Selected Topics in Mathematical Physics
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Module - 13
Lecture - 35
The Rotation Group and all that (Part II)

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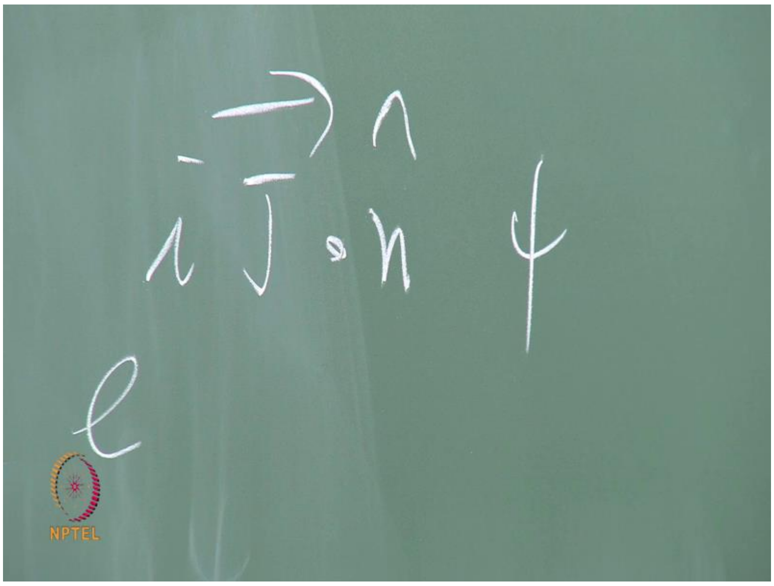

So, general rotation in 3 dimensions is parameterized in this form is written in this form J dot n psi. This is a short hand for this quantity here. Of course, n just contains 3 numbers n 1, n 2, n 3 such that the some of the squares is 1, but j 1, j 2, j 3 are 3 quantities which are themselves here represented by 3 by 3 matrices or cogonal matrices. But, they transform under rotations in a specific way.

Student: ((Refer Time: 00:44)) g 1, g 2, g 3 if we reduce a 1 columns, so we are seeing that a poly y matrices is sitting there. So, what is the, why poly y is sitting there?

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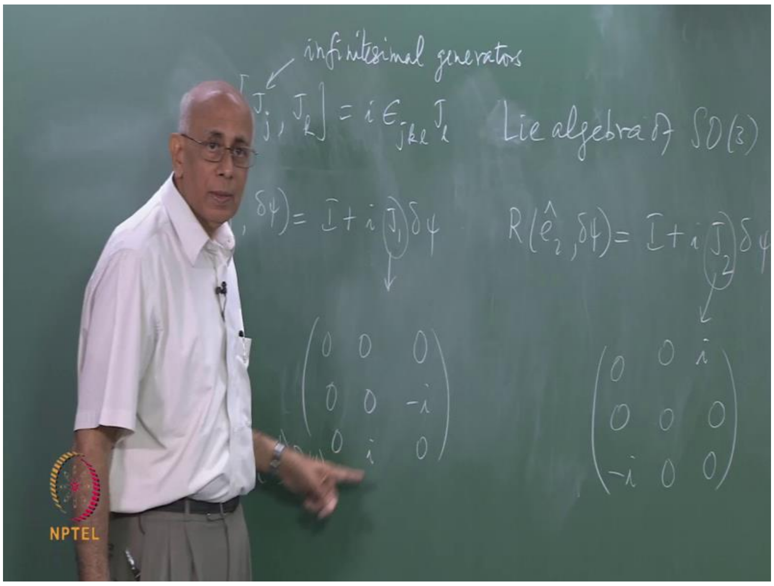

Good point, he says look there is a sort of poly matrix sigma 2 sitting here in this case. So, this means already is giving your hint that there is some connection between 2 by 2 matrices namely the poly matrices and the generators of rotations. So, this connection is the very crucial one you know become clear as we go along.

