Selected Topics in Mathematical Physics

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Module - 02

Lecture - 06

Calculus of Residues (Part IV)

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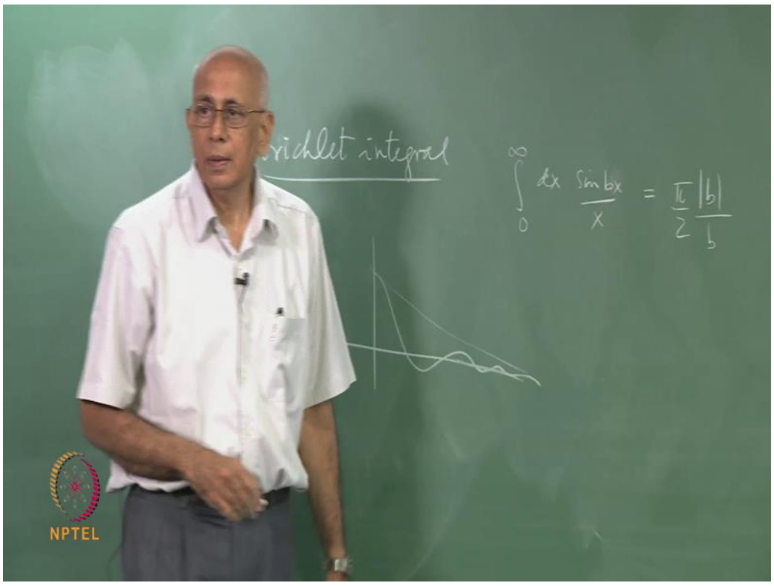

Next is to show you how you can evaluate the one more integral. Next to show you how the singularities is handled, and this is a famous integral it is called the Dirichlet integral. It is one of the Dirichlet integral and it is the following. It is integral zero to infinity d x sine x over x is not it. This integral happens to be finite, this definite integral happens to be finite. It is not absolute integral because if you put the modulus out here, to took the modules then this sign is always positive mod sign is always positive, and the term and this function being bounded, the integral is disappearing is decaying like one over x and off course d x over x diverges logarithmically at infinity. So, this integral is not absolutely convergent. There is no problem, but the origin because this thing here goes to one at your origin. So, there is no singularity at the origin, so it is the removable singularity, but at infinity you have a problem, if you took absolute values.

On the other hand, it turns out that because sign x changes sign become positive and negative in the limit they contributions will actually see to it with the whole integral is finite. Again positive and negative contributions adjust themselves, cancel each other in such a way that the answer is actually finite. It has to be brought this function it is actually the sign x over x it periodically vanishes it does this. And this envelope is going like one over x, but these positive and negative contributions add up is to give a finite answer.

And what is the value of this integral?

Student: pi by two.

Professor: It is pi over two. Actually it terms out that if I multiply this by constant let us call it b some positive constant b, the answer remains pi over two it does not change, for all values independent of b. The reason is very simple you can actually put b x equal x prime if you like and then d x over x is the same as b x prime over x prime. So, the whole things comes independent of b because the limits of integration remains unchanged.

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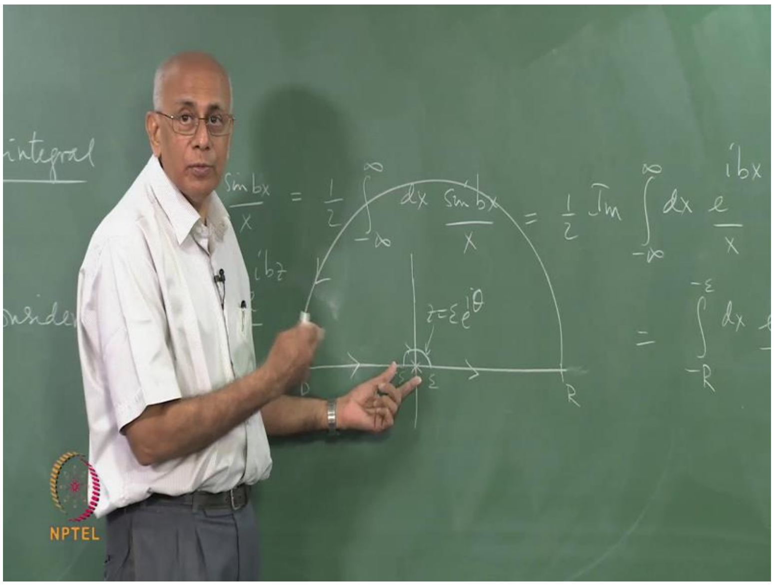

