Selected Topics in Mathematical Physics
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Module - 02
Lecture - 04
Calculus of Residues (Part II)

You are familiar with recursion relations in coefficients linear recursion relation. So, you got set up coefficients C 1, C 2, C 3, C 0, C 1, C 2 etcetera. Let us say that satisfied recursion relation typically at two step recursion relation that is like second order differential equation, this case difference equation which cannot who solutions cannot by written down by inspection.

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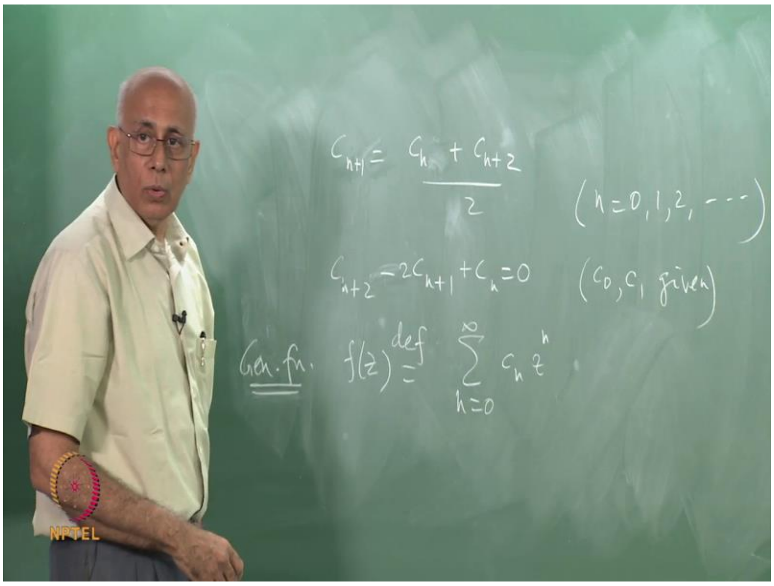

So, let us take typically example. Let's suppose C n equal to C n plus 1 plus C n plus 2 divided by 2, the arithmetic mean of n plus 1 equal to C n; n is 0, 1, 2, 3. It is a second order difference equation. So, to solve it uniquely, I need to specify some initial values. In this case, I need to specify two coefficients C naught and C 1, and then the rest of them all determined by using this recursion relation. The question is what is the general coefficient C n, in this case. Well, in this case, we can guess the answer what do you think it is going to be pardon me.

Student: (( ))

Professor: Yeah, it is an arithmetic. So, each value is the mean of the values on either side of it. So, it must be the linear function which means must be form a and plus b and then of course, you put a C 0 and C 1 you can what a and b also, it is very trivial thing. It is equivalent of harmonic function in the different case, but let us see how to prove this using complex analysis. So, the first thing you do is to say that C n plus two minus two C n plus one plus C n is equal to zero. So, C 0, C 1 are give that is a recursion relation.

And then next thing to do is define what is called a generating function which we will be used very extensively in various other context as well. So, let us define f for z define this generating function has defined as n equal zero to infinity C n z to the power n. So, introduce this complex variable z, and use you are coefficients to multiply z to power n and define this function z. And then multiply both side of equation by z to the power n and sum over n equal to zero to infinity to get equation for f of z.

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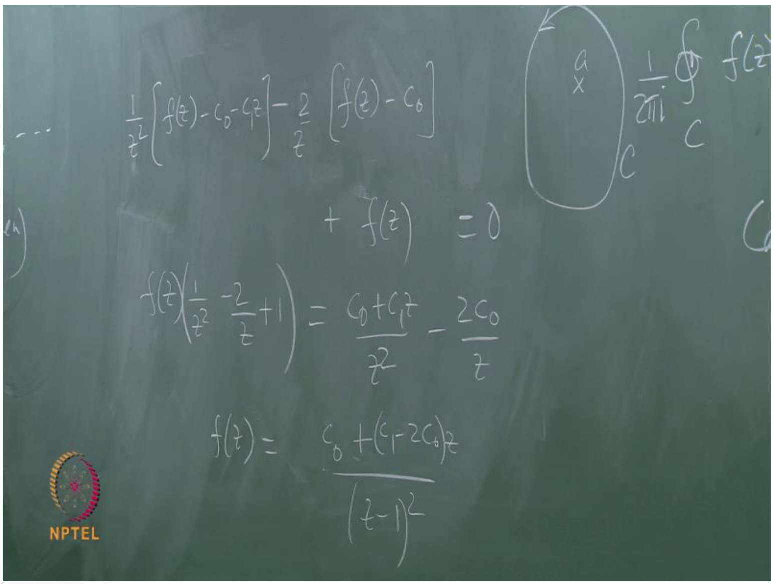

