Selected Topics in Mathematical Physics
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Module - 13
Lecture - 36
The Rotation Group and all that (Part III)

So, corresponding to each element of S O 3, you have two an exactly two element from S U 2, which map on to this element here. It is a 2 to 1 homomorphism. And these 2 matrices in S U 2 differ only by a sign, over all sign. So, several things immediately immerge.

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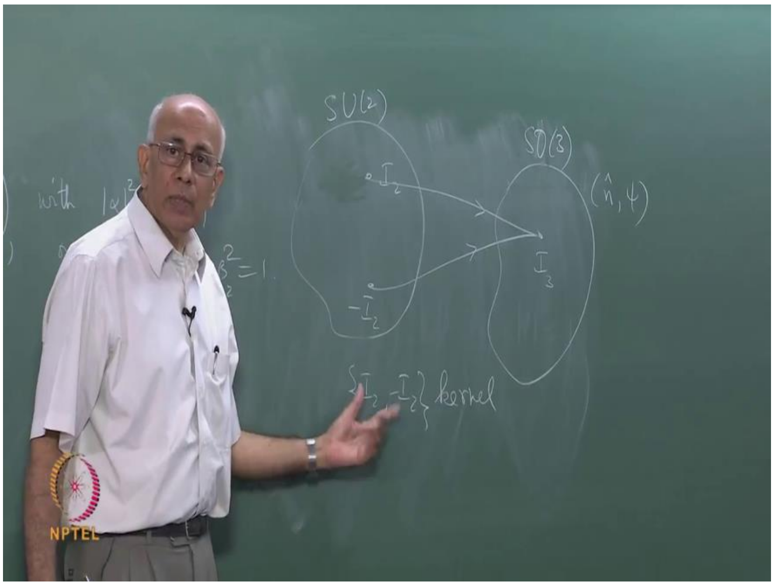

So, you got S U 2 here, this is the group S U 2. If you like, it is a connected group. It is connected and the reason, I say it is connected is because, the parameter space is a space of 4 real numbers satisfying this condition. And what sort of object is that, it is a hyper sphere. It is four dimensional sphere, it is a surface S 3. It is essentially S 3, so three dimensional object denoted by S 3.

Just as a S 2 is a surface of a balloon, it is a two dimensional object. S 3 is higher dimensional analogue of that and it is a connected object. Not only connected, it is a simply connected object. You can go from any point to any other point, without leaving the space. Every close loop on the space can actually shunt to a point, without leaving the space. There are no holes in the space, no handles or anything.

So, this is very important, that this is the simply connected space, the parameter space here. But, right now, we are interested in homomorphism, from that to S O 3. This too is a connected space, because the disconnected part of it, the once with determinant minus 1 do not exist in this set here. That is the other part, the improper rotations. This is the proper rotations in three dimensions with determinant plus 1. They are all continuously obtainable from the identity, by sequence of infinitesimal transformation.

So, it is a connected space and this space is parameterized by n and psi. And this space is parameterize by the same n and psi, appearing in different combination as to form a matrix of this kind. Because, I just wrote down what this matrix was, I said all you have to do is, to take e to the i sigma dot n psi over 2, write it as a 2 by 2 matrix. And that is this, you can verify that, it is an S U 2 element of this kind.

But, the space is connected here and simply connected. And we are not here seen, what the space there is going to do. But, what we do now is that, for every element here, there are 2 elements; that mapped on 2. In particular, the identity element I is a 3 by 3 orthogonal matrix, which corresponds to no rotation at all. So, it is three dimensional. Let me just put I sub 3 to tell you, it is a 3 by 3 unit matrix. That is no rotation at all. That would correspond here in S U 2.

To both the 2 by 2 identity matrix, as well as minus 2 by 2 identity matrix, because both these fellows would be mapped on to this identity element, out here. So, this is called the Kernel of the mapping. This set I 2, minus I 2 is the Kernel of this homomorphism, which takes you from this group to that group. It is the set of elements, which map on to the identity element, in the other group.

Student: 2 minus I 2 to 2 minus I 3 also, would it be minus I 3 they are...

Minus I 3 is not part of this, minus I 3 would mean the determinant is minus 1. But, this is the connected group of all those rotations with determinant plus 1, proper rotation. So, this map does not do that, does not take you there. This thing here, there is another structure to this a little more. That is, these two themselves form a group, I 2 and minus I

2, namely the 2 by 2 unit matrix and minus the 2 by 2 unit matrix. They form a group by themselves.

A group of order 2, because the product of I 2 with I 2 is I 2. I 2 with minus I 2 is minus I 2 and minus I 2 with minus I 2 is plus I 2. So, it is a cyclic group of order 2. It is isomorphic to the addition of integers modular 2. You can put all the odd integers in one bin, all the even integers in one bin. And under addition even plus even is even, even plus odd is odd, odd plus even is of odd and odd plus odd is even.

Student: Minus 1 is a part of S U 2 should not the determinant be minus 1 for minus into...

Minus 1 and minus 1 it is a 2 by 2 matrix, so the determinant is plus 1. So, it is very much in S U 2. But, that is all point, because this is even, these two fellows are sitting here, but the both map are into the identity transformation there. So, if you put I 2 on both sides and put odd or sigma in between, you are going to get odd or sigma. Weather, you put plus I 2 on both sides or minus I 2 is does not matter. Since, these two fellows form a group by themselves.

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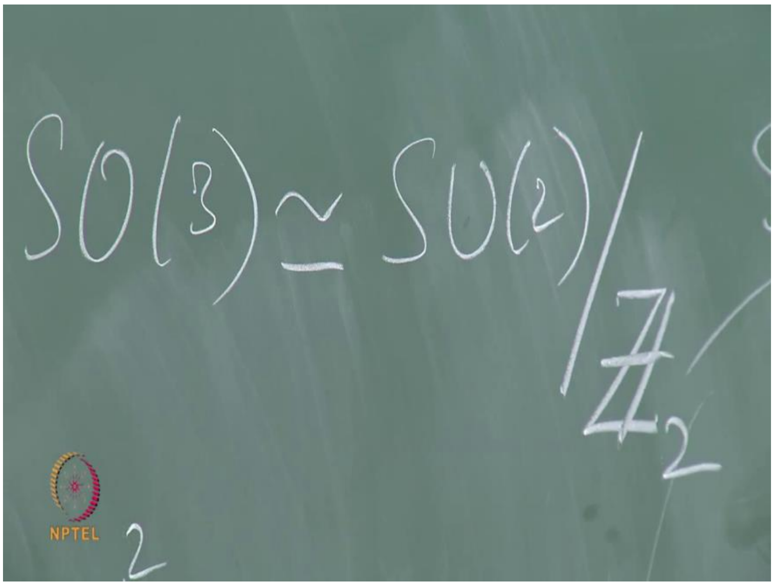

The technical way of saying this is that, this group S O 3 is isomorphic not to S U 2, because isomorphic means 1 to 1 connection, but isomorphic in the sense that, it is S U 2 up to a sign. So, what it technically is called a quotient group. This is S U 2 quotient with this cyclic group of order 2. So, up to a sign, this is modulo, this said 2 element, which and these two are in 1 to 1 correspondence.