So, for all positive values of b the integral is pi over two and for all negative values of b is minus pi over two, because sign b x is than a negative quantity minus sign mod b x. So, this is pi by two over b sign of b b over mod b is better way of writing it. So, it is a discontinuous function floated as a function of b for negative values it is minus pi over to and then it jumps discontinuously to plus pi over two. So, it is one of the famous discontinuous integrals and from this integral you can derive a very large number of other integrals. But what I like to do is to derive this by the way this result is easily derive from a elementary consideration, but I would like to derive it using contour integration.

And here is how you do it the first thing to note is is to write that is to that this is an odd even function. So, write this as half integral minus infinity to infinity d x sine b x by x. Now a laying the ground work for trying to correct a close contour somewhere, but you see the difficulty with the sine is that right away you can see. What I am going to do is to have have an integral long real axis, I am going to try to close it in the upper half plane or lower half plane get a close contour value and then try to argue that the contribution from that semicircle vanishes.

But I am in trouble, because sine b x is e to the i b x plus e to the minus i b x. So, x become z and has a positive real imaginary part then one of those to exponential blows up in the upper half plane, the other one blows of in the lower half plane. So, I cannot work with sine, I cannot do that at all instead what i do is to say this is equal to half imaginary part of minus infinity to infinity d x e to the i b x over x . But you got to be a little careful here by writing this, I am actually got into trouble, because at x equal to zero am in trouble here. It is true I take imaginary part I get sine at cancels, but the moment I write this, this integral itself make no sense because at the originate band it, blows up, it is not integral be at all.

So, I need to be a little cautious about this you know what I do is the following, I consider the following line integral. Consider integral over a contour which I am going to specify on to c d z e to the i b z over z over the following contour in the complex z plane. This contour runs among from some minus r to some plus r in this fashion. Starts here, runs along here and then because this is singular at z equal to zero, I avoid the singularity by going around it makes an indentation in the contour. Go around it and go all the way to plus r, and then close the contour that the semicircle in this direction that is my total close contour C. This is an entire function this is a simple pole at z equal to zero. So, this whole integral has a simple singularity simple pole at z equal to zero, but that is outside my contour, this is a close contour which lies entirely in the region of analyticity of the integral and by Cauchy's theorem this integral is zero.

So, I start by saying zero equal to this kind and now I write this out piece by piece, this is equal to an integral from minus r to this point. So, let us call it minus epsilon, this is epsilon minus epsilon d x, because remember on the real axis z is x d z is d x into the i b x over x plus an integral from epsilon to r d x x plus an integral from here to here around this contour. But what is z on this contour, on this contour z equal to epsilon e to the i e where epsilon as an infinite decimal positive number. So, d z is going to be epsilon a to the i theta i t theta. What is the range of integration of zeta, start here and go there, I started at pi and go to zero at the direction that is an important.

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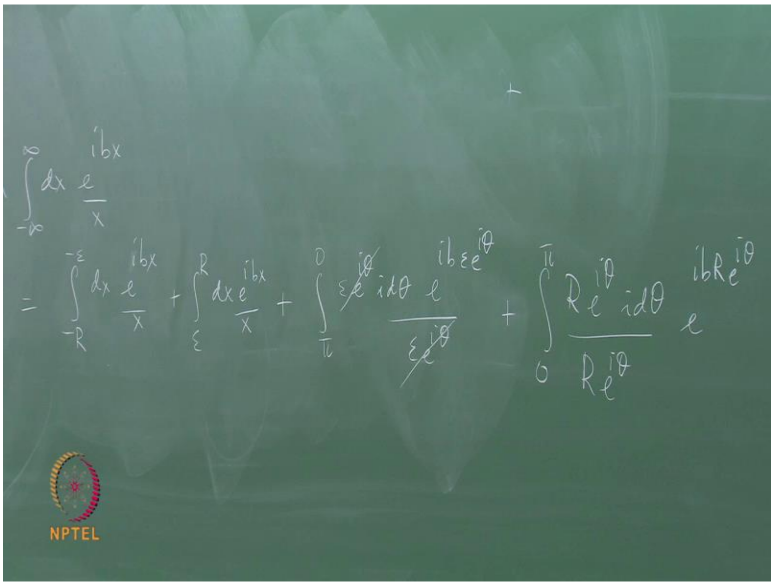

Started at pi and goes to zero, especially and then e to the i b epsilon e to the i theta divided by epsilon a to the i theta. Notice how this epsilon e to this cancels out obligingly I can go to the limit epsilon goes to zero and then this follow just goes to one out here and they answer as if minus i pi. So, this level infinite dismal portion cannot be neglected gives me a contribution minus i pi according to keep it carefully plus there is a contribution from this large semicircle on which z is R e to the i theta id theta divided by r e to the i theta this e to the i b r a to the i theta. And this integral runs this theta runs from zero to pi essentially. So, zero equal to that huge sum.

