Selected Topics in Mathematical Physics
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Module - 2
Lecture - 5
Calculus of residues (Part III)

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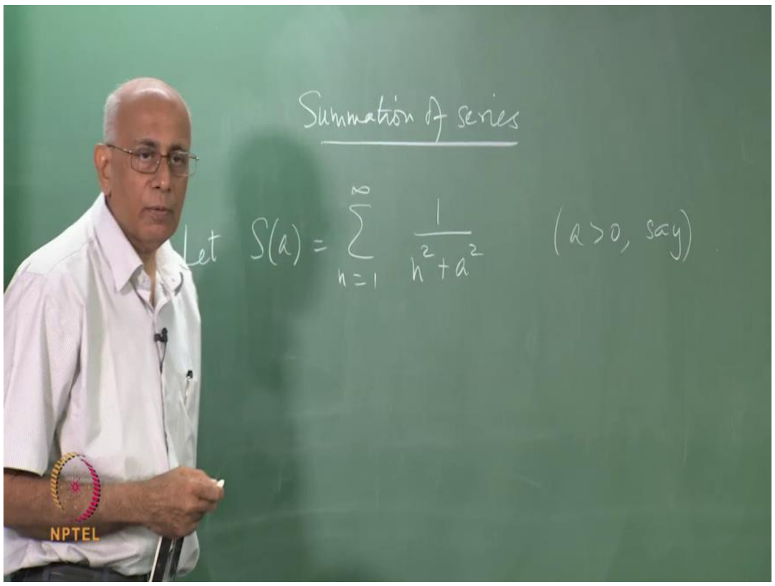

Now let us do the following. Let us look at evaluation of some series - infinite series, we looked at recursion relations. Let us evaluate some infinite series numerically by using again Contour integration. The most famous of this is the zeta function. So let's do that. One is called summation of series and what we like to do is to evaluate this series n to infinity one over n square that is the data of two and values phi squared over six. Now what we like to do is, what we like to do is not just evaluate this series, specific series, but evaluate also all the other even zeta functions, like one over n to the four n to the 6 and so on and so forth.

So I would like to use a little trick to do that and evaluate not this series, but little more general series than this. So, let us evaluate one over n square plus a square, where a is some real constant. Real or complex doesn't matter at this stage, but let's call it as real constant, some real positive constant so a greater than zero just to be specific say like to some this series as a function of a. So let's call it S of a equal to this. I am need to compute what this quantity is as function of a and get a close form answer if possible, and from that if I said a equal to zero, I should able to get zeta of two. Now one-way to do this is to observe that this is an even function of n. So if I make n go to minus n then nothing changes here.

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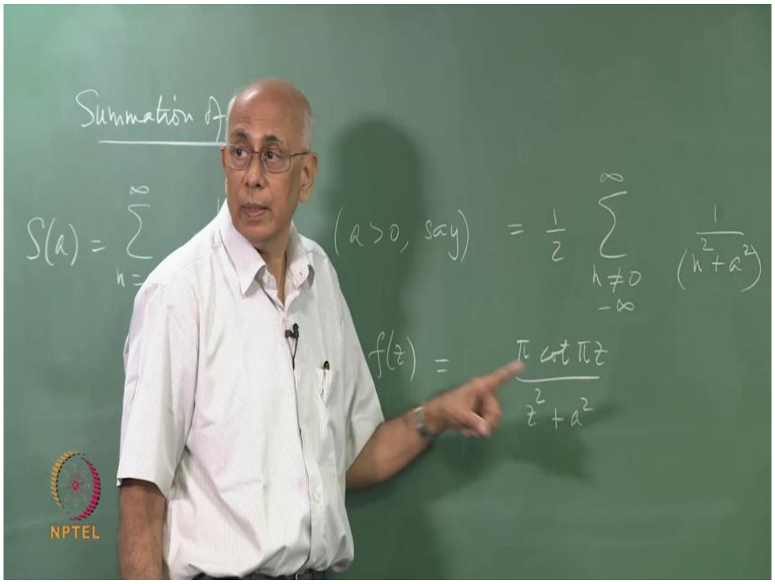

So I could immediately write this as equal to one half summation n not equal to zero and minus infinity to infinity one over n square plus a square. I just add all the negative power n contributions, in the same as the positive n contributions. We will see why I am doing this in a minute ok. The next step is to say ah it will be very nice if this term appeared as the residue of some function of z – the complex variable z, at the point z equal to n. ah I want precisely one over n square plus a square to appeared as a residue. So I need a function, which is analytic as nice analytic properties and has a simple pole at the point z equal to n, where n is an integer and the residue should be equal to 1.