So, if you ignore the sign of matrix, over all sign of the matrix in S U 2 and regard both the plus and minus sign has going to one element. That quotient group is the same as S O 3. This is 2 to 1 relationship and this the technical way of writing this. That it is a quotient group. This is actually a coset space; it is the set of cosets space of left coset of Z 2 in S U 2. But, since this is a proper sub group, none of it is any element of S U 2, let us call it g.

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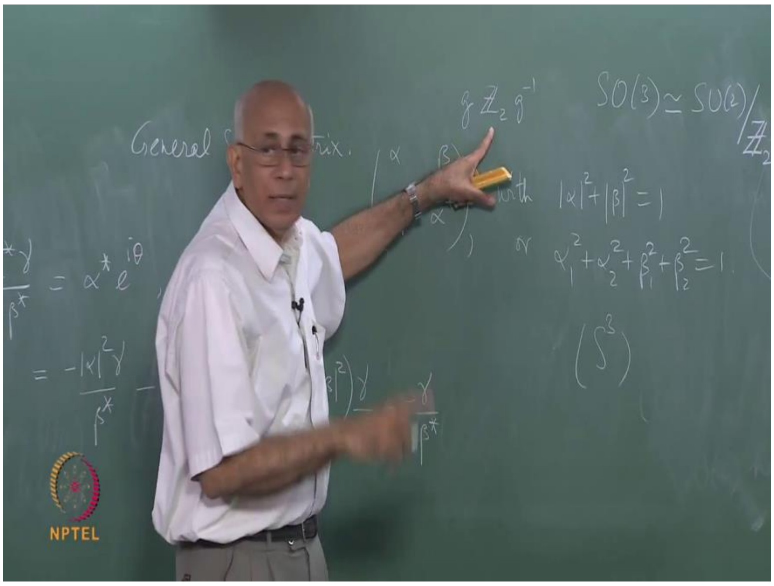

And then, you act on the Z 2 and do g inverse here, you end up with original thing, because this is just identity or minus the identity. And it is the normal sub group, this guy here. It is called the centre of S U 2. It is the kernel of the map in the sense that, those elements map on to the identity element, under this homomorphism. So, this quotient group, this structure makes things very interesting. It says that, locally in the neighbourhood of this identity, it looks like neighbourhood of this identity here, locally.

But, the global topology of this space and that space, can be very different from each other, as we will see. Locally, this going to look exactly the same, but globally this is very different or together. It is like, you have to torus and you know that the actual curvature on to torus is 0. But, of course you do not realize it. Locally, of course, it looks like flat. But, then, it goes around realize global topology is whole in the middle, etcetera.

Now, let us see what the parameters space of S O 3 is going to look like? And that is going to be a little non-trivial.

Student: ((Refer Time: 08:05))

[FL]

Student: S O 3 is simply...

No, it is connected, but we see that it is not simply connected.

Student: From the identity through in decimal rotation, you can build all elements.

Yes, but yet it is not simply connected. We will see how, precisely, exactly. So, locally the thing look like the topology it look like S 3, but globally it will not. So, it is clear that, once you do a complete circuit in a special way; that will be something funny going on. And we will see, what that is? Now, how do we parameterize the space? There are 3 numbers, so we should be able to parameterize it by three dimensional space. Let see whether we can do this.

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The first number is n. This is actually a set of 2 numbers, it is a unit vector. Now, n can be parameterize by points on a sphere, you can simply say latitude and longitude. Once, you specify on a unit sphere, you specify it n, all together. So, let say that with respect to some given reference frame, the direction is specify n. And now, I also would like to say what is the amount of rotation in the plane perpendicular to n. I am constructing a parameter space.

So, let us have the convention, that the length of this point from the origin gives me the psi, gives me the amount of rotation. And the direction of the point in space, gives me the axis of rotation. Then, what is the parameter space, it is a solid sphere now. Because, points inside the sphere would corresponds to rotation, less than the maximum allowed rotation. And what is the maximum allowed rotation about any axis, 2 pi.

So, what you need is a solid sphere of radius 2 pi. Such that, in that sphere, at any point you know the direction from the origin, that gives me the unit vector n. And you know the length of this point, distance of the point and that gives me psi. So, the nine way of looking at it would be the say, that the parameter space of S O 3 is a solid sphere of radius 2 pi.

Every point in a corresponds to some rotation, the origin corresponds to no rotation at all, the identity matrix. But, you see do not need that, because if I have axis like this, pointing up like this, this direction, this is the axis. And I make a rotation by pi and end up with this configuration. It is the same thing as rotating about the opposite axis, through an angle pi. And let me show you that, by actually drawing something on it.

So, let us make a little mark here and I put an arrow here up here, let us do that, in this side. So, this is the orientation, after the rotation, I start like this. The arrows, I can see it on this side. And I am going to rotate about this axis up here. The axis is pointing upwards and I rotate through pi in the positive sense. And here is the configuration of this object, with the arrow on this side pointing upwards.

But, I now rotate about the opposite axis, again through plus pi. So, looking from downwards, I have to rotate in positive direction; that is this. And it gives exactly the same configuration as before. This means, that I do not have to rotate through 2 pi, because I allow axis like this, as well as axes in the opposite antipodal direction, rotation the pi is sufficient. This implies that the solid sphere, does not have to have a radius of 2 pi, a radius of pi is sufficient good.