Now this thing here minus r to r with a symmetric neighborhood left out between minus and epsilon and epsilon is called the Cauchy principle value integral. So, Cauchy principle value is defined. If there is a singularity on the line of integration, and the symmetric neighborhood of this is left out and the infinity decimal symmetric neighborhood is left out what you get is a principle value integral. So, this equal to principle value integral minus r to r d x with i b x over x this thing. This p stands for the Cauchy principle value a little later.

When we look at this Pearson relation we will see the use of this concept here right now it is completely rigorous this integral exists there is no question about it. It avoids the singularity and then this portion, I am going to limits in which epsilon goes to zero. So, I might as well write it out directly this is a minus i pi then this portion this portion here this portion cancels against that and you see as long as you are in the upper half plane it means z has a positive imaginary part with a very large r there. And then the positive imaginary part gives you because of that factor r gives you a damped exponential which goes to zero in the limit as r goes to infinity.

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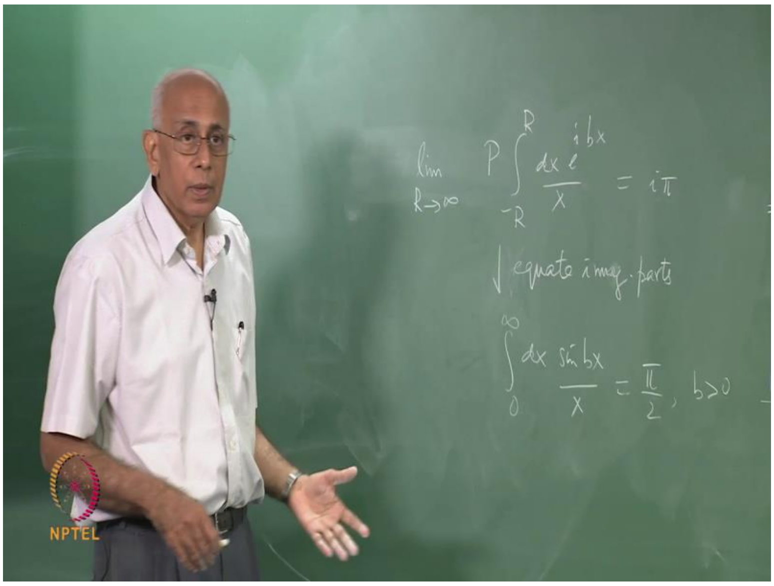

So, we could write this in this contour integral in the limit r goes to infinity tells you limit r stands to infinity principle value minus r to r d x e to the i b x over x is equal to i pi because this contribution goes to zero take imaginary parts from both sides. Equate the imaginary parts, it says principle value integral minus limit by the by limit are goes to infinity minus r to r is the same as minus infinity to infinity it is the short hand for that. So, principle value minus infinity to infinity d x sign d x over x is equal to pi, but there is no singularity in this at equals to zero in the imaginary part. So, I do not need that it is a same as a integral itself and this is an odd function. So, zero to infinity is pi over two.

I used the factor d was positive because i close it in the upper half plane and said this contribution went to zero because of this factor e to the i b is z that is only to be is positive. So, this means this is true for b greater than zero. So, be less than zero i, take out the minus sign first and then it is a same integral with a minus sign. So, this is actually told us this is value of the this continual integral. So, notice how I used contour integration to avoid the singularity add this well chosen zero here close the contour and use Cauchy's integral formula integral theorem to evaluate the integral real integral that I needed.

So, you can see from this example the power of this method provided you use at carefully and cautiously. It actually gives you a very very powerful results, this particular integral can be evaluated for elementary means actually as I said, but it shall illustrates to you. It is actually exactly how searching tables are to be handled, I could have done this by enclosing this in this fashion also could have done that nothing to stop me, but now the value of the integral is not zero its equal to two pi i times at this point.