What happens, I have summation n is equal to zero to infinity z to the power n C n plus two minus twice summation n equal to zero to infinity z to the power n C n plus one plus summation n equal to zero to infinity C n z to the power n that is equal to zero on this. This is of course f of z itself as it stand. So, I replace this by f of z and what this equal to had this been z to the n plus then I could said something, but it is z to the power n so obvious thing to do is to multiply divided by z. So, it is two over z and then transform one to an infinity instead of zero, so it is f of z minus c naught. So, this is just f of z minus c naught on this side and ditto here divide by z square and you get one over z square, f of z minus C naught minus C 1 z this whole thing is equal to zero plus this equal to zero.

So, this give you what f of z is we can solve for f of z, and you get f of z into one over z square minus two over z plus one that is equal to the right hand side you have c naught plus C 1 z over z square minus two c naught over z that is it. So, it says f of z equal to this is one minus two z plus z square. So, it is one minus z the whole square, and on that side, you get C naught plus C 1 minus two C naught z divided by z square, and the z square goes away on both sides, and you z minus one whole square. So, you give me a C naught and C 1, I put that in and I discover this generating function, is just the rational function, the ratio of two polynomials in z, but this is a representation of analytical function, it is a Taylor series here.

So, what does it implies for C n, it is a C n equal to the coefficients of f of z the power to n in the expansion of f of z, it is Taylor series. So, you could write it in several ways, one way is to say that it is derivative of f of z one over n factorial, n eth derivatives f of z at that point. That is bit of new sense trying to find n eth derivatives of this function, unless this is zero in which case this is just binominal; otherwise, it is new sense finding it. So you have to find out, what the inverse formula here is. This is equal to one over n factorial d n over d z to the power n f of z, at z is equal to zero. When the better way of finding out, what this function is what this quantity is then what would that be, I want to extract this c and pi an integral formula, what should I do.

Remember when I wanted to extract the pole, I multiplied by power of z, but now I want to do integration, I want to integrate f of z times something or rather, so has to extract this C n, I must divide by z, by what power?

Students: n plus one

Professor: n plus one, because this n will go away, and give me precisely this coefficient C n. So, you can also write this as equal to one over to two pi i integral around the origin d z over z to power n plus one f of z, so that is an alternative representation. Incidentally this thing is true for any Lorentz series even if this had negative power, this formula still true, but that this is not any longer. Once you have singularity at the point, it is not differentiable at that point, but this formula is true as the inversion formula, even for

Lorentz series, even there are negative powers of z or z minus a present, it is still true and all we have to do is now is to substitute this in here.

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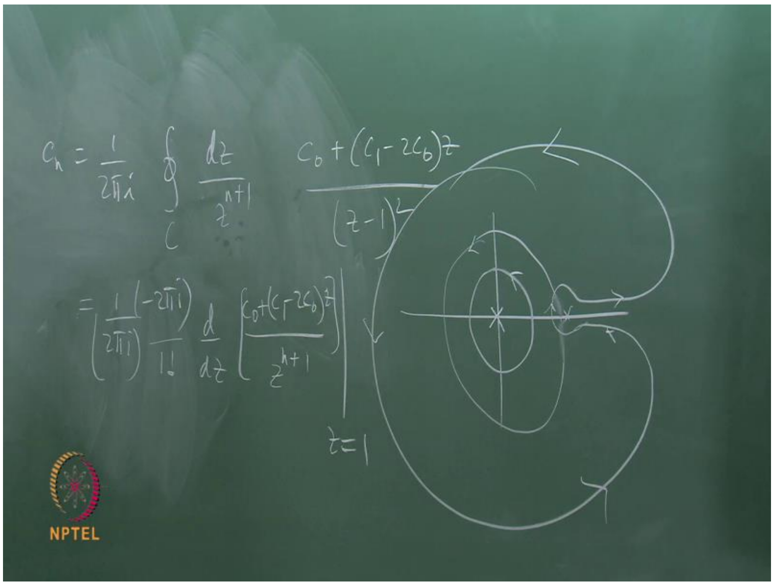

So, it immediately tells you that C n equal to one over two pi i internal d z over z minus z to the power n plus one around the origin of f of z which is C naught plus C 1 minus two C naught z over z minus one whole square. This is the integral we have to do. And now comes the power and rushing through this, but we will do this slowly and carefully we should try this out. What is the integral over. So, in the complex plane, you have a pole of order n plus one which is here and you have a contour which n circle that singularity and nothing else, but this function has the singularity somewhere else, where else it is singular.

