Selected Topics in Mathematical Physics
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Module - 03
Lecture - 07
Linear Response Dispersion Relations (Part 1)

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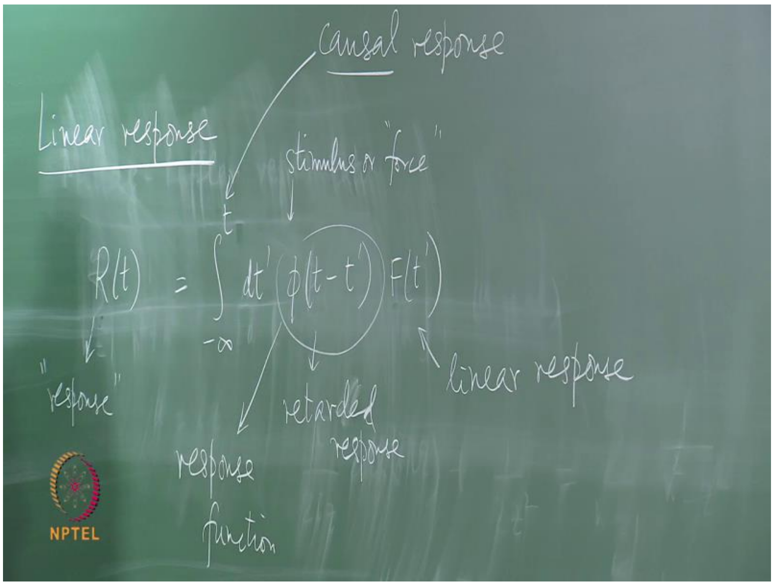

Today go to a physical topic and important applications of complex analysis, which has to do with response of physical system. So, will do a little bit of linear response will see where complex analysis helps. So, I will start with the description of general physical problem, and then we will specialize to in the case of dispersion relations, will derive dispersion relations after defining a response function appropriately. What have will mind is part of very large number of physical system where you disturb the systems by some stimulus and the system responds in some manner which will measure and so on, and roughly speaking one expects the great of the stimulus the great of the response.

This is an example of the linear response and then lots and lots of physical systems which at least for small values. So, suitably the smaller values of stimulus will respond the linier fashion. For example, Ohm's law you apply a voltage and the response of the system is current, the current is proportional to the voltage by Ohm's law. Similarly hooks law you apply a stress on the system and in the elastic region the strain is proportional to the stress to first part of the stress. Similarly, you apply an electric field on the dielectric and as long as going go to the region of dielectric brake down and so on. As soon as apply on small electric field, the small directing polarization of this medium which is the response. You apply a magnet field on a magnetic system, and it response by according magnetization like in a paramagnet for instance. These are all example of linear response everyone of these thinks and this very elaborate theory of this linear response.

What we like to talk about is the fact that if you apply a stimulus on the system, the response also depends on whether this stimulus has one kind of time behavior or another. For instance if it is an oscillatory the stimulus like an oscillating voltage apply to a circuit - electrical circuit like an LCR circuit, the responds could be quite different from what happens if you apply dc voltage to the system. So, as you know, physical systems would respond in manner which depends on the time dependence of the stimulus that you apply to the systems. And most general could you do we should taken arbitrary time dependent stimulus and ask what is the response to and that problem most easily analyze by saying a first state the stimulus break it up into frequency components by Fourier analysis. And then for each Fourier component, I will ask the what is response the system and supper pose all these simply because it is still in the regime of linear response, so that would be a logical thing to do.

Let us write down what the most general response for a system could be making certain basic physical assumptions namely the linearities is most important a put in a couple of other physical ingredients, which when you relax you get more general kinds of response, but this is the simplest system the situation that you can have. So, I am going to call the response of the system sum R of t and stimulus of the force that I apply on the system call it F of t in general response. And I am going to relate it to the force that I apply in system or the stimulus in a certain specific manner if the stimulus is F of t, this is the stimulus, or force the time dependent question is how does this depend and that. If it is linear it can at linear functional of this F of t, but of course the force applied over period of time the responds would start after you start apply in the force.