So, I would not have got zero on the left hand side i have got some finite answer, but this contribution itself would have changed in sign because it is now pi two to two pi. So, it would have become a plus i pi and they would give you a exactly the same result as before. So, you could have closed it in either plane. I could have started with e to the minus i b x and close it in the lower half plane and intent and indent at that here where you like consider does not matter provided the do it properly guarantee to get this answer it is quite unique.

This answer by the by ones you have this answer then you can actually do many other things with it you can also differentiate this you can integrate this etcetera over b from zero to some c and when you do that you want to get an integral one minus cost x divided by that to its finite. So, whole sequence of singular integrals can be red off from the single answer here, I will give the that is an exercise you to check out.

So, let me stop here, we will come back to this trick here. When we derive dispersion relations for physical response functions etcetera, let me stop here today. And then we take it from this point, next time in pi the pi i thought of another little point in answer to this question that was asked about what is a physical meaning of this residue that the thing is it depends on the application. Depends on what the application is one common application would be that we will discover that response functions to vary a stimuli will turn out to be dependent on the frequency with which this stimulus is supplied.

So, what will do is take a general function which is stimulus force acting on a system for a transform in to frequency components and then look at the response line, response component by component. Then it will turn out that the response function has singularities in the frequency as the function of frequency, whenever this come under the as you expect, if you look at an oscillator when push it oscillator with exactly resonance natural frequency of oscillator, you know the resonance this would not appear. So, divergence will appear at the resonance frequency.

Now the residue and this will typically turn up in the response in the susceptibility frequency, dependence susceptibility it will turn up as a pole of some kind. Then the residue at that pole will give you the strength of this response in some sense. The strength of this particular natural frequency what will be the contributions from this natural frequency to the response of the system. As a whole as you know, most physical system are not simple, how many cost letters are there. Many natural frequency and at each time each distinct natural frequency, there would be if it is a discreet spectrum, you would have poles the in the some susceptibility and the corresponding residues would give you the strengths with which this response occurs and then did not all be equal to each other. So, that is one way of seeing what is residue. In the sense is acting like some kind of strength coupling strength oscillators strength, there are other issue, there are other context in which you will see as for some more example you will see what the residues look like in various cases.

So, there is no simple answer, to single answer; obviously, it is to what is it actually physically correspond to, but it depends on the physical application. So, I thought that here we will next do very rapidly linear response theory and then I will explain analytical properties of the susceptibility itself as a function of frequency and things will become a little clearer so far. So far the singularity, we looked at are poles and at a pole an analytical function is divergence is a infinite. It blows up that point we will susceptibility come across other singularity called branch point where the function need not blow up, but the function would be multiple valued and those values would co inside a some special point and they called branch point and at those points.

The function is not analytic you cannot do a Taylor series, but the function does not blow up at that point we not come across those singularities as it is, absolutely. So, for such functions, it will turn out that the single copy of the complex plane is not sufficient in order to format. You produce single value function I need to take several surfaces place them together in a specific way to form what is called Riemann's surface for the function and this multiple sheet at structure the function will be single value that every point. So, we will see that in detail, we will understand incidentally there are other kinds of singularities which I did not talk about. The many many other kinds of singularity we talked about essential singularities, but even the example that I talked about here, namely this one over sine pi z which has pose at all the integers we could ask what is the nature of the singularity at infinity well.

If it is just e to the power z then I know it is an entire function, no singularities in the finite part of the plane. And I also know e to the z is not a constant it is a function of z therefore, it must be singularity infinity the question is what sort of singularity. Well I can bring the point at infinity to the origin by changing from z to w equal to one over z then this function looks like e to the one over w in the w plane and that has not essential singularity at the origin. So, I conclude that e to the z has an essential singularity at z equal to infinity now you could ask what kind of singularity does one over sine pi z has which has a pole at zero at 1, 2, 3, and minus 1, minus 2, minus 3 etcetera. What is it do at infinity, well these poles go to an accumulation point, you must ask what do they do actually in the Riemann sphere.