Students: (( ))

Professor: At z equal to one, it is singular at z equal to one. And what sort of singularity does it have at z equal to one, a double pole because it is square here with some residue etcetera we do not care. The residue will have contribution from here too, we do not care. So, there is a double pole sitting here and this pole of order n plus one sitting here. I am trying to differentiate rational function in times. Now if you try to do this integral directly I need to find the residue at z is equal to zero. So, I multiply by z to the n plus one and then differentiate n times, but I have to differentiate this ridicules function n times numerator and denominator at z is equal to zero which is a tedious task.

On other hand it would be so nice, if all we have to do first to evaluate residue at this point because at this is double pole, I differentiate only once. What should we do, we exploit the rubber band property of this function, this guy here, this is the contour, I start with, but I can deform this contour, it still true, this thing is still true, it is still the same thing. I deformed it little more make this even bigger, but (( )) I cannot cross this fellow, so come here. So, I keep the sense on the contour very carefully, it is still true, it still equal to this and keep bloating it up always paying attention to the factor that I cannot cross this similarity. Till eventually I am guaranteed that integral is precisely the same here ones, here singularity sitting here.

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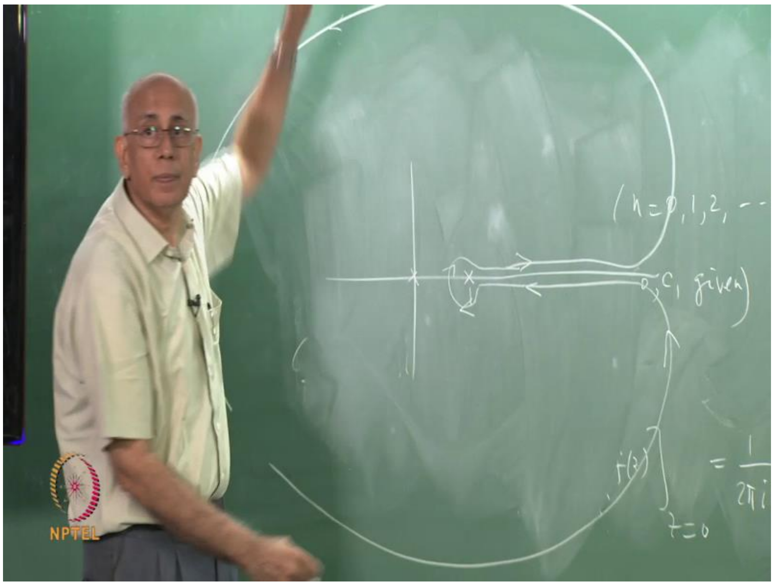

As on integral, which comes along does this goes all the way up arbitrary for away, and goes back and comes down once again. All the while, I keep track the sense, it is equal to that and I let this limit becomes infinite and what is the contribution due to that well z is to some capital R e to the i theta on that big circle. There is capital R to the n plus one here that is capital R there. So, let us count those power that in R that is R to the n plus one and then that the capital R from this guy one more power and then there is R square from this fellow. These guys cancel and you have R to the n plus one where n is zero, one, two, three etcetera.

What happens to this is r tends to infinity, goes to zero. So, this entire contribution disappears then no singularity anywhere on this line. So, you can slip this off and these two can contribution cancel each other. And you left with the small circle around z equal to one, but traversed in the counterclockwise, in clockwise sense, in the negative sense. So, this thing here is equal to this whole integral equal to one over two pi i minus two pi I, because it in the clockwise sense now times the residue of this function at z equal to one. But then it means we must take this function multiply by z minus one whole square and differentiate once, because it is a double pole.