So, certainly there is causality in that sense, you do not want cost to precede in effect precede the cost, you also wanted to be linear. So, most generally thing you could possibly write down, it some kind of supper position of the following kind. So, it be an integral in the general case could be some integral this continues vacation of the stimulus. From the earliest time, you can imagine which is minus infinite dt prime and then there is a F of t prime. So, it is a supper position of all the force you apply, multiply something here which is the response for unit force and that reason you multiplying by f of t, so that response for unit force.

Let me call it some phi, which will depend on the time at which apply to force as well as the time at which you measure to the response. And the question is where does the integral cut off, it cannot go on to plus infinity because today's response cannot depend on the force which is going to apply tomorrow. So, clearly it is gone to cut off at t itself it cannot possibly the anticipate. So, this is causality, causal response integral of here that is very important. The fact that is linear and f this factions that is depend on f it is response per unit functions.

So, this shows linear response and there is one more ingredient and put in physical reason and that is you say look does really matter, if the system is not changing underline properties inherent property as a function of time, it does not really matter whether it is the experiment today or tomorrow or anytime. All that matters is the time different, the time interval between the time apply the force and time you measure the response. The absolute time does not matter at all.

Provided system is not change or aging for instance if you measure the elastic modules of piece of steel, if the steel does not undergo any structural changes in between the do experiment today or tomorrow and so on would not matter. And if you did the same thing that flyover outside, the concrete fits that ages concrete ages and there for its elastic behavior changes when time and then indeed this F of t is function of t and t prime this fashion. But if do not allow kind of this ageing if you say the system in dependent of origin of time then this response can only depend on time elapsed, since the application of the force, so it is the function of t minus t prime alone, some kind of retard response.

So, this implies retarded response, everyone of these assumptions can be relaxed depending on the situation. Then we gets totally different, we are going strict do this. So, we look only at cases of causal retarded linear response and then the most general thing write down. Now of course, you could also argue saying that this force is the vector in general the mechanical cases, it could be an electric field, it could be a stress in which case in tensor of rank 2, and the response would also have similar character that is true. So, this could have many component that could many component as well and this response function in between would connect these two there, could also have large number of components.

For instance, if you took an electric field and you applied the electric field and three component and three dimension space this thing here is the polarization, which is also vector has three component. In the most general way, linear relationship between vector here and vector here is through a second order tensor here. So, it have a nine component objective here, you put in indices in all these places.

So, for simplicity, let us get it up is right now is write the scale as for the moment, but we can put it all the indices that will like. For instance this, if this where a stress it could have nine component symmetric and dimensions, this would done strain tenser it was also nine component here and this thing the complain tenser that would actually eighty one component, it would be rank four considered. But of course, we know that could be symmetries in the system and then these number is eighty one is reduced you do not have eighty one in a elastic modularized by for anything. It is much smaller number and if you material is for instance cubic crystal law something like that, there would be a small number exact crystal six or something and cubic crystal.

For example, we would have simple cubic just take simple would have shear modulus and then it would have longitudinal modulus, the transverse. If it is an isotropic medium, it justness too independent elastic constant, which you could choose it. We say bulk modulus and Poisson's ratio or if you choose on into to the young’s modules and share modules, but we not interested moment in the this is the general frame work in which think around measure and this discuss it this frame work this is called response functions the what you expect of this physically what would expect well lets struck on the moment to real functions here and real function here. So, real physical of force and real stimulus and this thing is real connect a between the two response functions.