But we already know such a function, we already know that if you look at this function pi cot pi z is function, has precisely that properties, because this thing here is equal to pi cos pi z over sin pi z, and it goes as z goes to n any integer n. This thing has a zero simple zero, so it’s got this whole ratio got a simple pole. And this goes to pi times cos n pi that is minus one to the power n divided by z minus n times the derivative of this function at z equal to n, which is pi times minus one to the power n plus regular part. So these factors cancel out and you have precisely what do you want, so this function pi cot pi z has a property that it has simple poles at all the integers including zero – positive and negative integers, and at each pole the residues exactly one, ok.

So let's consider, therefore, a function f of z which is equal to pi cot pi z divided by z square plus a square, consider this function. What are its properties, what do we know about this. Well, we know that it has simple pole at all the integers in z with residue one, but then this fellow also, so the residue is in fact the residue is one over n square plus a square at the point which is what we want. Ok, So, it immediately follows that you can write this as equal to half and then when I want to compute the residue theorem, I have one over two pi i times something is the residue.

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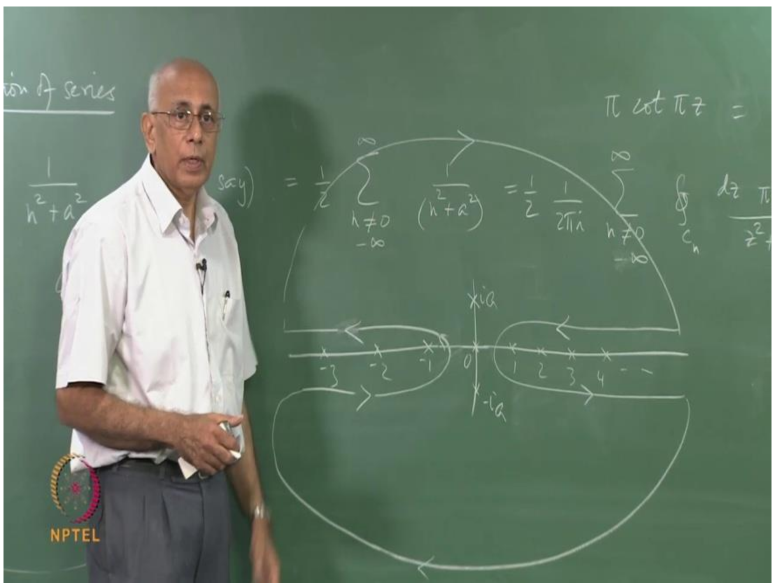

So I want to one over two pi i summation n not equal to zero minus infinity to infinity and integral d z pi cot pi z over z square plus a square. I have to now specify the contour, in the contour I'm going to specify is the following. Here is one, here is two, here is three and so on, and is the contour around each of these guys in the positive direction, leaving out zero, minus one, minus two in this fashion.

Let's call this is contour c three, this is contour c two and so on. And this thing here is c sub n, the small circle around each integer, traversed in the positive sign. You guaranteed that that's equal to this, because each of these integrals is two pi that cancel these times the residue of these functions at z equal to n, the residue of this part is one and this thing here is just one over n square plus a square.

So S of a is in fact equal to that contour integral, except that this contour contains many pieces now. But now you do the reverse tricks, you can open out this little more, you can bring it out here, you can bring this guy out here and joint these two pieces in this fashion, because the intersecting portion the two ends will cancel each other and you can merge these two in this fashion. And you can merge the next one and next one and so on and so forth.

So eventually, this whole things 1, 2, 3, 4 etcetera. It is a huge contour like this, going all the way to infinity going this side, and all of them traverse in the positive signs, so it's in this fashion. Likewise, you can merge all these pieces together, minus 1, minus 2 minus 3 etcetera. And this is another hairpin contour, which goes like this and this one again is in the positive sign, so it's traverse like this, that's why matters stand right now. ok What do you see, I can define after all what do I mean by an infinite sum, it some capital n and then the limit as n goes to infinity. So I can stop at some capital n and then let n go to infinity, ok that's what is meant by an infinite sum in any cases.

Professor: Now the point is, yes.