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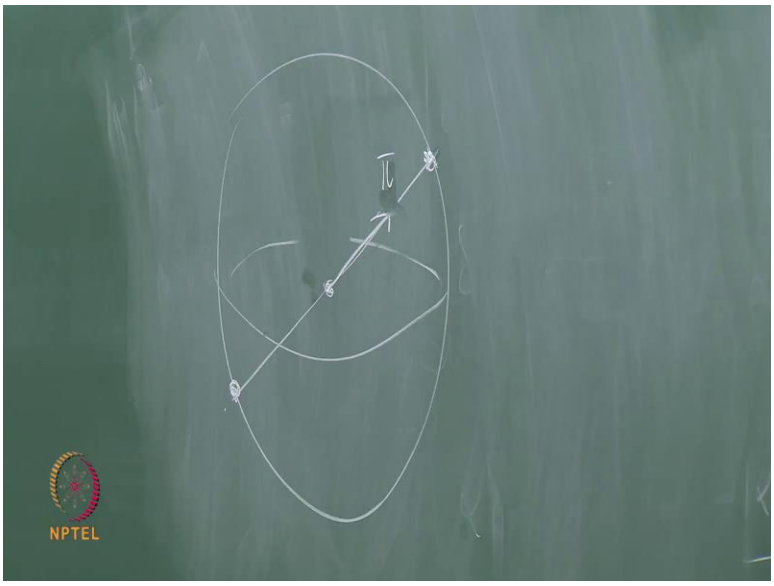

Once we accept that, then here is the parameter space. Here is no transformation at all, the origin. And then, the radius of this fellow is pi. If I said, let us take this point inside the sphere, at some distance less than pi. That would correspond to using this as the axes of rotation. And this amount here as the psi, the amount of rotation about an angle, about this axes n, but there is a catch to it.

This solo sphere is a simply connected object, there are no holes in it. It is a connected object, it is a simply connected object. But, the catch is that, once you hit the surface, this point here corresponds to a rotation by pi, about this direction. But, that must be exactly the same as a rotation by pi about this direction, which means that this point is mathematically the same as that point. They are glue together. And that is true for every pair of antipodal point of this sphere.

So, it is clear that this three dimensional object, it is got three dimensions. That is for sure, we cannot write it into three dimensional space. I cannot find a model of it in three dimensional space. You need a higher dimensional space, in order to imbed this object. They are lots of such surfaces, there are even two dimensional surface, which you cannot imbed in three dimensional space, like the cline bottle, you cannot imbed it.

So, this is one of those objects, it is three dimensional objects is bad enough. But, you cannot imbed it in three dimensional Euclidean space and make a real model of it. Imagine therefore, there is an invisible string, ((Refer Time: 13:38)) invisible string of 0 length, which says that this point is same as this point. This point is same as that point, excreta, let us imagine that. Now, the question is, what sort of is connected, but what sort of connectivity does this object have.

Now, how do you find the connectivity of any object, where there simply connected, doubly connected and so on and so forth. What do you is to draw is map loops on the space. Whatever space you have, you map loops on to the space. And you ask, can I shrink this loop to a point with continuously, without leaving the space. If you can and you can do this for every loop in this space, then it is simply connected.

If you cannot, then it means that, the space is not simply connected. And the connectivity depends on, how many different classes of such parts can find. Let us take a very simple example. And now, I am going to go back a little bit and give you this example. And then, we come back to the three dimensional rotation groups. Well, the answer here is turn out be doubly connected. There are exactly two classes of close parts, which are in equivalent to each other.

On the other hand, let us look at a simpler example easier to visualize, because it is a lower dimensionality. Let us look at the group of rotations in a plane. Let us got just one generator, it is just about the origin, this is one generator. And it is characterize by 2 by 2 orthogonal matrices.

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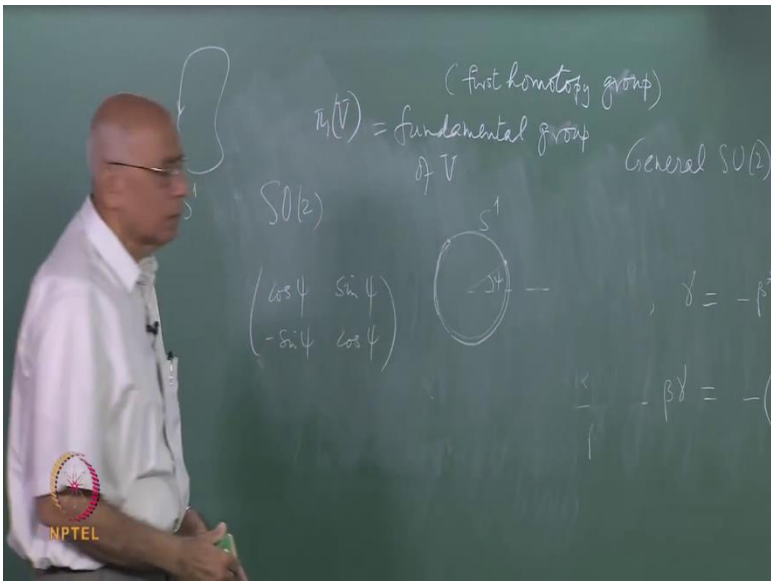

So, let us look at the group S O 2, represented by 2 by 2 orthogonal matrices. And a typical matrix of this kind would be cos psi, sine psi minus sine psi cos psi determinant is 1 and psi is any real angel between 0 and 2 pi. So, this set of matrices, that all orthogonal determinant 1, they form a group, S O 2 group. Now, what is the parameter space of this S O 2? Clearly, specified by one angle, which comes from 0 to 2 pi and that just S 1.

So, the parameter space is S 1, because with respect to any reference direction, you can actually tell me, this point here corresponds to rot angle psi. So, all the rotations here are parameterized by points on the circle, which is a 1 sphere, S 1. So, the parameter space of S O 2 is S 1. This we know. Now, the question is the pace is connected, the answer is yes, you can go from any point to any point, along the circle, continuously. So, it is connected.

Is it simply connected is a question? So, what you have to do is to take another loop, this is a closed part and map it on to this, mapping is topologically same as S 1. So, you are finding the maps of S 1 to S 1, this smooth map of S 1 to S 1 and asking, how many ways to do this. Well, here is one way, I can do it. I can start at this point, map it to this point. And then, I go round, so this map goes around and then, continuously goes back, that is a continuous map.

But, that can be shunt to a point without leaving the space. Anything that does not fully goes around this circle. Imagine that, this is rubber tube, the inner tube of a bike and this thing here is rim of a cycle, the wheel. And you trying to put on to that and you can stretch this as you please, but you cannot cut it. Then, there are ways, which you just to place it and shrink it to a point, a trivial map.

But, then you took a rubber band and stretch it all the way around and actually made it go around ones. Then, you cannot shrink that to a point without cutting it. So, it clear, that there is at least one class of maps, different from the trivial map, which can be shrunk to a point. And that is a class of map; you actually start at some point and go all the way around, you can do twice. This is an infinity extensive rubber band, topology.

So, you can do twice, you can go around twice and there is no way, you can change that to either 0 or 1, windes around the circle here. You could have done it in the opposite direction, because every close path has to be direction has to be given has to be oriented. So, you could have mapped into the opposite direction, corresponding to winding around minus 1 time and minus 2 times and minus 3 times and so on.

