.

(Refer Slide Time: 27:16)



This what was the basic trick the basic point is to note that this function base of functionally equation and that equation is simply that gamma of z plus one is z time gamma z that is all that needed. So, the moment, I wrote one over z that integral out there t to the power z instead of z minus one that was gamma of z b plus one in the being that z to the left an side in get this functional equation. Therefore, I can drive gamma or z plus two as in terms of a lower power z plus one gamma z plus one. So, z time z plus one gamma z and so on and so forth, and this functional equation is what enable has to do this.

Analytic continuation strip wise in this function. So, that is another trick you find a function equation you can try this town immediately for at sense suppose you took derivate of the gamma function what happens that suppose we took were this functional equation is creates that sitting here. So, its natural thing is not take derivate of the function directly, but first separate these to was a some and then take derivative. So, what should one do how would you convert the product or some take logs. So, define psi of z equal to d over the d z lo logarithm at derivative the log at make there be to if. So, this is equal to one over gamma what is the functional equation satisfied by this psi of z take logs on both side in then the different shade.

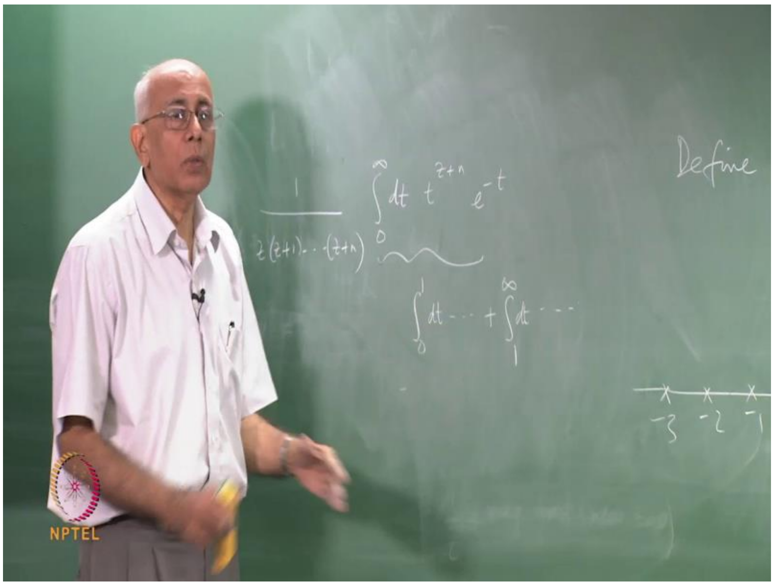
So, you get psi a z plus one minus psi of z equal to one over z take the derivate of the log derivate of this it one of the set and then a. I bring to the site to the left and sites. So, this is the functional equation satisfied by this psi of z function what is the singularity structure of psi of z why do you think it you see near the point z equal to minus n gamma also z has poles all these points right it is got poles at zero minus one minus two minus three etcetera etcetera. So, near z equal to minus n gamma of z is guaranty to be of the form minus one to the part n over n factorial z plus n plus a regular part. So, guaranty to be a plat form.

So, now, I do this I depreciate and then divide by gamma of z if I depreciate this part its again go to be regular, but I depreciate this part it gives me a z plus n hole square and then a divide by this portion by the gamma observe itself. So, what you think psi of psi going to look like that point this will imply that psi of z equal to again this regular part.

And I am going to differentiate this. So, thus the one be a minus one over z plus and hole square, but I divide by this whole thing divided by z plus n. So, one factor of z plus is going to cancel out and then is a minus to be an n factorial do not to be cancel out, I want going to left that is minus one is going to the left. So, its point to be minus one over than plus one plus arrange the point. So, what is the conclusion and psi of z the log at derivative of the gamma function is also numeric function it got simple poles at all the non positive in integer with the residue that is how ever the constant minus one at all this points finding this is different story finding. This is different story we need to know that carefully will do that we see what the Euler the constant has to save about it but this tells you the structure of both gamma or z.

Now the question we could ask it is just as write down yesterday and left the expansion for this function co sequence square or caught psi z as shown can be for this meromorphic function write down such a representation. While we have all pole, we have minus one to be n n divided by factorial z plus n the question is can we write down and explicit representation for it or not.

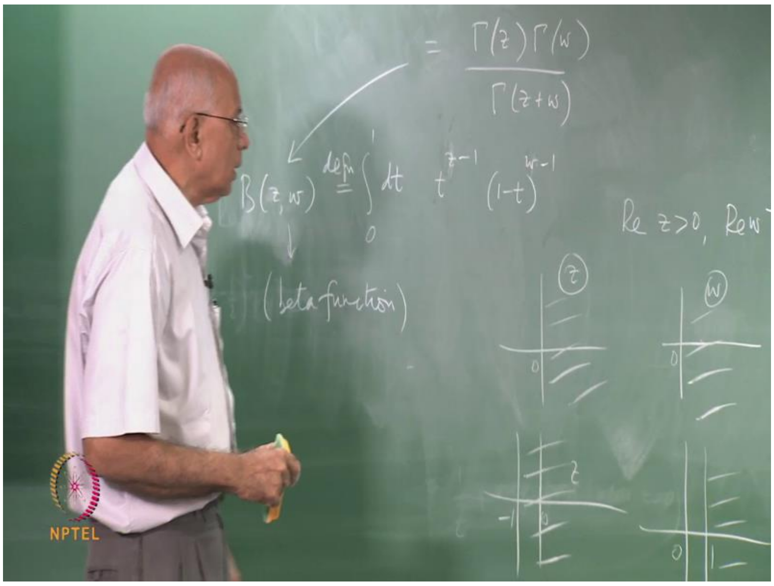
(Refer Slide Time: 32:37)