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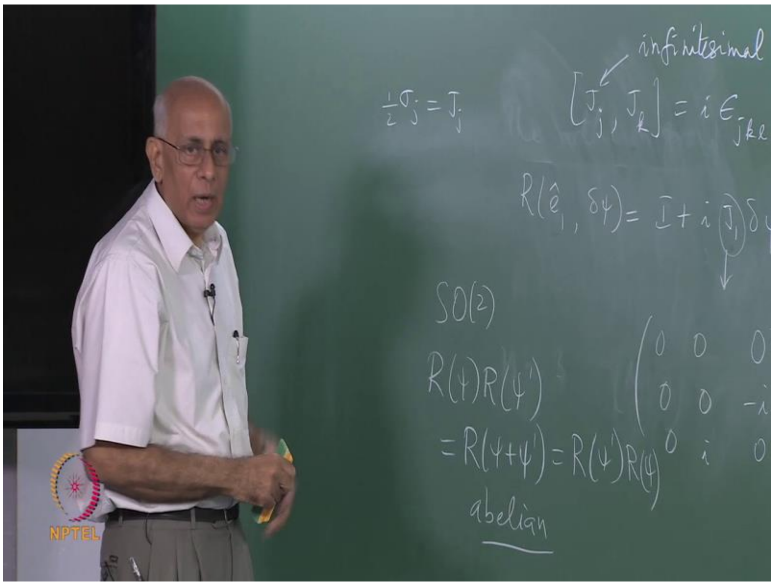

But, let me tell you right away that you can guess this by the fact that if you took the matrices half sigma J, and said that equal to J j half times a sigma matrix, these matrices also satisfy the same algebra that is easy to verify. The competitor of any 2 sigma matrices is twice the third sigma matrix, and I take the 2 out by put in a half sigma matrix and then it is ends up with exactly the third generator.

So, what it is telling you is that a very important point that as long as I say that rotations in 3 dimensions are represented are completely the information is completely encoded in these in this algebraic relations. Then the representation I choose for this these matrices should be irrelevant completely, and it is suggesting to you that there is a another way of writing rotations in 3 dimensional space not just as 3 by 3 or cogonal matrices, but perhaps as some special kinds of 2 by 2 matrices as well.

Student: So, frequently and we seen in a poly matrices x sigma, x sigma g twice i f 7 i j k then ((Refer Time: 02:39)) from there means this computation. I may write the 3 sigma matrices, and I put this down and I explicitly show that this satisfy this. So, this quantity here is a Lie algebra of S O 3. It turns out that Lie algebra has an infinite number of representations, you can represent these 3 generator by 2 by 2 matrices or 3 by 3 matrices or higher dimension matrices provided the satisfy this algebra ((Refer Time: 03:07)).

And the lowest dimensional representation that is non trivial is this 2 by 2 matrices, one by one will just commute as you can see. So, 2 by 2 is a smallest one, and that is the property of the rotation group that there exists a 2 by 2 matrix representation of these generators. So, it is suggests that there is connection between or cogonal matrices in 3 by 3 or cogonal matrices and 2 by 2 matrices of a special kind they will turn out to be unitary matrices, and this we want talk about explicitly. So, anyway I wanted to point out that you have this possibility of writing it, and that is not just notation as we will see later on.

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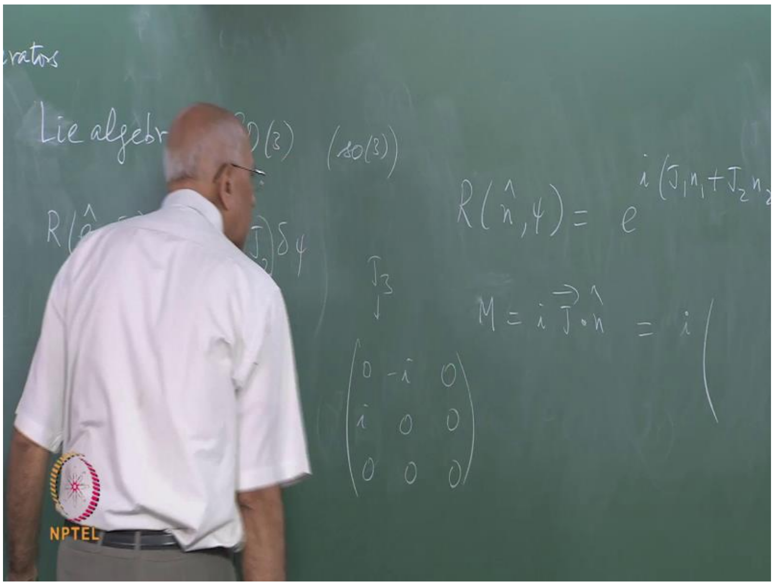