So, what happens then, it means that, all parts, close parts maps of a loop on to this S 1 can be put into classes and each class is specified by the integer. It says, how many times does it winde around and that integer could be anything 0, 1 plus minus 1, plus minus 2 etcetera. Now, the point is these classes are 1 to 1 corresponds to the integer. But, the integers form a group under addition, which means that the classes of maps close of loops on to this S 1. They themselves form a group and that group is called a fundamental group of the space.

So, whenever, you look at any space v and you ask what are all the distinct classes of maps of S 1 on to v. You call that pi 1of v and it is called the fundamental group of space. You would ask, while as soon as it means, there must be a group composition law, how would you find a group composition law. Well, there is a way, simple way of actually regressively specifying, what is meant by the composition of two parts.

Here, it is value obvious. If it binds around 6 times and you composite it with something which windes around 7 times, totally with something windes around 13 times and so on. If a compose box in a well define way and those things form a group, called the fundamental group or the first homotopy group of this space. And the statement is, if the first homotopy group has only 1 element in it, the identity element.

It means all close points shrunk to a point, then the space is simply connected. If the first homotopy group has more than 1 element, it is the non trivial group in that case.

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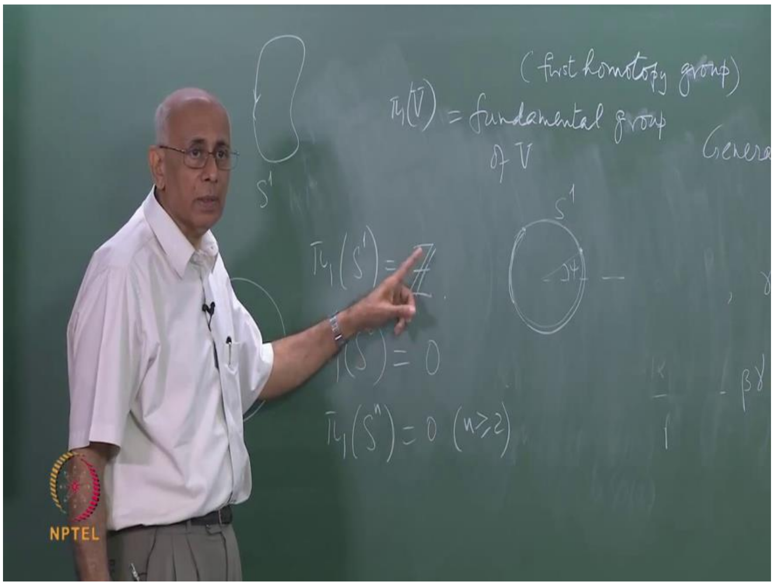

Now, what is the first homotopy group of pi of S 1 by equal to I mean isomorphic group. We can put all the element of this group in 1 to 1 corresponds with the set of integers. So, this thing here is the group of integers Z itself. You can do this for any space actually. Let us suppose, you take the surface of a sphere, a balloon, that is S 2 and I ask, what is pi 1 of S 2 equal to...

So, now, what I have to do, you should take the loop and try to map it on to the surface of a sphere in three dimensions. I take this loop and put it on the surface. Of course, I can shrink it to a point. There is no way in which I can circle this sphere, without being able to shrink it point, no matter what I do. Therefore, all closed parts on this surface can actually be shrunk to a point and so what is pi 1 of S 2. It is the trivial group adjust the identity element.

And one of writing this is to just write it as 0, the short hand for saying that, it is group with 1 element, it is a trivial group. So, pi 1 of S 2 is 0 and it is not very hard to show, actually that pi 1 of S n equal to 0 for n greater than equal to 2. So, pi 1 of S U 2 is 0. Simply connected, I am not proving it, but you can show, that pi 1 of S n is clear is 0. This is not true for S 1, because pi 1 of S 1 was a very non trivial group. This set of integers here. And the next question is, for instance. Let us give one more example, what is pi 1 of a 2 torus.

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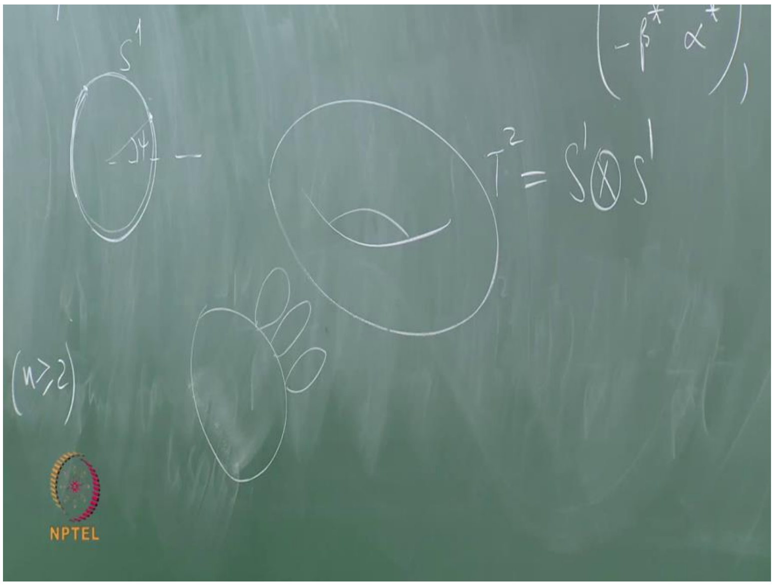

That immediately tell you what this business all about. This is the surface, we talk about the surface of a inner tube, it is a two dimensional object, surface of a donut. And the question is, what is the fundamental group of the space here? After all torus is formed by saying. I take a circle and it every point on it, I associate a circle per absence of perpendicular direction, join them all together. And then, you get this torus.

So, technically, the position of the point on the torus is specified by saying, with respect to some reference direction, along the axis, where is it 0 to 2 pi 1 angle. And then, in a given cross section, where is it respect to the cross section, so second angle. So, it is clear that, this is torus T 2 is essentially S 1 direct product with S 1. Specified by saying, two different angels independently from 0 to 2 pi. So, topologically, it is S 1 cross S 1.

So, what would pi 1 of space b, it would be Z cross Z, direct product of two sets of integers. Because, it would say that, well to specify any close path on this, we have to merely ask, how many times wind round this and how many times wind round that. And it specified by two integers, independent of each other. So, it immediately says, pi 1 of T 2 equal to Z cross Z.

Now, we come back, just a little bit more about S O 2 and that is the following. That is the valuable lesson.

Student: ((Refer Time: 24:36))

Pardon, it is connectivity is infinity, it is infinity, because infinite number of element. If it had just 2 different classes of parts, I say it is doubly connected. But, this triply connected and so on. This thing infinite connected S 1 as infinitely connected. But, you can do the following, I said out there, what did we see, we saw that 2 by 2 unitary matrix has the determinant, which is e to the i theta. And 2 by 2 unitary matrix with determinant 1, does not have the d to the i theta.