So, it is one factorial times d over d z C naught plus C 1 minus 2 C naught z over z to the power n plus one evaluated at z equal to one and that surely we can do. It is a trivial thing to do. And then of course, you have expression for C n general coefficient. So, you are see you avoided a whole lot hard work in this sense, always path of these resistance in this case, but you see the power of contour integration. How you are able to convert the integral over this n plus one eth order pole, which involves differentiation n times to evaluation at the other pole, which is just double pole at this point.

In particular is C naught is one and C 1 is equal to two so you start with one, two etcetera then this portion goes away and you just have one here and would differentiate this once. So, that is gives you minus n plus one the minus sign is cancel and you end up with C naught equal to one C 1 equal to two implies C n equal to n plus one a linear function as we knew all along. And of course, n plus one is the arithmetic average of either side of n plus n on one side n plus two (( )). So, this is, but you can evaluate this for arbitrary evaluation of C naught and C 1 in this case.

So, solving this linear recursion relation becomes fairly trivial. Once we know the generating function method provided, you realize that you never try to find the residue at this pole at this higher order pole directly, but you always open out the contour and find pickup the residue from the other poles. Then of course, it becomes a straightforward (( )). So, you might be used to solving this recursion relation by assuming function for them, assuming answer for the them that is like finding writing down trail solution based on superposition of exponential for ordinary differential equations with constant coefficient. But that this is different analog of it, but you do not need that do not need that with generating function right away to find the exact solution no matter what.

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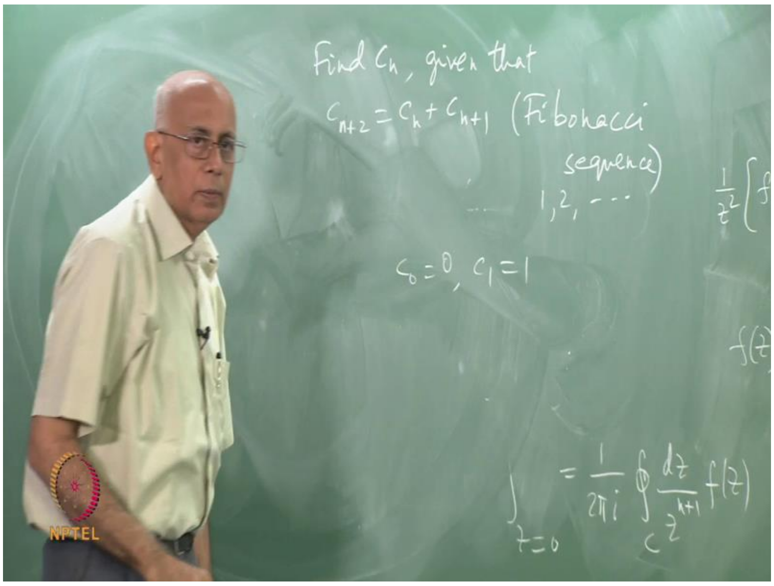

So, I read you to find I am going to give several exercises. In particular you can amuse yourself by writing down for this thing C n plus two is C n plus C n plus one this is the Fibonacci series. Find out what C n thus in this case. So, you have to give me what C naught and C 1 are. So, typically functions start with some simple value. So, C naught equal to zero, says C 1 equal to one, yes incidentally this Fibonacci series which are familiar with per each term is sum of the preceding two terms. What do you think is the growth of this C as an increases. Do you think it will be a power on n, clearly the C n increasing with n.

The question is as the increasing like some power of n not something. Suppose you took this Fibonacci series and you try to sum this series to finite some number of terms by the method of differences what would happen in this case. Well, you familiar with the method of differences which you learn in school. You have sequences series and then you take the first difference between the terms, second difference and you wait till you get constant on right. And from there, you work backwards. What happen if you take in the Fibonacci sequence, you take the differences, what you get, you get the same series over again; you take the next differences, you get the series once again. Now this means that whenever you have taking differences is like differentiating, you have a function which you differentiate and you get the same function back. So, what sort of function is it, exponential, it is an exponential function, it is cannot be power.

If I have function n cube function n cube, then the next difference will be n square and then n and then n to the zero. But if I have e to the n, how many differences that you take will remain the same thing here, so that is why the method of differences fails for the Fibonacci sequences. And this immediately tell you that this relation itself tells you that there is no way that C n is going to increase like any power of n, it is going to increase exponentially fast. And the interesting question to ask this how fast what this exponent. So, if goes like e to the lambda n for large n, what is this lambda that is an interesting thing to find out. So once you find c and explicitly of course, you have the answer directly, so I leave you to do this. And what we would like find out explicitly is formula for c sub n from which you can read off how fast this exponential growth occurs.