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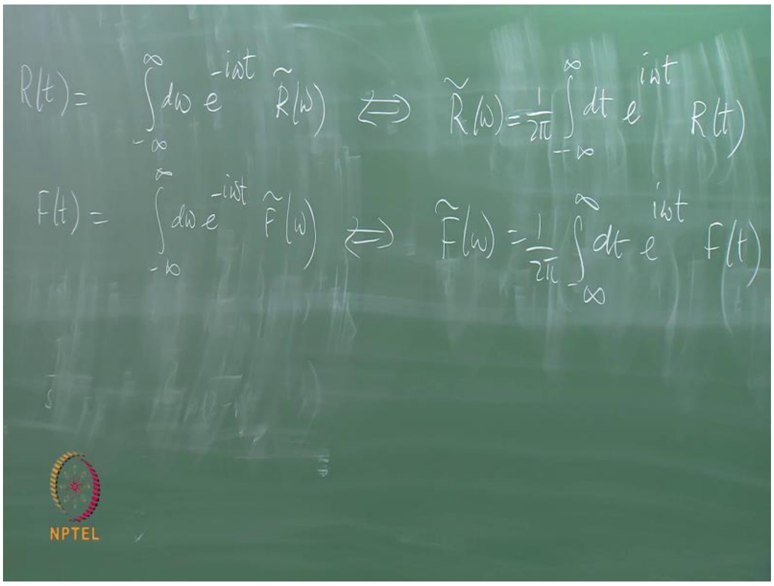

Now, what we expect of this is by plot, let us called t minus t define it could say if plot as function of tau then I expect that is the force is stress as expect high response and then as the time delay increase is response due to this force decreases. So, in general, I would expect, but this five o t tau and navy the d k especially otherwise d k in some fashion. It could I saw oscillate is nothing to stop from it could become negative, it sometimes it could become possible. So, all these are promoted could that if could these thinks what I do not expect take most increase as a function of time in generally expect time it respect the effect of what happen day before yesterday on you today.

It is happen less than the effect of what happen yesterday in what happen in today. So, I expect to be some kind of d k, it is function of tau could go to zero, could go to constant, we do not care now given this what can be say next while the most general form of r of t you can write down could be equal to is the emptive of this response at time t corresponding to the frequency omega and do not use omega to the frequency i guess you, but just call as frequency when I choose in time convention, but as one if i a transform convention as a people, but will choose one and stick to it is function choose minus here. So, this will off course imply that till of omega equal d t it is the plus that why sure this minus that is plus and then r of t and we have to careful because delta function has one over two. So, i put here is this is this is this now there are and square root two point here, but these are the sort of people write j f for i since there beyond reach of humanity is still write j is still write j engineers write j electrical engineer write j write write do this probably very sensible because use i for something else for kind, but facieses you like capital life kind the reason being then became intensity like o t have confusion notation, but I stick to this stick to this here i guess they also write two five new out here that is the then you could same thing force f of t is equal to d omega it is minus i m t.

Now plug back to this. So, let us put that this for representation into this and look at what happens. So, on these side if i fue a transform this i get a d omega is for plus minus minus i w t of timb of omega equal to on the other side of minus infinity to infinity d t t minute t into minus infinity to infinity d omega is minus i w t that is important t point f to omega all I did was right the faure representation if each of these function of time.

A nana enter change order of integration k provided these thinks converge fast in a inform you can enter change the order of integration if i do that yes f of t cube absolutely y a minus infinity that is very important thank you. So, this becomes equal to minus infinity to infinity d omega f till the omega i cannot full out the t prime outside because still got integrator. So, minus infinity up to t d t prime f of t minus t prime e to minus i omega t prime.

But let us write that a lets change variables lets change variables to t minus t prime. So, let us put that already said that here t minus t prime is tau say and t prime that is the variable of integration. So, d t prime is equal to minus d tau and this becomes equal to integral minus infinity to infinity d omega f fill omega and integral well t prime is minus infinity tau is infinite and then t prime is t tau zero, but this minus sign here in the. So, this equal to zero infinity d tau five of tau and then is minus omega t prime which is e to the minus I omega t minus tau and off course you can full this out here.

So, this integral become equal to integral minus infinity to infinity d omega e to minus i omega t f till the omega times plus i omega. So, that is what this thing become equal to now let us collect terms together and look at what happens. So, we fine integral minus infinity to infinity d omega i w t times r till w minus bring that this side minus and integral from zero infinity d tau i w tau p of tau that is a function of omega because a tau is integrated over. So, function of omega multiplied by f till that omega and the whole thing is equal to zero k now i argue as follows this thing here as omega runs from minus infinity to infinity forms complete set a functions orthonormal said a functions of tea that is why you can do a fue a transform like. So, when two is equal to each other the functions are equal to each other, because of to transform to each other at each frequency this implies because to the manners i omega t is complete orthonormal.