Student: ((Refer Time: 08:16))

Professor: ah, I made a mistake. Well, each of these guys goes this is the positive signs always, so this is how it is here. And this one is again ah yes, absolutely, absolutely, this way, you are right, thank you. it goes in the anticlockwise or positive signs. So now what I do is, I join two pieces together; So I add that big semicircle in this fashion. Ok So this goes out and comes back on that side and similarly here, this guy goes out and comes back in this fashion.

Student: ((Refer Time: 09:08))

Professor: The question is hm?

Student: There be poles at plus or minus i e.

Professor: Pardon me.

Student: There will be poles at…

Professor: oh yes, oh yes, there are poles, the pole here at i a and the pole here at minus i a and a pole at zero. They are all simple poles, because this as simple pole at the origin, which were left out in the sum, this has a pole at plus minus i a definitely, so they are three of the poles sitting here. Ok, So all of them are simple poles, but the point is this contribution and this contribution in the limit has capital n tends to infinity is adding a well chosen zero, because the integrant, you have to make sure of that. You have to make sure that the integrant actually vanishes at these points, because you are adding these ((Refer Time: 09:55)), it's better be a zero that you are adding in the limit, It is because this quantity goes like Capital R, this goes like R square, so this is one over R and what does cot pi z d? You have to well very careful here, because cot pi z is of the form cos over sin so it is e to the i pi z plus e to the minus i pi z over the same thing, e to the i pi z minus e to the minus i pi z.

So, whatever zeta, have remember that z is actually imaginary over here, so these could be blowing up exponentials, but the interesting thing is no matter where you are in the complex plane. If some exponents blows up, this another one compensate and down here. So cot pi z remains bounded always, unlike it will pi z which will blow up, it goes to positive real values, if the minus pi z will blow up, z goes to negative values, if the i pi z will blow up, z goes to negative imaginary values and so on. that doesn't happen with this this cart. It's always bounded and since this is always bounded and this guy goes like one over r the guaranteed that these contributions actually vanish in the limit, hm but once you glue it on this is a closed contours, and you can now do the reverse tricks and shrink it back. You can shrink it as far as it will go.

And what is this going to become so it is going to become this is a pole here, pole here, pole here. And it's got to enclose it in this sense, in this fashion, so it's in the positive sense. But that is the same as saying that it's enclosing this guy in the negative sense, this going to the negative sense, this going to the negative sense, all of them enclosed in the negative sense. Look at now, you can pinch of the intermediate portions and have just the three residues from these points. So this S of a is given by that complicated expression, but it's just the residue at zero plus i a and minus i a.

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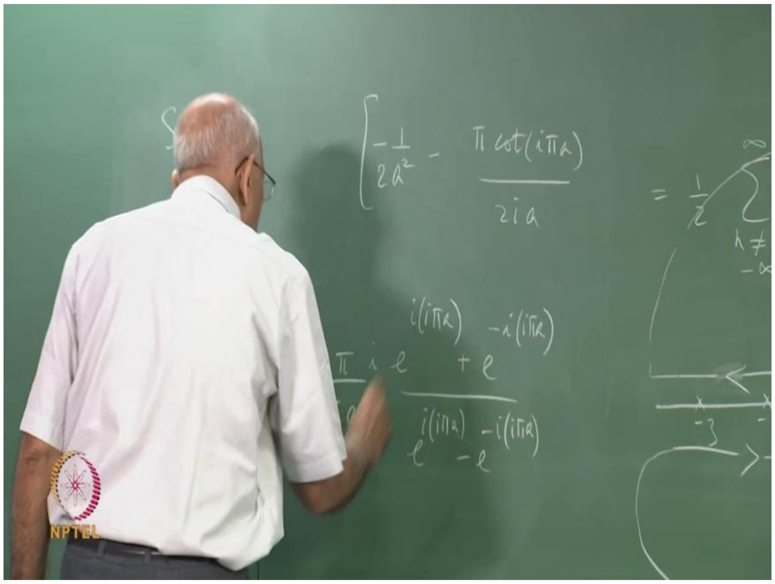

And now look at it’s fairly straight forward now, because S of a equal to one over 4 pi i and then each of these guys is in the negative sense of the minus two pi i times the residue of these /// things. So let’s put a pi, the residue at the origin. But the residue at the origin of pi cot pi z is one – always, because this thing here has simple poles at all the integers with residue one, so that’s the one there. And then you put z equal to zero, you get one over a square from the first term hm plus the residue from z equal to plus i a, so everywhere except the factor z equal to plus i a, you put in z equal to i a. ah So this becomes pi cot i pi a divided by two i a, because all I did was to write it like this, z plus i a z minus i a. So when I’m looking at the residue at plus i a, I multiplied by this factor and put z equal to i a, I get it two i a from this plus pi cot minus i pi a over minus two i a. Now I multiply by this factor and put z equal to minus i a, so that gives me minus 2 i a.