And some sense, you specifying a S U 2 matrix and adding on top of it and e to the i theta factor. But, what e to the i theta, it is the representation of the two dimensional rotation group, because the two dimensional rotation, instead of specifying it by an angle or point on the circle. I could say, this is the unit circle in the complex plane and specified by e to the power i theta or e to the i psi.

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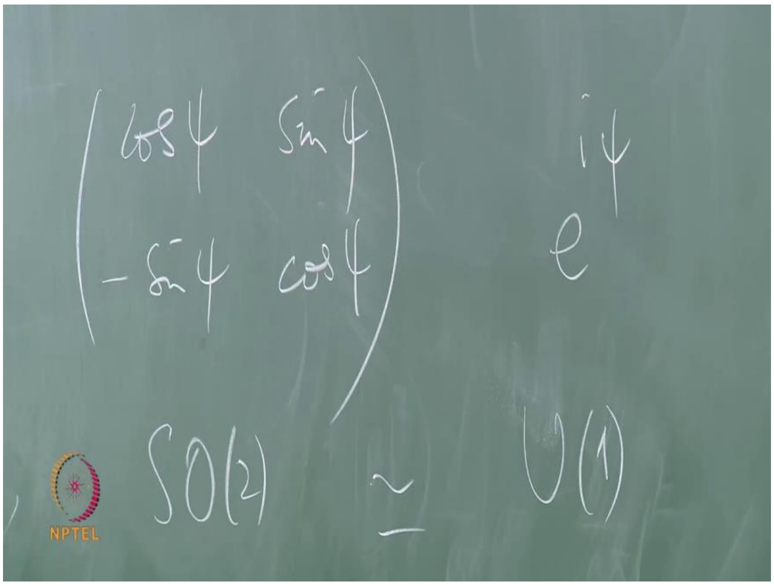

So, instead of this matrix, this is a valuable lesson, instead of the matrix cos psi, sine psi minus sine psi, cos psi, which is specifying the element of S O 2. I could say, let us associate with this rotation, the complex number e to the i psi. I just associate it with this matrix, I associate that, with every such matrix i associate this number. And I use the fact that e to the 2 pi r is 1, identically.

Then, it is clear that the 1 to 1 correspondence between 2 by 2 orthogonal matrices and points on a unit circle in the complex plane. But, points on the unit circle also form a group, it is a group of 1by 1 matrices. And the unitary matrices, because e to the minus i theta is e to the inverse of e to the i theta. So, these fellows form a group U 1. U 2 was fail in nontrivial, but U 1 is just 1 parameter. And these two are essentially the same.

There is 1 to 1 correspondence between them. But, then these are points on the circle. But, you could say, I can do something else. This space is not simply connected, it is infinitely connected. The pi 1 of this S 1 is infinite is Z is a set of integers.

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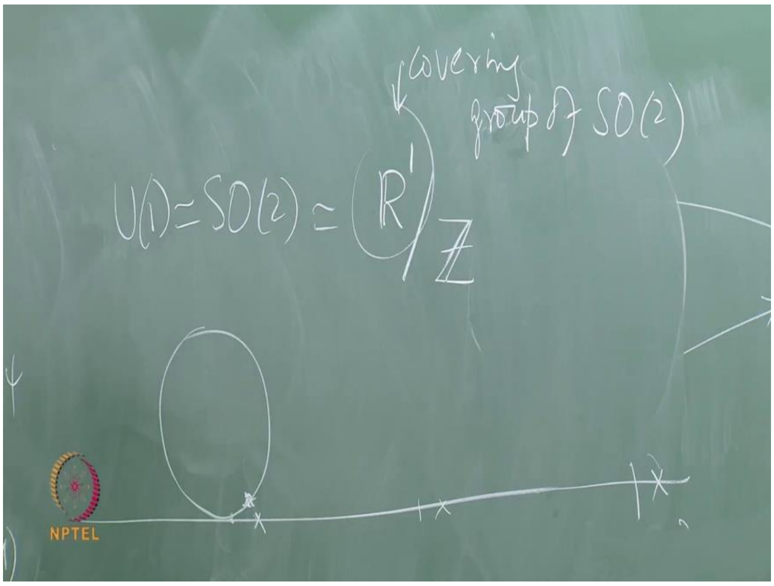

But, I can do the following, I can say I can take this circle and I roll it on to the real number line, I just roll it out without slipping on to the real number line. So, what does that mean start with here, I put it here and then, I roll this circle as soon as it hit is on the end of the circumference, this point get start getting mapped again, once again. So, a point here will get mapped, for example, this point will get mapped, it will get mapped here, it will get mapped here and so on and infinite number of times.

So, it is clear that the real number line provides a kind of cover for S 1. But, the mapping is 1 to infinite number of copies all separated by circumference of the circle, which you can take to be 1. This means, that the space S O 2 can also be written as quotient group this 2 can be written as R 1. That is the real number line, modulo and integer. So, this is in fact, what this structure, just as I wrote S O 3 is quotient of S U 2, modulo Z 2. Here, to, I can say S O 2 is R 1 modulo Z.

R 1 is called the universal covering group of S O 2. Because, it is simply connected and infinite number line, the connectivity is simply connected. Every close path can be brought back into a point. So, this guy here is covering group and does the theorem says, every league group has got a universal covering group, which is simply connected, whose parameter space is guaranteed to be simply connected.

In the case of S O 3, we not yet shown, it is doubly connected. We will see in a minute, that is doubly connected and the universal covering group is S U 2. And now, we can see that, the relation between these homotopy group, if I write, what is pi 1 of this guy, this is S 1, by the way. So, pi 1 of S 1 is Z, this guy is simply connected and just reflect it here, pi 1 of any discreet group Z self you can see trivial.

So, immediately you realize that the idea of universal covering group is extremely useful. Because, it ignores the complications in the global topology of this group space and focuses on the portions at are nice. It is simply connected space can be associated with it. Now, let us go back and see what happens in the case of S O 3, were the connectivity is also a doubly connected.

Student: ((Refer Time: 30:32))

R 1 is a lee group; R n is a lee group by itself. So, this thing here is not only a space, but also have topology is also group is a lee group. And it is simply connected.

Student: ((Refer Time: 30:50)

There is a homomorphism and the covering group, the theorem is universal covering group of a lee group guarantee to simply connected. So, all connectivity properties will come in this quotient. That will tell you, how complicated the connectivity is. So, you will see the S O 3 example, immediately. In S O 3, what happens, the parameter space tend out to be a solid sphere of a radius pi, with antipodal point on the surface identified with each other.