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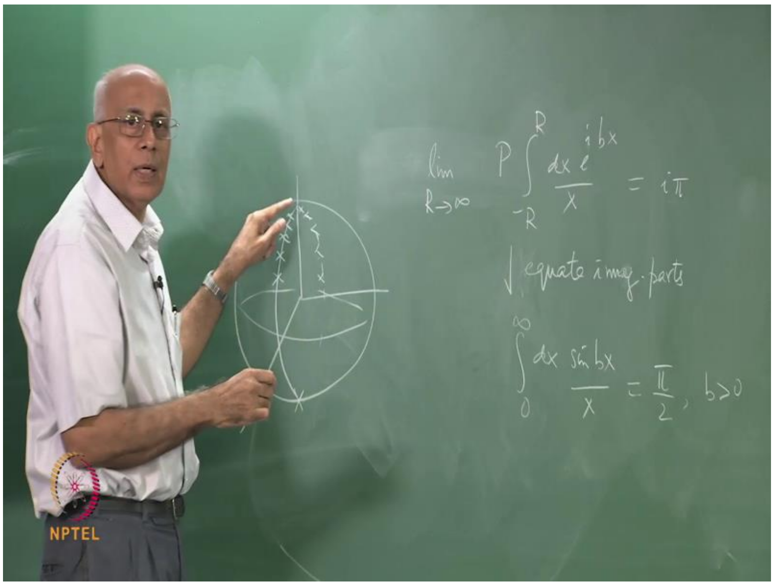

If you recall the positive relaxes correspondent to zero longitude, the negative relaxes correspondent to hundred and eighty longitude. On the other side, now you are saying that at zero there is a pole. So, there is a pole setting here and then there is a pole at z equal to pi and then 2 pi etcetera z equal to 1, 2, 3 etcetera etcetera. So, those pi's would be poles sitting here, and then here, here, here, here, etcetera till you approach this. And similarly that be poles here and they getting closer and closer as you approach the point at infinity. So, what you have here at the point that at infinity on the Riemann's sphere is not a simple pole it is an accumulation point of poles.

It is no longer an isolate singularity, because if you get arbitrary close to it there are then many poles, which are getting accumulated infinite number of them and joining up at the point at infinity, so that is a difficult singularity it is an accumulation point of poles. So, you can have very harry objects like this very complicated looking singularities at infinity, but most of the time the functions that we deal with we will deal function which have multiple value and which for which you have to define a suitable Riemann surface etcetera. We will that carefully we take carefully, but all the functions we use so far.

They have got finite number of poles on infinite number of poles, but we not dealing with contours which have enclose the finite number of poles, no limit points etcetera etcetera. We only dealing with these examples, but we will come across more complicated examples. Now one question that arises naturally is suppose you given a function who's only singularities in the finite part of the complex plane are poles of whatever order such a function is called a meromorphic function in the finite part of the complex plane. This function has at best on the poles might have no singularities at all in which case it is an entire function, but it could have poles in which case it is a meromorphic function.

And the next question that arises is can we write a in the neighborhood of a pole. Of course, for a meromorphic function, you could write this function as a singular part which involves negative powers of g minus a that a is a position of the pole. And then a regular part, but one could ask suppose I tell you all the positions of the positions of all the poles and I tell you what the singular parts are then can we represent the function in terms of it singular part plus some portion which is regular.

And the answer is yes, it turns out that from meromorphic functions, you can write the function down everywhere in the complex plane as the sum of a singular part plus an entire function which off course as a singularity. It could either be a constant or as a singularity only at infinity, and the classic case. Of course if it got just a finite number of poles like one over z minus one plus one over z minus two or something like that that is it this is already such a representation, but if it has an infinite number of poles then questions about conversions appear and the classic example of such a representation.

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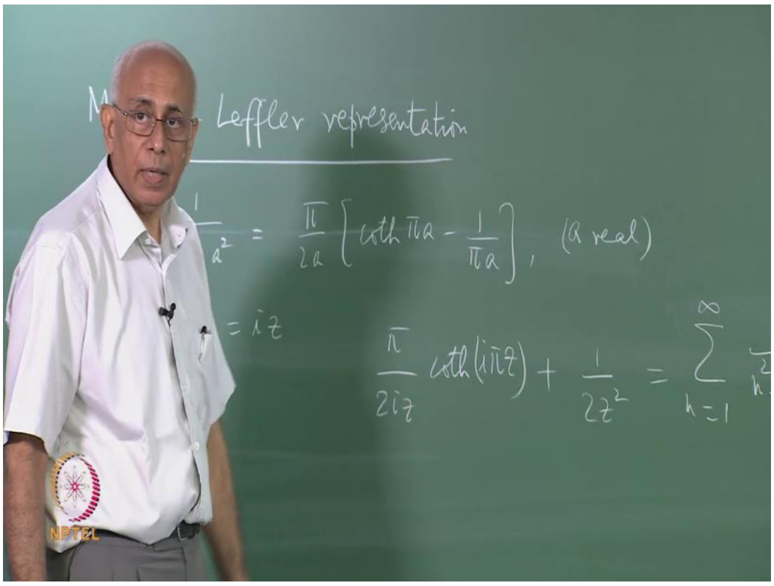