But the cotangent is an odd function of its augment, so it’s just put a factor two here, and I can get rid of this. And now we can start reading out what is happening, this is equal to I should be careful now ah so this gives me a minus half, so one over two a square minus sign and then the minus of, so it become a minus and the half goes away in this fashion. So it is equal to minus one over two a square minus pi over two i a, and then cot i pi a that is equal to cos that is e to the i times i pi a, so let’s do this slowly, so I get making no mistake, e power minus i times i pi a divided by two and then the sign is going to be e to the i i pi a minus e power minus i i pi a divided by two i, so it is going to be pull and i on top. And that i will cancel against this i.

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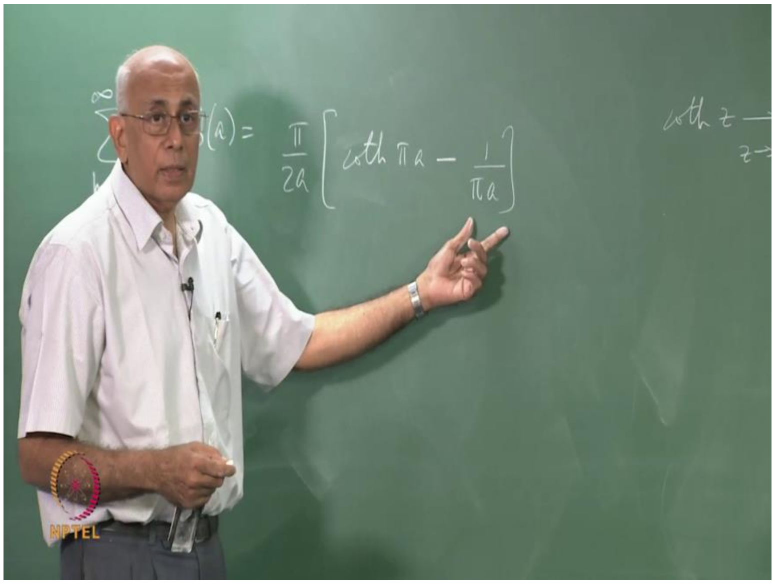

And the next thing is this is e to the minus pi a this is e to the plus pi a and the same thing happens here, e to the minus pi a minus e to the pi a, but the minus sign here and this minus sign cancel against this. So let’s pull out the one over pi over two a times once these two minus signs go away, this is equal to cot hyperbolic. So this is cot hyperbolic pi a minus, I pulled out pi over two a, so I put that back, one over pi a that is the answer, that is the closed expression for arbitrary, for real a, but we can now analytically continue in a itself to all sorts of (( )). So it gives a closed expression for summation n equal to one to infinity one over n square plus a square. It is a very famous function, by the way this function appears in a physical context, you familiar with this, where it is appear. Pardon.

Students: (( ))

Professor: It is a (( )) function, it is appears in a very simple model of paramagnetism. So we know all about this function properties, for real augment, but this is now valid for all sort of (refer time: 17:00) augments, complex augments included. What does cot z do as z goes to zero, what do you think it does, cot hyperbolic. Well, it is cos over sin, but cos is regular, it is (( )) at that point. But the sin hyperbolic is singular, it has a pole, it has a zero. So this cot hyperbolic has a simple pole. Just as cot z as a simple pole at z equal to zero, so that cot z as hyperbolic and z equal to zero.

So the question is what is the leading term, we need to find out and then it must cancel against this, because I am going to let a go to zero, I know when a is zero, this is finite. You know the value of this is pi square over six, which is what we are trying to prove. So I need to know what is this guy do at z goes to zero. The leading term has to be proportional to one over z, because it is a simple pole in that point. What will be the next term in the regular series, proportional to what power of z. Will it be proportional to z to the power zero, or z or z square what do you think.

What I am saying is this is equal to some residue some residue divided by z, plus regular part, and this part will always have this behavior a n z to the power n, n equal to zero to infinity any of the origin. My question is what is going to be the first term proportional to what power of z.