Set of functions t in minus infinity to infinity it is selves co operations are the equal. So, the for a transform of zero is zero therefore, it implies co efficient by co efficient this must be equal to zero otherwise you cannot supper like zero. So, that is how normally do fill transforms in the co efficient in reason of (( )) that your equating the components of vector function space of in the individual components in a basis certain ortho normal basis. So, this implies immediately this equal to watt for each real a never.

And this is sum function of omega. So, let us call that give it a name x w f till of w and this x of omega is called the generalized susceptibility its frequency dependent the susceptibility now what is telling you it is telling you that if you put this think here you re call namely in one you are in high school you land the relation like the magnetization on his proportional to the the magnetic feel m divided h is equal to the susceptibility what really say is magnetization per unit plot force for unit magnetic field is your susceptibility.

And now expect this unit is in the frequency. So, at each frequency per unit this of omega measures the corresponding response at that frequency. So, this is what the dynamic susceptibility or generalized susceptibility does and how it is related to a response function it is given by this. So, that is very very crucial that relationship says x of omega is equal to integral from zero to infinity it is the integral of the response function it is definite integral. So, might as a call it d t remember the this function is define only for positive value of this argument because he t minus t prim it is response function it is start is equal to zero and then continue forever beyond that.

So, that function is defined only from zero to infinity in its arguments and in deep susceptibility also in walls the integral only in that range m in this happen why did the happen why did skeet cut off at zero causality because of zero came from the limit in the response causal response when a change variable and that got cut off a t rather than plus infinity because of causality. So, this is responsible this thing here an effect of causality is this a fule transfer is this a fule transfer of the response functions no, because of fule transfer would be minus infinity to infinity functions rode even defined in the like to be

I could define it i could define it by continuation tomorrow anything and then off course and have to specify what dose of i talking about will come to that i try to point out later on were this f of t actually comes from in the in a response this f of t turned out to certain correlation function which will have meaning even forty lesson zero depending on the time reversal properties of the operator concerned. So, at the movement we do not know that this is it this not a plus transform because there would be e to the minus s t this is i omega. So, it is neither nor a plus transform you cal fuel transform a one side fuel transform, but it is create certain problems, because of the this because if it plan transform he could define it even if this function when to content at infinity, but now that' go to problem a here.

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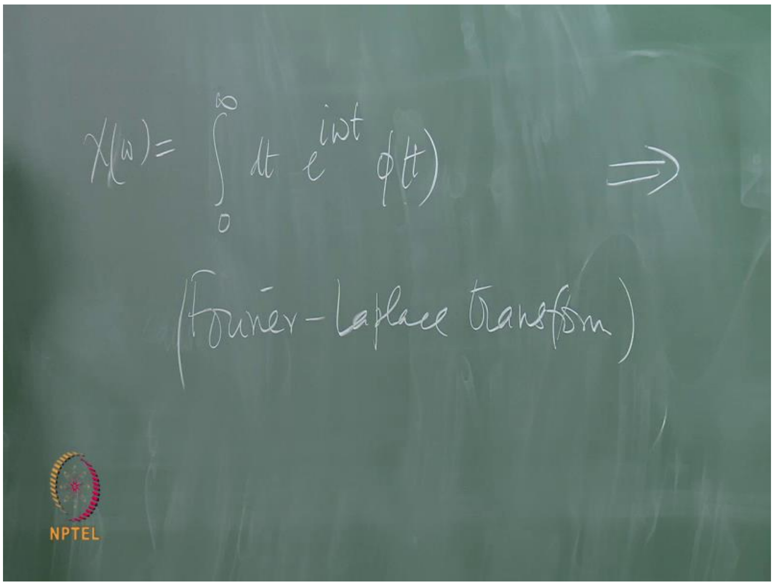