Now, how going to do that by this requires is little bit of work remember that the original integral was zero to infinity d t t to the power z minus one e to the minus t in this fashion. And then I did integration by parts systematically tell about this stage and an i pulled out the residue at least point here. So, what one has to do need the points z equal to minus one is to recognize that somewhere along the line this integral that i have there that z plus n. So, let us look at that t to the power e to the minus t and then there is this factors one over z z plus one at to z plus n, it is factor that gives trouble at the origin in here, and it gave trouble do to this zero setting here. So, what one does is to great into an integral from zero to one d t etcetera plus integral one to infinity d t in this function. We could a broken it up at any point here, but I am going to leave it you exercise you have write disorder to ask you to do this detail you show the this break up will need you to the meromorphic expansion for the gamma function. So, this is the portion that is actually going to be regular part.

And the all the singularity is going to come from this portion this integral that is the précis division that you have to make no other point will give the example representation. And we see how that other things we want to come back to this formula we going to write down what we spoilers what is this part especially add z equal to minus n similarly for the gamma function. So, we try and write down more analytic properties of this in a amount you show you product representation for this gamma function related to it, there are other functions there are kinds of integrals is a finites. All are data function which is actually the Euler integral the first kind when the let me write the down introduce at then will be come back to it and that function is defined in the following way again

(Refer Slide Time: 34:51)



You define it through and integral and this time the integral runs zero to one d t t to the power instead of z minus one some n minus one one minus t to the power n minus one.

And this is and is the definition that m and n are non negative integers zero one two three etcetera, but if it zero it blows up if, so m n in to start with a positive integers. Now this factor is going to start diverging do to the lower limit of the integration if n it is the value zero and if n goes negative, it is where this factor is going to start divergent. If end its the value zero, because if this limit upper limit of integration here again you can try to improve matters little bit first a fall instead of m comma n. You can define this of two complex variables.

So, let us write this immediately as z and w call the z call this w and where would the integral converge where would this conversion it converge as long as this exponent bigger than minus one the real part of this exponent. So, the clearly real z greater than zero it convergence as long as real w greater than zero in these two complex variables. So, as it stands this provides a representation for this analytic function in this region and this is by the call the data function.

So, in this two complex variables in the plan two complex variables down he has to w and it converges in this region and this region and like to improve matters a little bit. So, again I do the same thing, I integrate by parts. If I want to improve this to real z greater than minus one I integrate this by parts, but what happens to this integrate this by parts increase the power, but what happed to this that is get different sheeted in the power d decreases. So, you can a find an a alternative representation after one integration by parts integrates this in different shade that which will actually we good in this region in z, but in w it going to sink in w it will be got only this region.

If you tried the other way it would just reverse roles. So, it is clear the beta function cannot be improved its a region of analytic cannot be made bigger in both variables together by this stick of integration in a part. They are struck completely, but other ways of doing this will come across the contour integral will derive a contour representation for it which is again till you how to do this in general. But the at another way and that should be define that is to show that is beta function is related to the gamma function and then we know everything about the gamma function. So, we can use that i will show subsequently that this function is actually equal to gamma z gamma w over will derive it identity between the two function once we have that then all the analytic property is completely know once again.

So, will take this up little later after we talk about branch points, but mean time I want to show you that the integration by parts does not always work then you have this kind of problem then it sub restricted, but in the case gamma function it worked. So, then stop here today then will take it up from next time, any questions? While there a general tricks a few general tricks, but the hole you have to do this you have to know something about the analytic property is a, at the function a, where it is a valid. And depending on what representation you have or we need you look for we look for functionally equation we looks for (( )) of what is a interesting is that the enough permanents of equation.

So, I will make a few comments on the after words after we do little more analytic continuation about uniqueness of such continuation and the this idea permanents of equation. So, when you have an equation between two analytic function in some region then you analytically continue these functions that equations is going to be true still provide both side can be continued and that is because you can separator one side from other in call at zero and that off course analytically continue to every point. So, in that sense the continuation of these functions will continued the satisfied the same equations and one important lesson is that we know deal with a difference equation special function in. So, on you must think of them a function of a complex variable both the depended and the independent variable a complex variable. In fact, even the parameter in the equation should be regarded as complex variables and then you get full picture of a we will do will talk about a general function in the nice show in the works. So, you not only.

If you have think you use to p l of cause theta that l is zero one two three etcetera and cause that as some number between minus one anyone. So, the first thing is we can define a p l of z which is a complex variable not restricted to minus one to one not just the real access see that all that have complex plane second p if an do is to be make l a complex variable. So, real a function of two complex variable and that is the way one should think about it then it has it analytic structure becomes very clear.