Student: Z to the one

Professor: Z to the one, why, why not z to the zero? You are right, why not to be the z to the zero term?

Student: Because if you do (( ))

Professor: Yeah it is true. Why can you say what is the reason you can say right a way there cannot be a z to the zero term. What sort of function is cot z, even or odd function?

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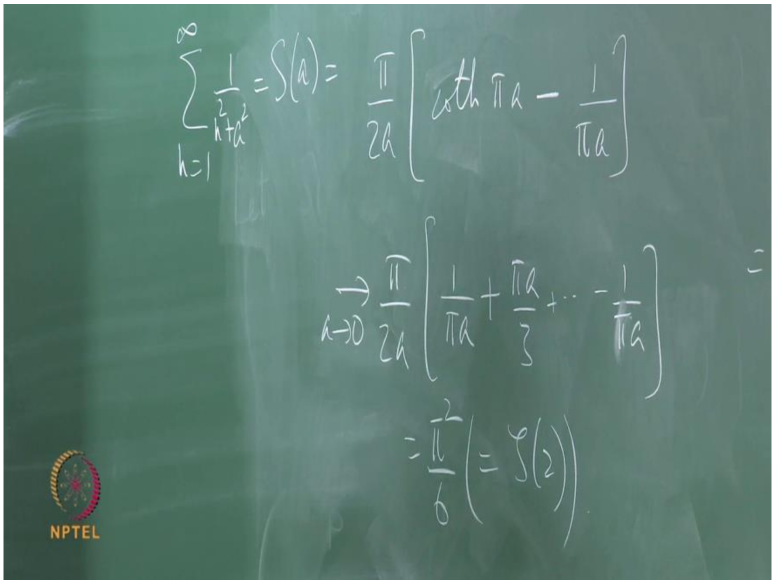

Student: It is an odd function

Professor: It is an odd function, so it cannot even power of z. It is an odd function. So the first term must be proportional to z. Let us find out what is the functioning is, well this thing here is cos, you can sit down and compute the power series term, but it really cos z over sin h z which is equal to one plus z square over two plus dot dot divided by this guy here is z plus z cubed over 6 dot dot. So take out of z, so one plus z square over 6 etcetera. We want the leading term proportional to z, so this thing you take it upstairs and it becomes 1 minus z square over six, the whole thing divided by six. And now you combine these two fellows all you have to do is to add half and subtract one-sixth, that is equal to one-third. So the whole thing goes like one plus z square over three plus dot dot divided by z is equal to one over z plus z over three plus higher order terms. So the leading term is z over three.

By the way in this theory of paramagnetism, this three will appear in the denominator, and you have this susceptibility in the (( )) law proportional to three K t in the denominator that three is number of dimensions in space in that application. So this whole thing goes like pi over two a, a goes to zero of one over pi a plus pi a over three plus dot dot minus one over pi a (refer time: 21:00) which is equal to pi square over 6 that is f of two. So rightly gives you the (( ))ugh at a equal to zero, it says rough to the value pi square over six.

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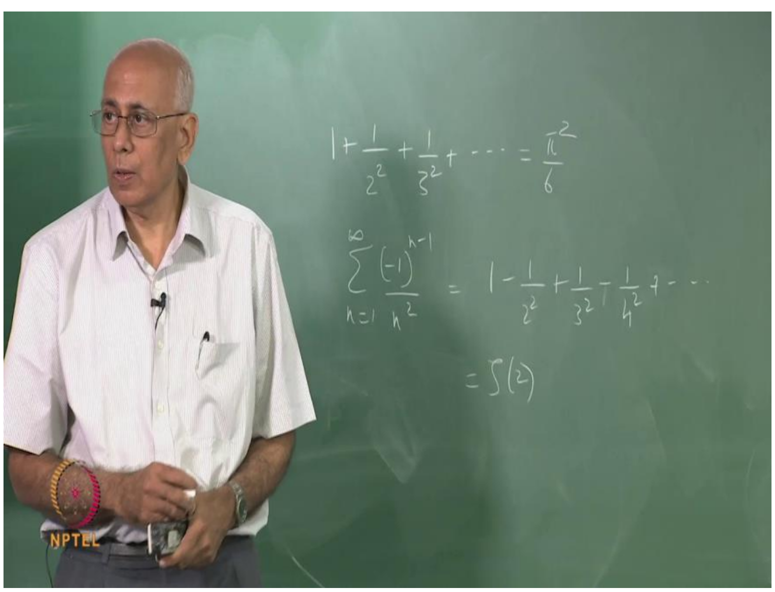

Let’s go a little further this, what would you do, if we had instead of this series, suppose would have this series, n equal to one to infinity minus one to the power n over n square plus a square. What would you do then, there are series which alternate after all then sign. Whenever it doesn’t alternate, this pi cot, pi z was very nice, but what would do you do, if we had minus one to the power n, sitting there. Clearly the way to do this is to look for an analytic function, which has poles at all the integers, but the residue at each poles is not one, but minus one to the power n.