Let us look at this response function, I will see were takes us first and go to assume this is an important assumption, but wave we look at this. I am going to assume that this integral actually exists, that it is finite axis it is fuel a plus transform we have to be a little careful about this generalized accessibility is fuel alphas transform of the response functions that much no soon there is axis for real values of omega in particular we are going to assume it excise zero of frequency in we at dc this an assumption can relax we can settle it, but write now let us assume that we can this taken excise we go ahead with it and can be say about it immediately the first thigh realized is that this is the complex member because it is real part and imagination part if i f t is really this sign part and co sign part and flows immediately from this the fact that you have sign here and co sign there for really and imaginary part you have really part co sign and sign respectively this immediately employs that real part of kind of minus one omega is equal two real part of kind of omega another words a real part is a symmetric function of omega because it is in cost omega t and imaginary part is an odd function for real omega at the movement omega is real this symmetric is very, very importance.

The next question is what do the things mean what do the really and imaginary part means will see as we long that one of the two part of will represent dicepective effect and other part will represent reactive part just like in a l c r s circuit you have a resistive part reactive part same things is go happens here which one is which is depend on the system and the kindly of the, but most common example of come across the imaginary represent discipline real part at the reactive part all responsible by the way for of this kind all of them k function below proactive index that when you index in the lab for a flex of a glass something like that in actually there fricative index that very high free frequencies of optical frequencies on the other hand different frequencies sees different kinds of different values for the fricative index.

And along with fraction along with transcription there is always absorption and the real part of the fractions index will represents dispersion part will represents absorption similarly all others aspects the directive concept represents the directive function of the imaginary part does the mandatory society and. So, on for go together the first think of the this to go together in this part little later will see fraction function see therefore, it cannot be purely real cannot be purely real go to have both parts the next think u have is that exists of real values of it would certainly if instead of this function here you have a damping factor in addition. So, this is a exists for it also imply that also imaginary by greater than or equal to zero exists.

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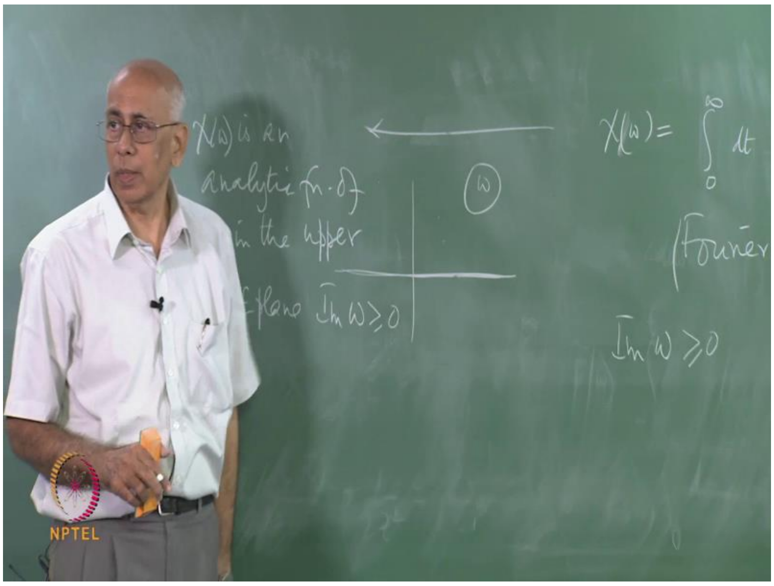

Because if you move of in the omega plain all that think of it as a frequency play in this is the region of real frequencies into the complex plain as a define a complex function of complex function by this proceed real by the definition then that exists it continues the exists if the integral exists here the function

It is certainly exists when omega summaries here, where it is got it damping factor in addition. So, what is that mean it implies that kai of omega kai of omega is an analytic function of omega in the upper of half plane including the real functions what is the no excess does not know this is not help you gone out as its kinds this representation by break down in particular of the imaginary values they should blowing out and the integral may not system. In fact, in general not exists in the lower of play expect in special cases. So, all manner of dangerous this luck here you do not have they are at the movement, but it out here itself by functions what can we say in that well it.