But now, let us go ahead and see what this matrix looks like, what does i J dot n look like or i times this. So, I let us put M equal to i J dot n, let us call this matrix M equal to that no vector, stands for this and what does it give you? It gives you we need to write this non explicitly, so there is an i and then J 1, so I need the 3 J (s), I need J 1 also, I have got J 1 here, I got J 2 here, and let me write J 3 here this matrix is 0 minus i 0, i 0 0, 0 0 0. So, what is J 1 n 1? Here J 1, so there is an n 1 here and then n 1 here this point. So, it is 0 0 0.

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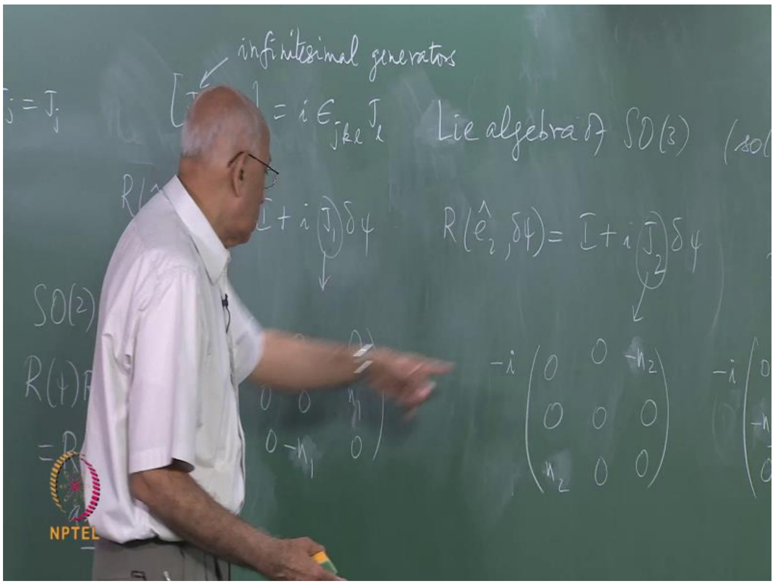

Let me write it write there, so that you can this thing I need to multiply this by n 1. So, let us take out an i or let us take out a minus i, and then this is a 1 and a minus 1, and I am going to multiply this by n 1. So, there is an n 1 sitting here there is n 1 minus n 1 sitting here. And then J 2 n 2, so I take out a minus I and this becomes a minus 1, so that becomes a minus n 2, n 2 here and then the third 1 is minus i becomes n 3 minus n 3, and then may multiply by that i outside this fellows will give me 1 completely, and I have this matrix down explicitly all I got it do use to add these 3 matrices.

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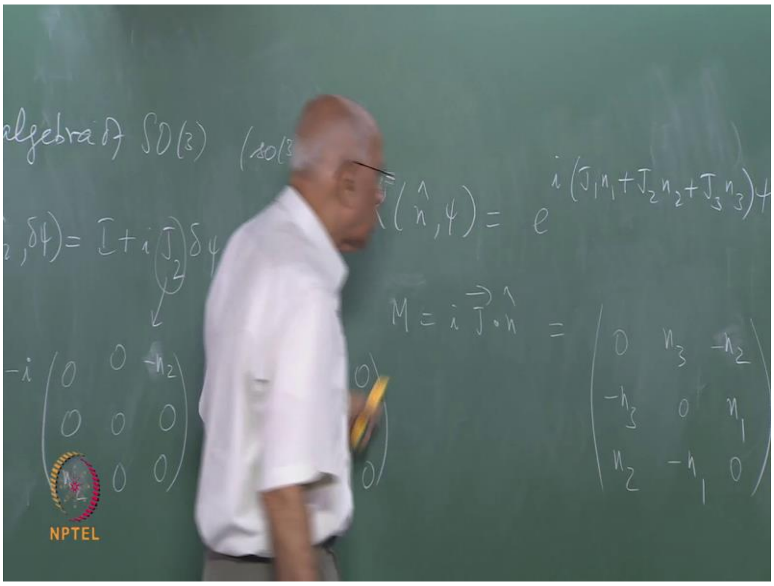