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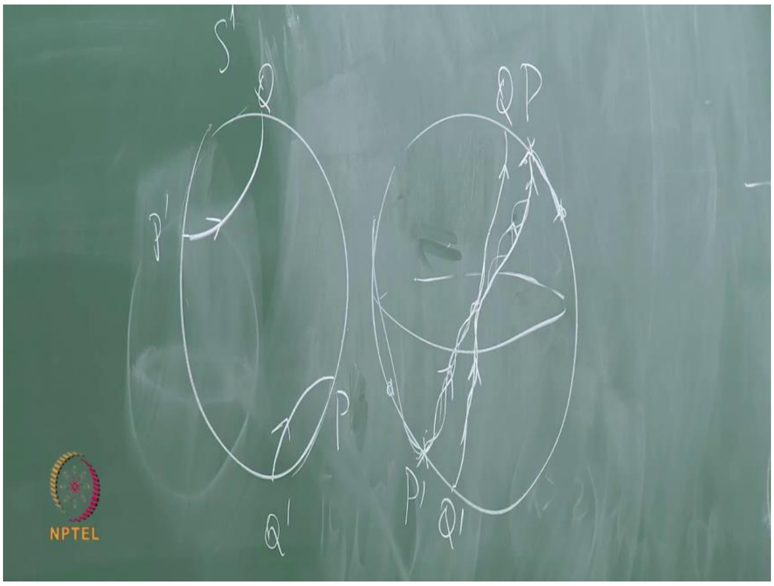

So, you have this space and then, the statement was that point is same as this point. But, the point just inside is not same as that. Those are different rotation; a point slightly less than pi less than going up to the surface, would be a rotation like this. A little angle less than pi and in opposite direction little angle less than pi and these two are not the same. It is only; if you make it fully pi that you end up with exactly the same thing.

So, points on the surface are identified with each other, points just below the surface are very different from each other. Now, what is the closed part look like, well in the solid object certainly a plot like this, it can be shrunk to a point. But, you can also do the following, I start with no rotation and make a sequence of rotations and go there. But, that point is same as this.

So, I come back here, that is a closed path, back to the original orientation. That is a closed path, but this cannot be shrunk to a point, because if you try, if you move this, the way to shrunk to a point is move this here in to this. So, let us call this P and P prime. I move this fellow here and bring it up to here, but this fellow dorges me and goes there. So, there is no way, you can close this part and you cannot shrink this to a point.

The only way, I can shrink it to a point is by doing the following. I start here and go there, I come here, I go here and then, I go again out, I word it Q and start here and come back, this is Q prime. This is now shrinkable to a point. Because, was now the trick is, I start moving this P towards P prime or toward other direction does not matter. But, now I can regard the part this point, I separate these two guys.

When, if I start moving P in this direction, P prime moves there, I keep Q and Q prime fixed. So, after a while, this path looks like this. And P was here, I got move to this point here. And part was from Q prime, I move P even closer in this direction not P prime sorry, this is P. I move P even closer, this fellow moves there and each of them is shrink it to a point as it appears. And these are the only two parts, possible, only two kinds of parts possible, those which are obvious close loops and those which start at the origin and go right through from diametric one point to antipodal point. But, further rotation repetition of this bring you back to no rotation at all, which means that, there are objects in the space. In the representations of the rotation group, which will not come back to the original value, if you rotate the coordinate system by 2 pi. If you rotate it by 4 pi, it is guarantee to come back. This is the origin of double value representation spinner representation.

We need a little more mathematics; we need little more to show, what are the Eigen values, what is the cassimere operator here, etcetera. That the standard thing about angular momentum algebra. They turn out to be the various representation are labelled by a little number j, which is non negative takes on the value 0, half 1, 2, 3 half, 2, excetra. And those, which take on integer's values, correspond to scalar, vector, tensor, excetra, representation of the rotation group.

And those, which correspond to half odd integers, correspond to so called spinner representation. These objects will change sing in order 2 pi rotation and come back to the role values after 4 pi rotation. And ultimately, that is of the origin of, that is at the root of spin half. That is at the root of integer spin and half integers spin, which then in turn by something called the spin statistic theorem, can accept with the existence of goes on from the one hand and comes from the other hand.

So, finally, it is a property of three dimensional space; the parameter space here. I already wrote down the relation, saying that, S U 2 is isomorphic; S U 2 quotient is with Z 2 is isomorphic see the origin of Z 2. There are two such parts.

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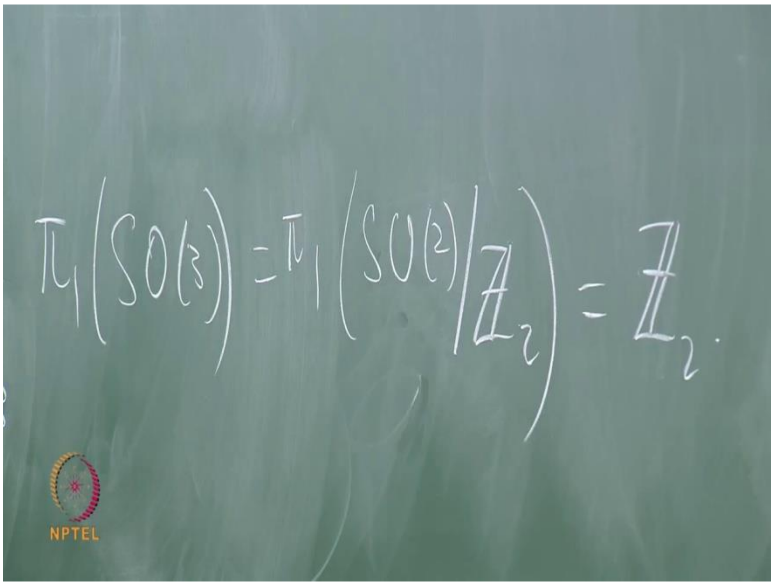

Now, we can write of this space by the way, this solid sphere of certain radius pi with antipodal point connected to each other. This thing is called the projected space and is written as R P 3. That name is not important; I just call it pi 1 of parameter space of this group here. This is the same pi 1 of S U 2 quotiented with Z 2. But, this fellow is simply connected. So, this is equal to Z 2 itself.

Two such classes of close parts and you saw explicitly, what these two classes were.

Student: ((Refer Time: 37:12))

In physical terms, it says, you got to make two such close parts. So, what is it correspond to say I go all the way pi and then comeback from pi to 0, I made a complete rotation by 2 pi. I have to do it twice.

Student: ((Refer Time: 37:42))

No, it is just that normally, if you have an object like vector or scalar or something. Then, the rotations of a coordinate axis by 2 pi brings the coordinate axis to the original orientation. But, this object will not change at all. But, there exits objects called spinners, with change signs. And for them, you have to make coordinate rotation by 4 pi in order to bring it back to the original sign.