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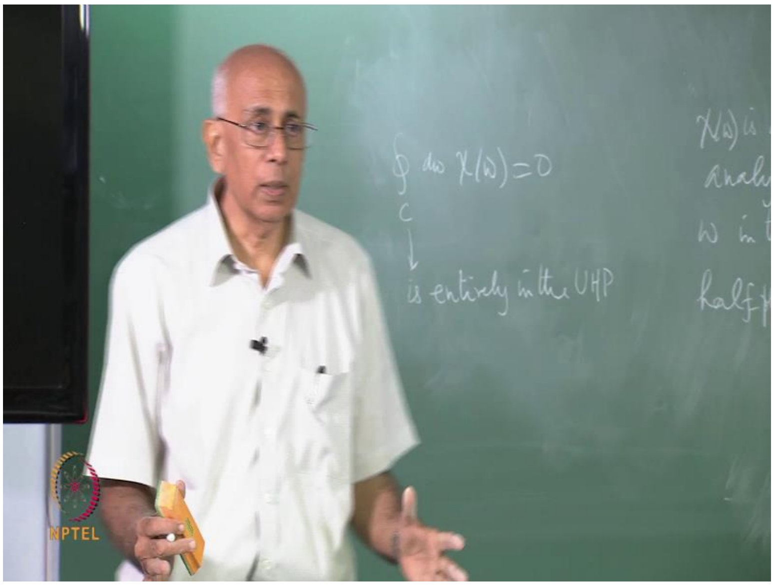

First think we can say i that over any close to contra c the omega pi of omega equal to zero if this c is entirely in the upper of which include the real excess. So, as long is a two does not go now i do not no anything about what kind of a omega is this is a zero by or he is integral does take this omega is that does not say much my take this omega if that take that function integrated over like this and zero and that is and help much a target to try, and show that this analytics of omega of kai of omega enable to right the value of kai of omega is frequency is real physical frequency in terms of all over other real frequency and that is called of a version relation I want to show that that is is implied by this analytics property to that and end that is do the column lets define.

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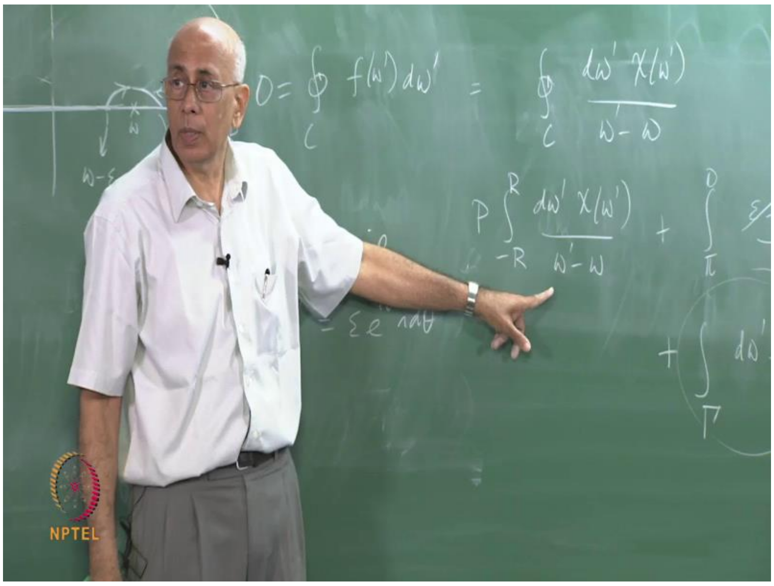

This function f of omega equal to kind omega type of omega prime over omega prime by omega where omega is some fixed real value define it in this fraction omega prime now what are the analytical prime is f of omega prime well they depend on the analytical prime which means of analytic the upper of of omega prime, but these also pole at omega the simple pole that part on the real excess else where it completely analytics. So, if i look at this functions in the omega prime plain at this value omega thus a simple pole at a fixed value fixed value and other than that its completely analytic everywhere on this line except for this pole as well as software.