So, this quantity turns out to be 0 n 3 0 minus n 2, and then minus n 3 all diagonal are 0 of course, then n 1 and then there is n 2 here, and then the minus n 1 that is it. So, that is what the matrix m s, and we can write down what M the general element of m s very easily. So, let us do that, I notice right away it is an anti symmetric matrix. The 1 2 elements is n 3, the 3 1 element is 2, and the 2 3 element is 1.

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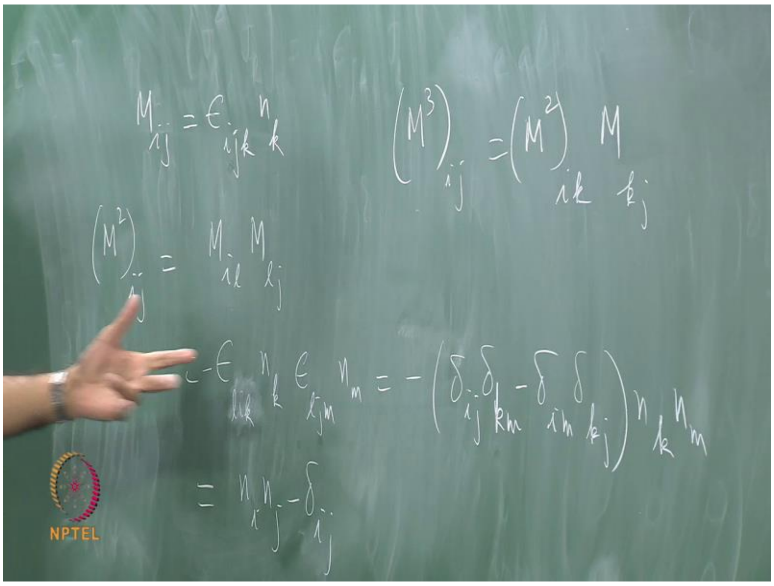

So, it immediately follows that M i j equal to epsilon i j k n k that is it right away contracted over k. So, what is M square i j? So, M square i j equal to M i l m l j. So, this is equal to epsilon i l I need some other symbol that is call it k n k epsilon l j m n m. And I use the identity which tells me that this think here once contracted is just a set of delta functions, so what that give you. So, let us put l i k with a minus sign l i k, so this is equal to minus and then you have delta i j delta k m minus delta i n delta k j, and then n k n m, and I use ((Refer Time: 08:44)) delta to simply things. So, this says M square i j is equal to well, out here I am just going to get the delta function nothing more than that n k n m.

So, this is contracted here, and that is one there is a minus sign and this fellow gives mean n n i n j. So, it says n i n j minus delta j and that is it. So, what is n cube equal to and it is M square times M, so M cube i j equal to M square i k M k j, now we know both the quantities here. Now, this fellow when it contracts this guy remember that there is an anti symmetry this thing is totally anti symmetric will give you 0. So, you end up with a result which is equal to essentially minus this guy itself. So, minus M i j or M cube equal to minus M that is wonderful, we need to find e to the M psi and I am telling you that M cube d equal to minus M, which means that the exponential collapses completely.