Student: ((Refer Time: 38:18))

No, not directly, the fact that square root does not play role here. That is not doing it. That is one particular function with branch points, square root branch point, but that is not the origin. The origin is deep; it has to do with the connectivity of space of rotations has to do with that. So, we see that, the S O 2 is infinitely connected with parameter space. But, S O 3 is doubly connected.

Now, the question then I rise is, what about higher rotations, rotations in higher number of dimensions S O n. What would they start doing, well, fist you write angular momentum algebra for those objects. And then, we have to ask, what happens to a case of rotation and so on. The algebra itself is fairly simple, it is not hard to do, well write that down and come back.

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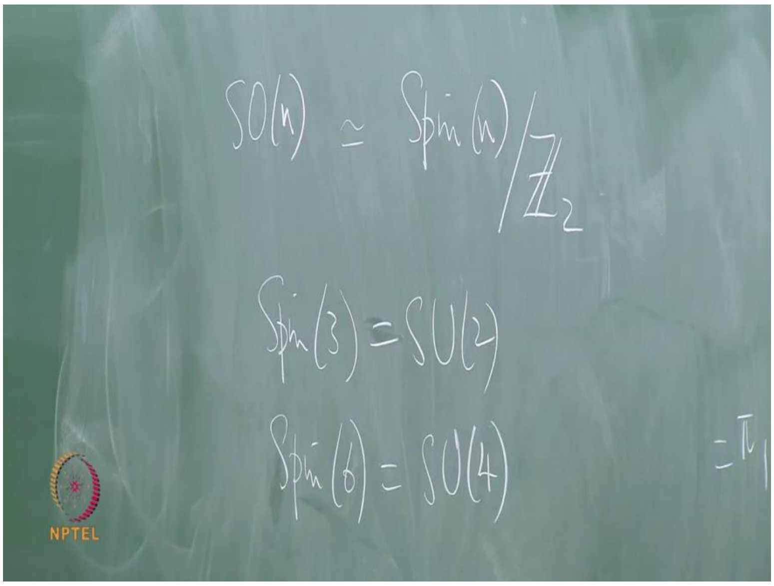

But, let me say directly that, the space S O n. The parameter space of this is much more complicated as you can see the n times n minus 1 over 2 parameters for each such element. That space is also doubly connected also happens to be exactly doubly connected. Then, the next question is, what is the covering group, what is the group analogue of S U 2, what is the group that will play the role of S U 2. And the answer is that turns out to be, this thing turns out to be group called spin n.

That is the name for the group spin n, which is simply connected. It is group, because same number of parameters n time n minus n over 2 as S O n and it is simply connected and the quotient again is Z 2. And it is so happens, that spin 3 equal to S U 2. In this case, it is so happen, that is not true for other groups in general. In fact, only other case, where there is a connection with unitary group, I think is the case of six dimension.

So, when you have S O 6 here, 6 dimensional Euclidean space. That it turns out spin 6 is S U 4, the group of uni modular 4 by 4 matrices, which are unitary, because there are no connection of this kind otherwise. By the way, since we are on it, you could ask, what about the parameter space of unitary matrices themselves. Well, let me just make one small comment on it quickly.

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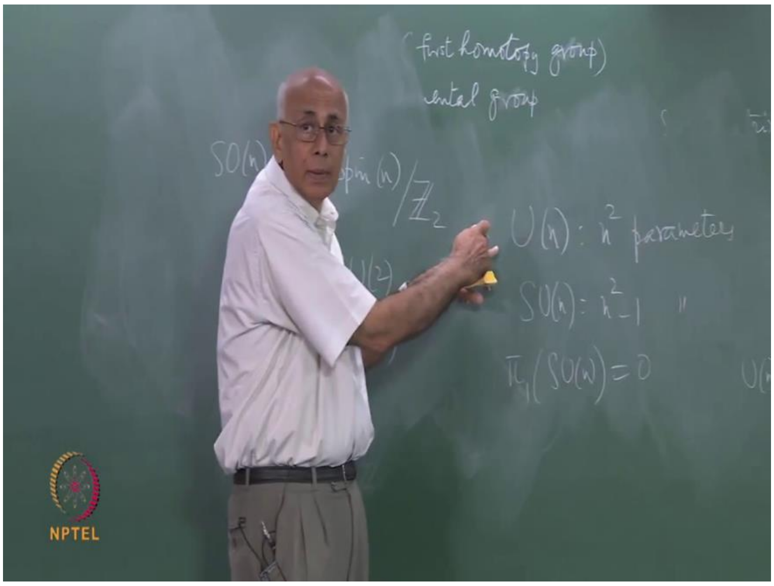

If you have an n by n matrix, there are n squared elements and if there complex numbers you have 2 n square parameters. But, you moment you put unitary, then U n this group has n square parameters. And then, if you say the determinant is 1, you put in one more condition. So, S U n has n squared minus 1 parameters and indeed we saw that n was 2 4 minus 1, parameters made it homomorphic to the rotation group S O 3.

The question is, what about the parameters of these guys, well here this goes to theorem, it turns out that pi 1 of S U n equal to 0. That is S U n for whatever n is simply connected. S U 2, you saw explicitly where these 2 by 2 matrices alpha alpha star, beta minus beta star. But, you want S U 1, it is a set of 1 by 1 matrix, which is unitary and have determinant 1. It is 1, it is a trivial group, it is just 1.

Because, e to the i theta must be 1. That is it theta is 0 and that is the end of the theorem. So, S U 1 is totally a trivial group, but you turns out S U n also is simply connected. And you ask is there any connection between these guys, because there is after all a U 1 sitting in between there is an e to the i theta factor here. And indeed it turns out, that S U n. Let us write it as S U n, U n quotiented with U 1.

Once again modulo and e to the i theta factor you have S U n. So, these two are connected in that passion. This fellow is simply connected.

Student: ((Refer Time: 43:10))

Well, in general n by n matrix has 2 n square parameters. Now, you impose that condition, that it is unitary and then, you have one parameter less. You have 2 n square parameters less. We saw that in general 2 by 2 matrix you had 8 parameters, but the unitary matrix has only 4 parameters. You had enough conditions to do this. So, it reduces from 2 n square to n square. And then, you put in the condition that determinant be 1, it reduces 1 more condition. So, the question is, the relation I want to point out is quotient relation.