So, now apply of the present take the formula of this and take some course on to c and the integral f of omega prime d omega prime o c is zero, but this course of this is integral c the omega prime kai of omega prime over omega minus omega now let us do the usual trick of the in this. So, i am go to far as i can go which means i can brought this to lie on the real excess i indent the then i close it the semi circle usual manner and lower that can to this integral h m. So, this is some minus r this is plus r this pont of omega minus x r that is omega plus r which means that an integral from minus r to r the omega prime kie of omega prime over omega prime minus omega.

The principal value due that i am going to limit obstruent a zero a going to the leave out symmetric neighbor here, and then all the way in a war that is the principle value or it is a singularity i already explain the principle value done us plus the contribution in small semi circle run in this fashion on that semi circle omega prime is omega epsilon is eight to the omega. So, on this semi circle omega prime is function of omega plus epsilon e to the omega. So, the omega prime equal to omega the epsilon data divided by omega prime minus omega and that is equal to epsilon plus the integral of this whole think over the huge semi circle. So, let us call this capital gamma or some plus this the etcetera.

That the whole think is equal to zero now we play the usual we let are go to infinity we go to epsilon cancel dot we go to in the last happens both side of the sequence this side of the zero provided kie of omega prime vanishes of omega prime goes to infinite in the upper of plain this term this semi term will also go to zero and the reason is on this semi circle omega prime is r in i data and look at the and looks like there is a capital of from here thus a capital of here the cancel of put and this term vanishes if kai of omega prime and goes to zero how slowly if that happens then in the limit has r goes to infinite that portion goes to off.

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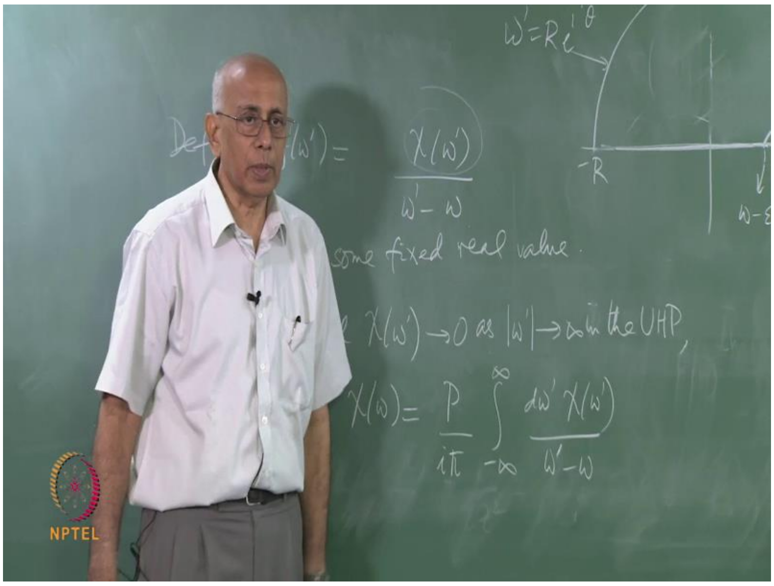

So, provided provided kai off omega prime goes to zero o and as is to infinite in t he of plain provided this is true to the limit of this integral plus this is equal to zero, but what is this equal to this pi set of abstonic equal to zero as f form of omega and then this integral is pi to zero is this is an pi minus sign kai its already kai provided that is true the end of the kai of omega equal to principle value over i pi integral minus infinite infinite the omega prime kai of omega prime over omega prime there is no prime of complex omega no this is a real frequency they started with that this is integral all over frequencies. So, we make this little discussion with the to explain that the upper came to the real excess the connection between functions on the real excess on the real frequencies.