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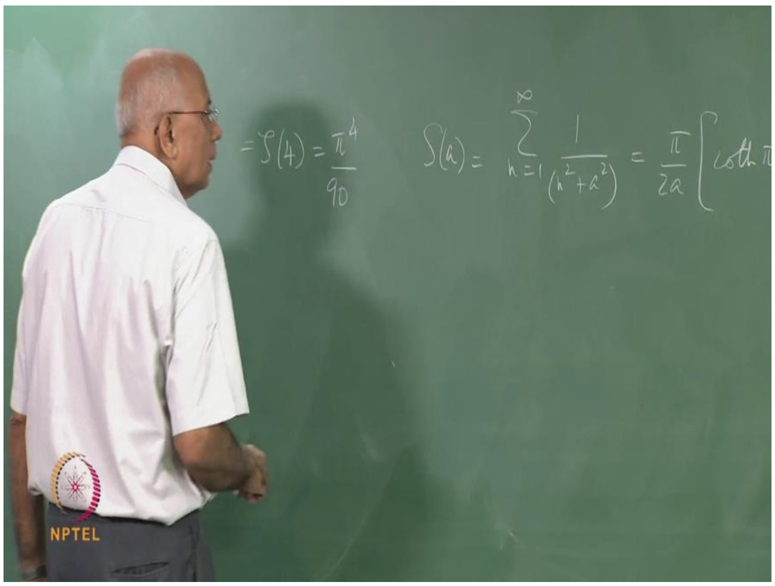

On the other hand, if you did this what would happen then, what sort of analytic functions would you get, this would become at four, plus a four. And if you did that trick, as we did there, what would happen now. How many poles that this function have other than the pole at zero of course, so when do the contour tricks and you pull out test of few contributions, you would have a simple pole due to this at the origin. All the poles at minus one, minus two etcetera outside of the contour in this side, but you would have to 4 poles due to this kind. So at all the 4 through it stuff unity in multiplying a, you would had poles. And you have to evaluate the residues at those points, so that be one here, one here, one here and you have to evaluate those residues. So either you have to evaluate five residues at simple poles or you evaluate a fifth order pole at the origin, I think it is easier to do this format.

Can you think of any other trick by which you can find one over n to the power four, do you think of any other trick? Given this, s of a, so you have to constantly look for tricks. Now you know the answer here, pi over two a and some cot hyperbolic in (( )) function. Given this can you think of anyway by which you get at this.

Student: (( ))

Professor: Pardon me.

Student: Differentiate with respect to a.

Professor: Differentiate with respect to a, exactly. You that other parameter, differentiate with respect to a. Then this become n square plus a square the whole square apart from some other factor minus signs and so on, and then you put a to zero. You can differentiate this with respect to a. You differentiate once again you get one over n 6 and so on. So from this S of a, by continue differentiation with respect to a, then letting the limit a go to 0, which you have to do carefully, you can evaluate the zeta function at all the even positive integers. This terms out to be zeta 4 and so on. What is the zeta of n approaching as n becoming large, what is the zeta of k (( )).

As k becomes larger and larger positive, what is the series approaching, what is this approach?

Student: One

Professor: One, it is approaching one, exactly, just the first term, everything else is going to 0 very rapidly. So as you can see, you had pi square over 6, how big is pi square around 10, so it is like ten over 6. Pi 4, how big is pi 4?

Students: (( ))

Professor: Very close to 100, 90, 100 whatever. So those much closer to one in this side from (( )) is the term, because these are positive terms, so this thing here tends as k tends to infinity to one. And like to point it out, to plot this, the function of k, at one it is infinite, and of course it tending to a. It is rapidly approaching, very rapidly approaching value one. We will write down what this zeta of k is or zeta of z is the function of z and then analytically continue it to the left of z equal to real z equal to one, little later of course.