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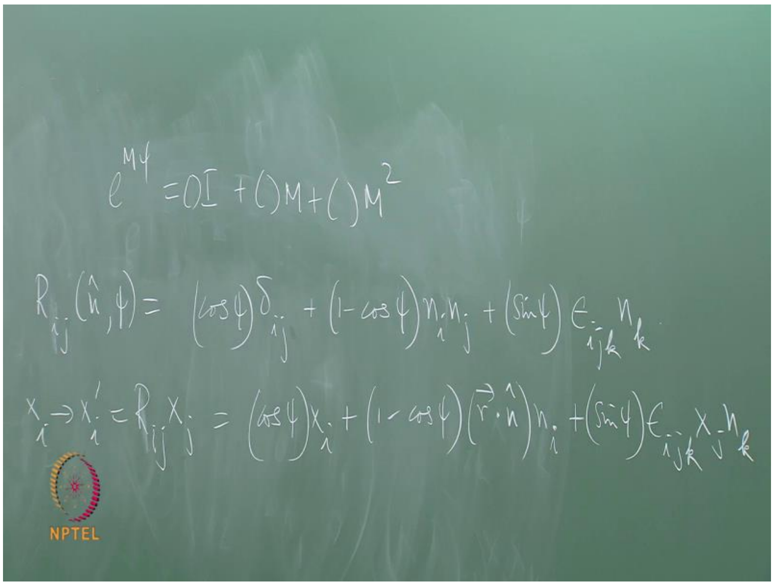

And e to the power M therefore, e to the M psi will be equal to the identity matrix some number plus something times M plus something times M square, because M cube will come back here and you can sum the whole thing. So, this means that we have actually achieved the task of exponentiating this 3 by this special 3 by 3 matrix. As you know you can write down the exponential of any 2 by 2 matrix fell easily by just expressing it in terms of the sigma matrices, but 3 by 3 that is not true there is no general formula, but in this case there is essentially, because these are you know sort of reducible representation that are sigma matrices sitting here.

So, when you do that you discover the so called finite rotation formula. So, I leave this to you as an exercise to you to show that R i j in the general case this is for n psi is equal to the following is equal to cos sign and delta i j, it is the first term plus 1 minus cos psi times n i n j it is the second term plus sin psi times epsilon i j k. So, at the end of the day that an extremely simple expression which involves as you would expect the parameters psi and the components of n here, and this is what the i j with element of this 3 by 3 are cogonal matrix R i j looks like.

Now, you can ask what does a vector do under this transformation is clear that the vector does this X i goes to X i prime equal to R i j X j, so you can write down what this guy is by simply putting X j out there and what does it give you. So, the first term if you put an X j here gives you just X i once again, so it is cos psi times X i plus 1 minus cos psi times X j with an n j that is a dot product, and then there is a n i. So, X i with n j is r dot n that is a scalar times n i plus sin psi times epsilon i j k X j n k that is what the i component of the transform vector position vector r cross.

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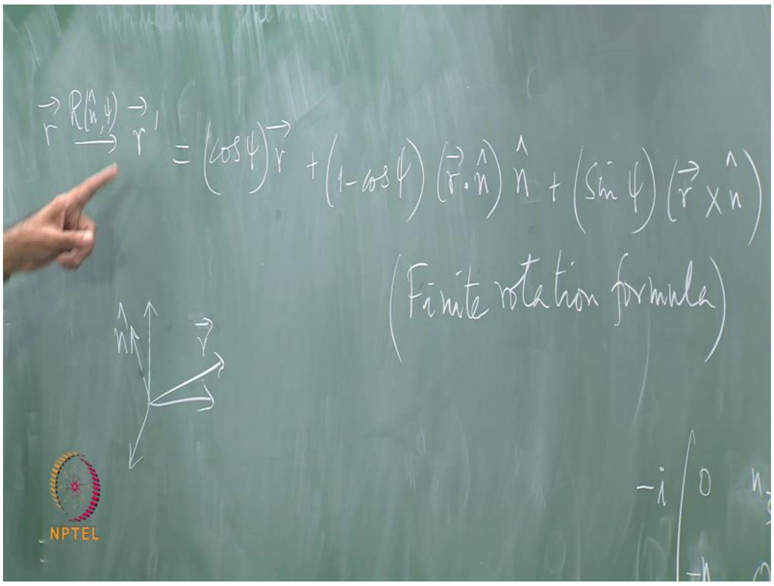

So, we can write that down in vector form immediately, and since this is true for each of the indices, this immediately implies that under the rotation any position vector r goes to n r prime under this rotation n psi, and that is equal to this psi something which is a from it is a linear transformation. So, the first term is cos psi times the original position vector itself plus 1 minus cos psi multiplied by r dot n times n vector the unit vector here for which the i th component has been written there plus sin psi times, and do you recognize this epsilon i j k X j n k is the i th component of the cross product r cross n.