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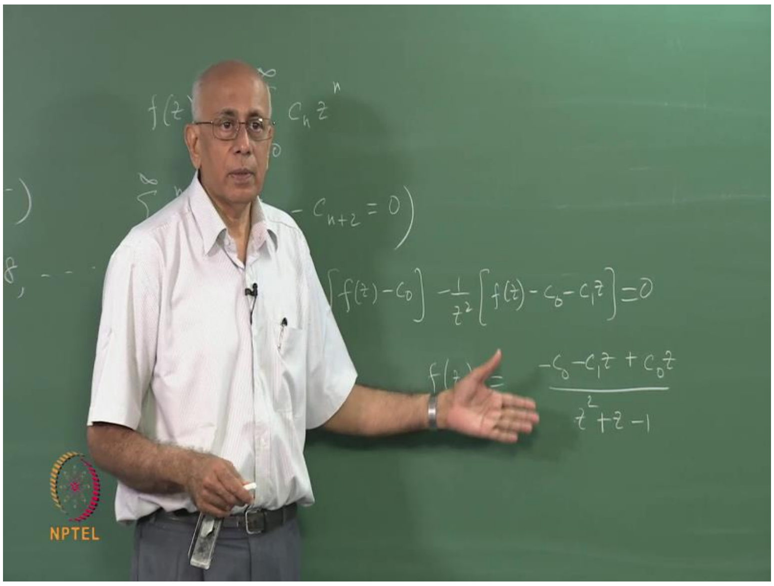

So the way to do this is to define the generating function. So, define f of z equal to summation n equal to zero to infinity C n z to the power n, which will converge in some region about the origin, some circle about the origin, we still have to determine this and this radius of convergence is going to be determined by the properties of the C n. So, will be see in minute where it converges. Once you do that multiply both side of this by z to the power n and sum from zero to infinity. And then of course, you immediately get since C n minus plus C n plus one minus C n plus two equal to zero multiply both side by z to the n and sum from zero to infinity then you get f of z that is the first term.

The second terms is z to the n plus one over z and that type of z minus C naught of course, because you have do not have the zero term there. So, class one over z f of z minus C naught and the last term minus one over z square f of z minus C naught minus C 1 in that is equal to zero multiply two by z square. It is said F of z times z square plus z minus one that this portion that is equal to on the right on the side, you have minus C naught minus C 1 z and then this terms use you class C naught z and dividing two by this. You immediately get f of z has rational function z square plus z right plus z minus one that is it. So, the problem now is very simple; in this case it is completely trivial we do not need to counter integration because this fellow is after all just does the quadratic function and all we have to do is break it up into partial fraction and then use the binomial theorem.

And you can compute what the n coefficient is, this C n is, but let's do it by counter integration for arbitrary C naught and C 1, you can plug it and work it out. So, this is typically what is going to happen, whenever you have a recursion relation with constant coefficients not n dependent, but constant coefficients then this f of z will typically be some rational function of z ratio of two polynomials in z. And then all you have to do is to find the coefficient of z to the n in power to series of this rational function about the origin. So, what does this implies, let us look at the special case C naught is zero, C 1 equal to one for instance then let's make it little easier it is becomes equal to minus z over z square plus z minus one, I put C naught to zero, C 1 equal to one.

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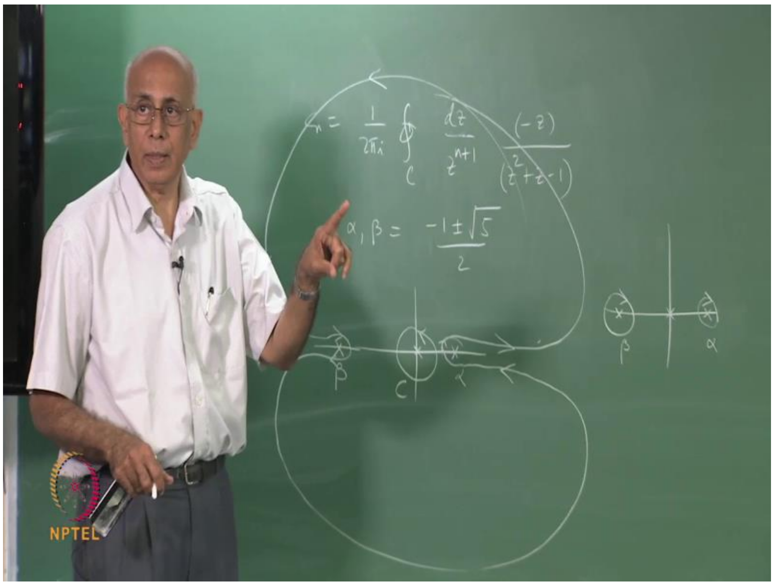