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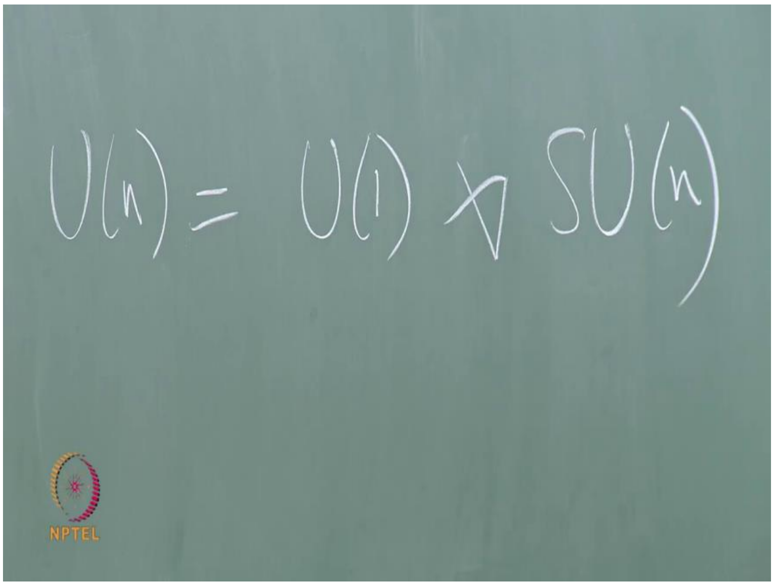

Is simply that, U n is something called semi direct product. So, this is U 1 many ways of denoting the semi direct product with S U n. So, what is the pi 1 of this guy going to be, you can see sort of unitistically from this. This part is simply connected, that is not going to do anything, this part is same as S 1 and pi 1 of S 1 is Z. So, this 2 is infinitely connected with fundamental group equal to Z group of integers itself.

So, when can play this game, actually write ask for the fundamental groups of various lee groups manage for various lee groups. And then, from that deduce very valuable information on global properties of the group R. But, infinite properties can determine by the generators. Now, let me write down, what the generators are for S O n.

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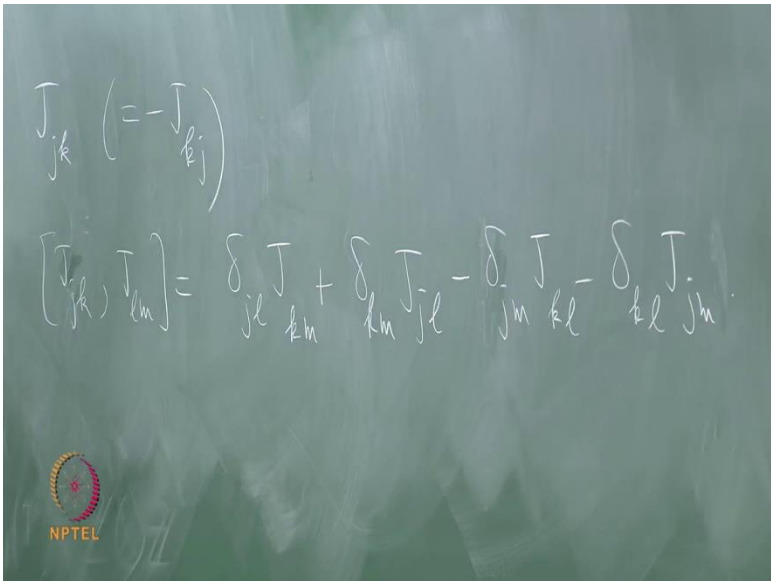

In general, you have n time n minus 1 over 2 generators. So, the general generators are in the form J, j k for example. But, j and k run from 1 to n, but it is anti symmetric, just n times n minus 1 over 2 independent parameters here. And the algebra is very straight forward the lee algebra looks like J, j k, J, l n equal to a chronicle delta J l, J k n plus chronicle delta K m, J j l minus the opposite order. So, J n chronicle delta, delta J n, J k l minus delta K l, that is it. So, that is the liner combination.

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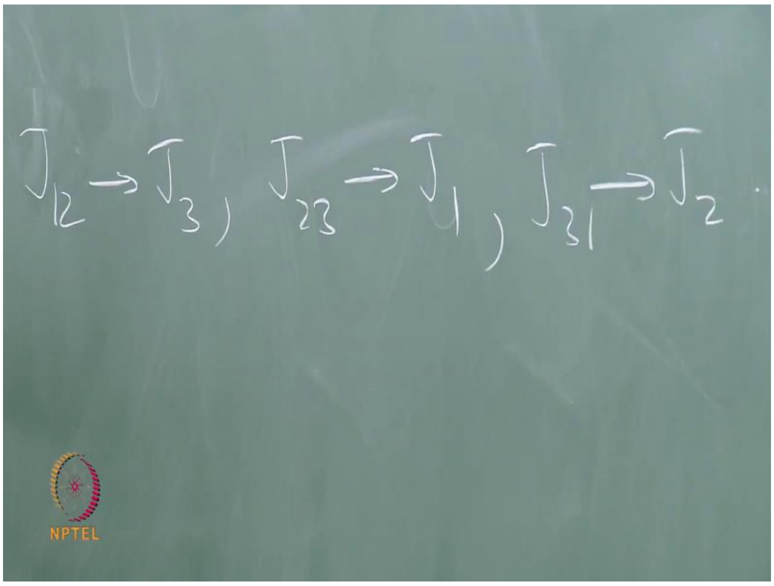

Now, you could ask what happens three dimensional case. Well, that is exactly same as the except the identification J 12, I called J 3, J 23, I called it J 1, J 31, I called J 2. That is, because of Z in three dimensions. You have three planes and three axis, you can ambiguously convert it in to this. So, that is the general algebra here. Now, you can start to this point and play for the games and if instead of in four dimensions.

For example, if you have 1 time like dimensions and three space like dimensions and instead of looking at S O 4. You look at the group of Laurence transformation is S O 3 comma 1. Then, you write instead of delta here, you put metric tense here and you get algebra for generators of homogenous Laurence group. If you putting the elements here of the simplistic matrix. Then, you get this for the simplistic groups and S P 2 and etcetera.

That is really a topic in group theory, we would not going that, but the main points I want to convey, where that the lee group have a very remarkable structure analytic structure. They are given by the algebras, they generate the lee groups. Generally, exponenting these generators, the generators obey a lee algebra, which carries all the information about the group.

And in general the global topological properties of the group would require you to find it is fundamental group and it is homotopic group. But, you guarantee, every such group has a universal covering group, whose that topology is very simple. This is simply connected and whose structure will give you the local structure of the group itself, we are talking about.

The rest takes you to the little more detail, which I do not want in the moment. But, this will give you hope fully some little flavour of, what the theory of lee groups is all about. There is more lot to it; one needs to understand the role of symmetry, both in classical and mechanics and in classical dynamics as well as quantum mechanics. For which these serve as a starting point here, so we that I can rewind up say feeling.