Which pi the phi, it is called the Mittag Leffler representation or series if you like the classic example of this is the function cotangent of pi z. So, let us see how that can be derived a several ways of doing so, but I am going to sort of work backwards from a result which you already know and the result which we already used or derived is the following. Remember we showed that the sum from n equal to one to infinity one over n square plus a square plus a square was equal to this famous function cot h hyperbolic pi a minus one over pi a. And I said a was a positive constant were real constant, so that this did not blow up any point.

On the other hand, regarded as functions of a both sides are analytic apart from some singularities on either sides and certainly one can continue this. And we are going to talk about analytic continuation in detail we going to write this we can write this expression down as a functional relation between two functions of a this quantity a. So, you could make a a complex variable and then you can equate this two sides. By this permanence of such equations functional equation.

So, let us do that, let me set a equal to i times some complex variable z, say then this immediately becomes pi over two i z cot h hyperbolic pi i pi z minus the pi cancels one over two z square, one over two z square. But when because it is n i there i square it is minus one. So, that is goes away this quantity must be equal to summation n equal to one to infinity one over n square minus z square is a square is minus z square here. Now wherever these two functions on either side make sense this equality is valid the, but what is this quantity it is cot h divided by cinch and that is equal to a and the numerator you are going to get cot h and the denominator you gone to an i sign pi z. So, it becomes the whole thing becomes minus i. So, let us write this out it is minus i pi cot pi z divided by two i z plus one over two z square equal to summation n equal to one infinity one over n square minus z square

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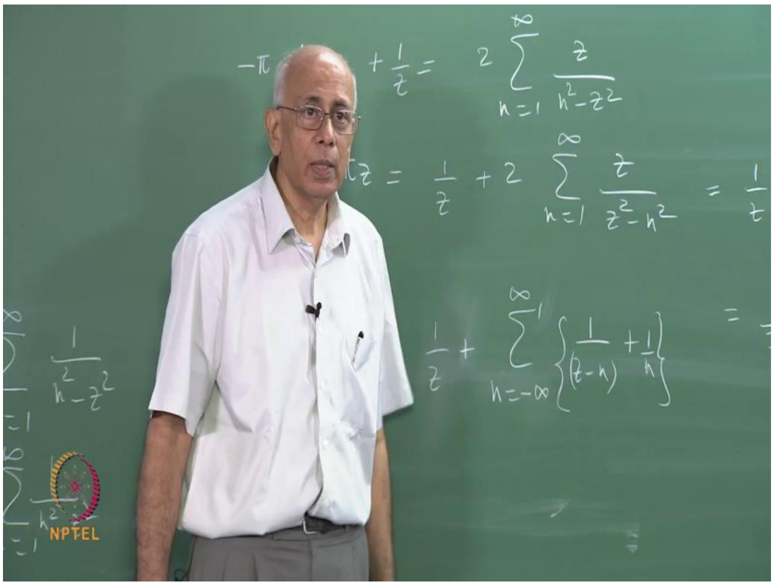

So, let us solve this for cot and then this i goes away on both sides and i get minus pi cot pi z and multiply by two z. So, this is plus one over z equal to twice summation n equal to one to infinity z over.

N square minus z square and there was a minus there. So, i want to get rid of that. So, i will write cot pi z equal to pi cot pi z. So, this becomes a minus it becomes plus on this side plus twice summation n equal to one infinity z over z square minus n square. So, i will observe the minus sign there x plus. So, that is a representation for cot pi z now what are the properties that we know about this cotangent we know that it is coth over sign and therefore, since a coth and sign are both entire functions we know that this function has simple poles at all the integer values of z including zero and this makes it explicit out here you can see that.