So, C n which is the n eth derivate of F of z at the origin divided by n factorial, can also be written as one over two pi I the contour integral over the small circle around the origin of d z over z to the n plus one times f of z. But f of z is now minus z over z square plus z minus one and this is a quadratic function. So, it is got two root and what are these roots, the roots are alpha, beta equal to minus one plus or minus square root of this square plus minus four times that so that is square root of five divided by two. And what does this whole things look like. Well the integrant has n n plus one n eth order pole, because it is the z up there.

So, it is an n eth order pole at the origin and then there is a pole at root over five minus one over two which is somewhere. In this side that is alpha and there is the pole at minus one minus root five over two that is here. And the contour looks like this, this is contour c, this is the quantity we have to evaluate. But I like keep saying evaluating this residue directly means differentiating this rational function n times at the origin. We do not want to do that instead we open this contour out. And as long as you do not cross singularity by Quasi theorem, you can actually open this contour and distort it as you please. Provided you are in the region of analyticity of this function, but what is this function like?

It is rational function, it called pole here, pole here, pole here and nothing else, no other singularity anywhere else in the play; it is actually analytic everywhere else. So, you can open this contour out has big. So, make it really huge like that provided of course, you do not cross that we all the wave that and you can cross this and class contour on this fashion retaining the sense of the in principle we could let this radius go to infinity provided of course, you have to vary about the behavior the function at infinity. But at infinity, this is c o like capital r radius of this a fear this is a n r to the power n plus one. So, this over this as an r here r to the n here sitting here and r square sitting here.

So, the whole thing is actually going to zero has r tends to infinity, definitely denominator power grater then numerate power. One of the thing you are when you do complex integration is that when you are expand this look at z at very large semi circle in either half plane. You have to be careful to remember that d z it self's caries of factor r because this is r. So, this one power coming from here and there is n power here and then there is things here. So, the wholes things was to zero the limit. So, in principle this is z this is the same as that contour integration is same has that. But now what happens all we have to do is to remember that once this set equal to zero what is left is this portions which would traverse and this sense and that portion which would traverse here being careful go around this integral.

So, if i can expand this little bit this pole here and what you do with this coming allowing in like this and going back and these portions causally each other this no singularity that all, but that lies you with this convergence exactly this same way remember the sense. Now it is in the positive negative sense it would in this this too this integral to like this came down. So, this this little circle to is to wards in the negative sense in clock wise sense. So, there is one more here to words in the negative sense this singularity remains were it is this is beta and this is an alpha and lets this. So, what is happen is the this contour integral has become the sum of two contour integral one around the point alpha other around the point beta both traverse in the negative signs. So, the residue the answer is minus two pi i times the residue and these points.

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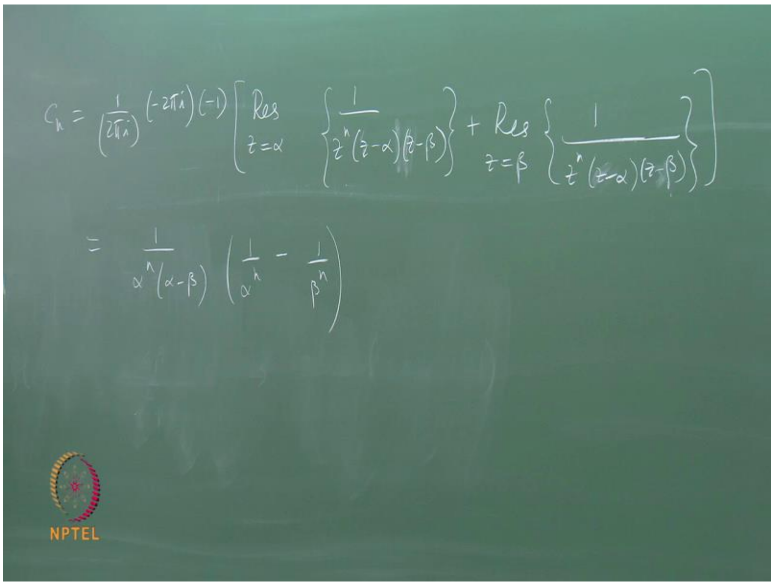