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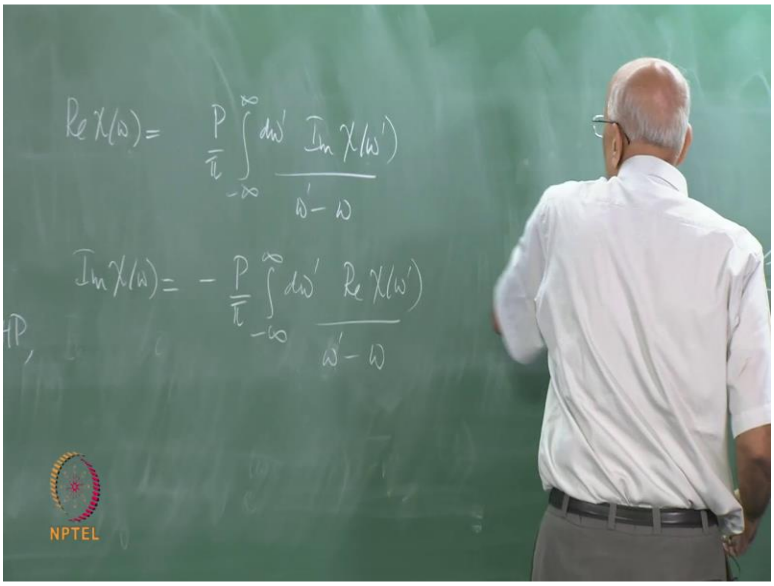

This principle value say is leave out a symmetric neighborhood of the pole at omega prime equal to omega well all we need to now thus take to real imaginary part on both sides and the new immediate see real part of kai of omega is equal to because i pi at move that abstains minus over pi times that and just time to make that may the mistake in the sign. So, this is minus i pi and moved to the and side pi goes to pi to zero goes to pi to zero pi pi. So, then it is becomes know the right hand side no no its it is this is a minus the right hand side pi pi in the downstairs vanish this. So, what is this say it says the real part of this in equal to to the pi this is the symbol minus infinite infinite prime imaginary part the tie of omega prime over omega prime, because this be a real part plus times imaginary part pi and minus pi gives plus one, and conversely imaginary part of pi of omega minus this over pi. So, these follows of the couple to each other in this very interesting way, if you have two functions of real variables they f and g such that f the integral one over the pi the principle value g here g is minus this pi here, and f and g and that to hilbert and where hilbert transfer, this is called a hilbert transforms and real and imaginary part.

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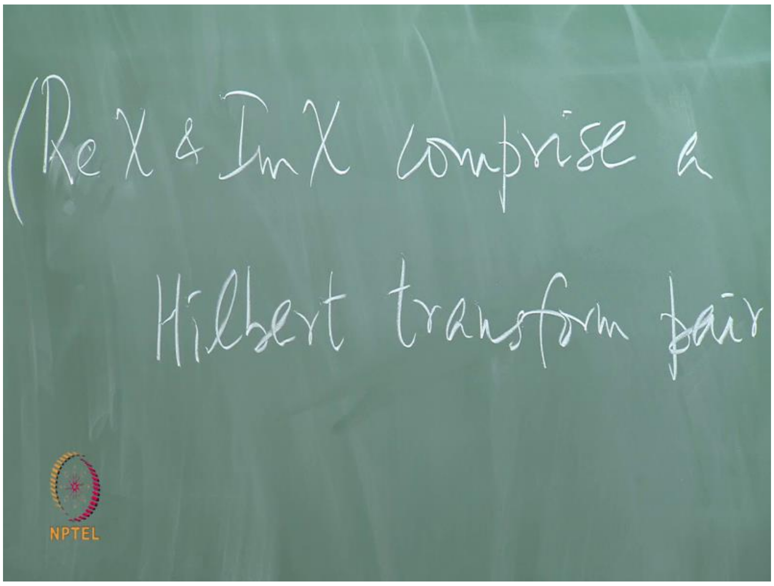

Real kai and imaginary kai comprise its interesting exercise is not a very difficult to a on a little careful to substitute this in there then we should get a function it should recruit this once again. So, once a put in area double integral terms out u can do that and the delta functions attraction providely with that another representative for a delta function incident that is not a all together exercise training that on rules. So, it is a little bit like recall in a forms what happen when the cornel of the transfers from be rot f of t was equal to omega minus i omega t and f of omega omega was d d t the plus omega t the cornel change in sign. So, this is something similar to that what made sure they got the forms of the original functions itself that is how this say this minus functions sign does pretty must it will ensure that to a delta function appropriately.