This function here it is z plus n and z minus n. So, at all the non zero integers there are simple poles and then the pole at zero this function here this almost looks like such a representation that it is essentially equal to if you like some over poles, but this mittag leffler representation should isolate all the poles. So, you have no business to write it has one over z square you should really write it as the pole part and some over all the poles explicitly right. So, way to do that very obvious this is equal to one over z plus summation n equal to one to infinity. For write this as one over z minus n plus one over z plus n then off course the numerator two z and then denominator this guy. And now what is tempted to say this summation one to infinity z plus n. I will change n to minus n and then of course it becomes one over z minus n, but n is gone to run now from minus infinity to minus one.

So, it looks like we are almost there this is the sum over the poles at all the positive integers and this sum over the pole at all the negative integers, and this is the pole this is the pole at z equal to zero. This looks like such a representation, but not quite because it is clear that it is not such a representation unless I can write this sum and this sum separately as to separate sums only then would it be a sum over all the poles individually. On the other hand such a some will diverge because if I just remove this and write one over z minus n then this goes like one over n. And therefore, it blows up goes all the way to infinity and the series not the convergent know.

Similarly this series to is not convergent. In other words am not allowed to take this square bracket and separate and write it as two different sums. So, here is a case where you have to first sum these two terms inside the square bracket for each n and then do the summation over values of n. So, that clear because of conversions problem how do we get rid of it very very easily what is this thing do as n becomes very large it goes to minus one over n and this goes to one over n plus one over n.

So, let us simply re write this as one over z plus summation n equal to one to infinity one over z minus n plus n minus n over n, so plus one over n plus one over z plus n minus one over n that is perfectly legit to it. All I do is to add and subtract one over n on either side, but this portion can be sum separately because asymptotic behavior of this for large n the one over n part cancels and next term is proportional to one over n squared. So, this summand goes like one over n(s) queued.

So, therefore, it is convergent all the way to infinity and ditto for this portion here. So, we immediately have achieved what we wanted we have pi cot pi z equal to one over z plus summation ten equal to one to infinity one over z minus n plus one over n in this fashion plus sum one to infinity this guy. But I can change n to minus n in that sum and it then becomes equal to this on this side. Therefore, this summation is n equal to minus infinity to infinity with a prime here to show that n equal to zero is out here, it is not included in this sum. And of course if I put n equal to zero inside the summand then this goes away and you just get a one over z out here. On the other hand, you got to be very careful if you put z equal to zero this thing cancels here. So, you could take this inside and just write it has an unrestricted sum minus infinity to infinity. So, the whole thing just becomes this is that correct did I say something I would rather do that I rather leave it like that at one over z plus a prime may be saying something which requires better examination whatever, but this is expressive.

Now it says that all that cotangent of pi z pi cot pi z is a meromorphic function. It has only poles in the finite part of the plane it has poles at all the integer values offset and those pole pieces are all sitting here every one of them sitting here this portion is not singular, but it is necessary for you to write such a representation. In fact, the most general way of writing a meromorphic function, the mittag leffler representation would be to write it as a sum over the singular terms plus a portion which will ensure convergence, which could in general be is a dependent itself plus an entire function outside.

But in the case of co tangent that entire function portion is missing it is not there and this part alone this function here, which regularizes the series is a constant does not happened to. So, happens it does not depend on side, but it is all the same, it is a mittag leffler genuine representation. So, this example shows you got to be little careful in writing it even though it looks like you got this here it is not the representation and this is in t that if you remove this you need to put this extra piece here in order to this. So, we work backward from this function to get that, but you can also start from a different view point and go back and derive this relation here.

Now an interesting thing emerges immediately if you differentiate both sides with respect to z then this side becomes pi squared over sign's squared pi z differentiate cotangent with a minus sign with a minus sign, I am going to kill out here and what happens is this becomes a minus one over z squared. And this becomes minus one over z minus and whole squared that does not get differentiate, and now you can add the n equal to zero term also along with this and this. Therefore, becomes equal to summation n equal to minus infinity to infinity one over z minus n whole square.

I want you to be surprise by this relation it is a remarkable relation because here are algebraic functions z minus n whole squared this was a double pole at all the integers including zero and it is just a sum over those double pole terms and nothing else in that turns out to be sign function function. So, this series, this series here is a starting point this or that the starting point for a large number of other developments in this. But you can see how a complex variables enables you to get hold of such powerful representations here something which is and the left hand side transiently function right hand side, just a sum of rational functions and the two are exactly equal to each other. So, this formula should you should over it for a while see how remarkable this formula is there many many other such instances and lead to what is called Eisenstein series, they emerge from this, this is from one static point for such developments.