So, all we have to do is to say all we need to is to C n is one over two pi here and then minus two pi here on this side, and then the function was minus z out there and then z to the n plus one. So, really there is a minus one and then it is the residue and z is equal to alpha of the function one over z to the n z minus alpha times z minus beta plus residue at z is equal to beta of the same things one over z minus n z minus alpha. So, it is just the sum of these two residue in this fashion, and that easily evaluated because it simple poles at both alpha and beta, these are simple poles.

So, what you need to do is to multiply for instance the residue at z equal to alpha multiply this function by z minus alpha and take the limit as z goes to alpha. So, in just say get remove this factor and said z equal to alpha everywhere else. So, this becomes equal to this cancels out and you get one over alpha to the power n and then this is alpha minus beta that this portion, and here you have to said z equal to beta, so you get beta minus alpha. So, whole thing becomes one over alpha to the power n minus one over beta to the n and that is the answer that is the explicit expression for C n. All we need to do is to substitute these values, let's do that and let us see what happens.

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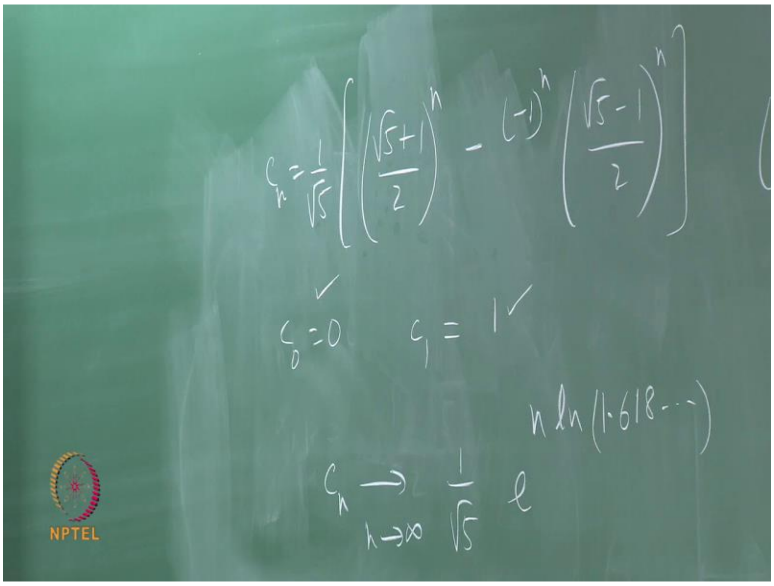

So, C n is equal to one over alpha to the power n, so that is two over root five minus one to the power n, this one over alpha minus (( )). I am taken that inside what is alpha minus beta, it is this with plus sign minus same thing it is the minus sign. So, just square root of five. So, this just C n one over root five. And then that is the first term minus beta is same with minus sign. So, let us put the minus on to the power n and then two over root five plus one to the power n here in this fashion, that is close expression. Let's rationalize this. So, this becomes two over four and then root five plus one goes on top. So, it root five plus one over two to the power n and then same things happen here except root five minus one goes on top, root five minus one over power n that is the expression for C n.

Now we know C n has to be an integer, we always know. So, these square root of five must cancel, all the way round for instance, let us just check if C zero was equal to zero. Well C zero, this becomes one that becomes one at cancel, so C naught is zero as expected, that is works out. C 1 is one over root five and then root five plus one over two and then there is a minus sign here. So, it is plus root five minus one over two once again and it is again equal to one to root five or root five is equal to one, so that is works out.

And similarly, C 2, C 3 etcetera, once these two are very sure that the rest of this will be right. So, equal to integer that what is remarkable. For positive integer n, this is equal to integer all the square root five z cancel out, so what does C n do at very very large values of n. So, C n goes like n tends to infinity goes like an exponential. So, this is one over root five outside and then this terms dominates over that because this is smaller than that and take out this factor then that ratio tends to zero as n tends to infinity. So, it is really this guy and this is equal to e to the power n log root five plus one over two. But of course, root five plus one over two is the very famous numbers right its golden mean right. What is the value of this number? It is one point six one eight something something something etcetera, this guy is like two point two four, so that is three point two four over two when is like one point six two or so, one point six eight, it is correct.

So, it is increases exponentially. So, this is the positive number. So, it increases exponentially fast and we know exactly how fast. This number by the way this root five minus one over two, it is a point six one eight whatever it is the remarkable properties. Because you can write it in some sense, the most irrational number you can think off. Among all the irrational numbers, it is the most irrational numbers, because as you know every number can be written as a decimal, which either terminate or recurs in which case it rational or goes on forever without recurring, and then it is a rational. You can also a write any number as a continued fraction, you can write every number as an integer part of fraction and fraction can be written as a continued fraction.

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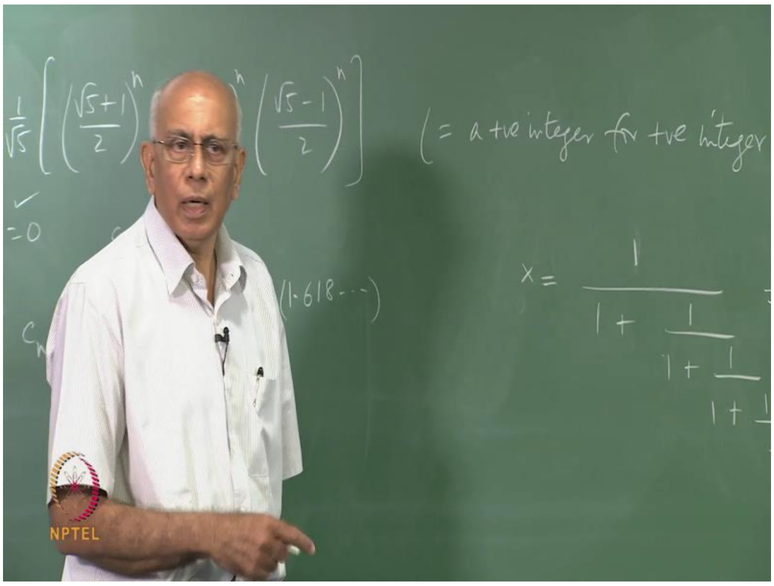

For instance, you could write in a number between zero and one, you can always write a number x as equal to one over some integer a naught plus one over a one plus one over a two plus one over etcetera keep going like this. By the a(s) are all positive integers, this is a standard form of writing a continued fraction in number. Now of course, if one of these a(s) turns out to be much bigger than unity, then you can approximate this continued fraction by truncating it at that point, and those are going to give you rational approximation to any given number. As you know you can make rational approximation of arbitrary accuracy for any irrational number and one way to do this would be to either truncated decimal expansion or continued fraction expansion. This is how Ramanuja for instance found an incredible continued fraction, which gave pi to a large number of decimal places several decimal places.

One of these numbers, I do not remember which one, the second or third of these numbers turned out to be the two hundred and ninety three that is so large compare to one that you know give you an extremely good approximation. Now given that the more terms you have to keep the more irrational number is if you like the harder, it is to approximate and I mean the longer you have to go before one of this a(s) ends up being much larger than one. It is clear the number some senses more irrational more difficult to apply proximate by a rational. What is the worst case in scenario the worst thing would be a not a one, a two are all equal to one. If everything is equal to one then you cannot stop some sense.

So, you put all these equal to one, one, one, one etcetera and what is the value of this number well it implies x is equal to one over one plus x which will imply of course, that x square plus x minus one equal to zero. But that precisely precaution in the static equation you solved out the routs are impact minus one plus or minus square route of five over two in this case you want alpha because you want this way of positive number this thing. In fact, the rational root five minus one over two. So, it is called the golden mean it has a very large number of interesting properties and. So, we will talk about continued fractional little later some other time. So, if you give me any such precaution relation two steep prefer precaution relation or three steep whatever it is then principle you can solved it by using these using the generation function, and inverting it using the this trick. So, is this clear how we evaluated this integral stretching it out.