So, the third term is r cross n in vector form explicitly. So, this says that if you took in 3 dimensional spaces, if you took some arbitrary vector in this form r, and you took some axis of rotation n. And you rotate at the coordinate system by an angle psi in the plane perpendicular to this unit vector n, then this r goes to r prime which is a linear combination of the original r with the cosine here, and then a portion along the axis n, and the third part which is perpendicular to the plane form by r n n.

This is called the finite rotation formula, and you can easily check that if you had no rotation at all you get r prime equal to r, because these two terms disappear all together. And if r should be along n then nothing changes this part thing vanishes identically, and if r is along n you can easily see that points are not affected at all no matter what size.

So, this thing here came about by actually physically exponentiating, this e to the i j dot psi n times psi explicitly found out what the formula is.

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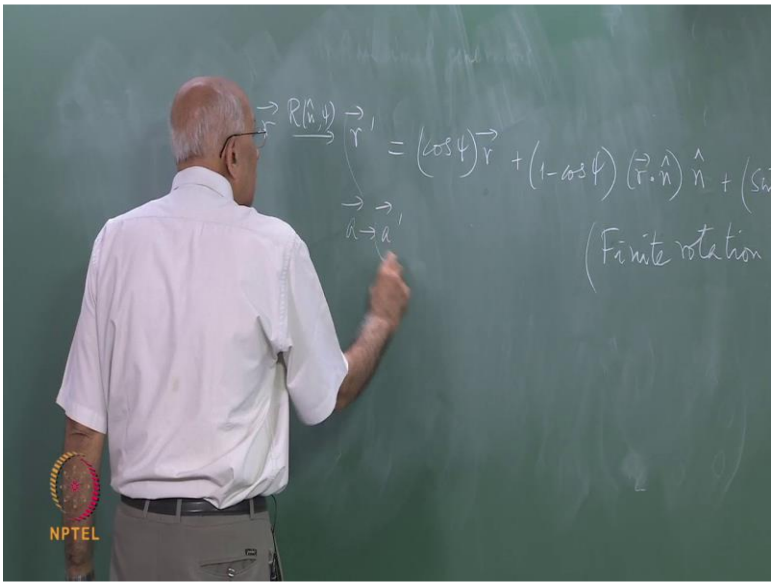

Now, the point is any vector, any vector any set of 3 quantities is a vector any see number ordinary vector is a vector, if under rotations of the coordinate system. The vector a goes to a prime which is equal to cos psi times a plus a dot n n plus sin psi a cross n. So, if this set of 3 quantities satisfies this finite rotation formula then it is a vector by definition. The next thing you could ask is and this is a certain point as j satisfies this rule at all not quite, because j is an operator it is matrices, so we got to the little more careful about how j transforms.

It two times forms like a vector, but this a not the transformation rule for operators, this is only for ordinary vectors whose components are all scalar numbers ordinary numbers. So, what we will do next time is to start with at this point, and ask is there another representation which we got a hint about namely the unitary representation for the rotation group in terms of 2 by 2 matrices. And then we will see which properties are independent of the representation all together, we will also talk a little bit about as I promised the topological properties of this rotation group in the parameter space.

Student: ((Refer Time: 17:32)) R in an ultra sin it forms a trial at...

Yes they form a trial, but not an orthogonal triad necessarily, because r and n need not be perpendicular to each other. So, in some sense what else can be answer be, this is all it can be it is a linear transformation, it should go to the identity when psi 0, then when psi goes to minus psi should have an inverse transform etcetera, etcetera all those properties has satisfied by this expression. So, one can give a kind of hand wave in physical arguments to say that it is got to be this, this it is caught to be a portion proportional to r, proportion to n n, proportional to r cross n, because these 3 will form a triad and 3 dimensional space in general. And then you can kind of figure out what they coefficients are etcetera by heuristic arguments, but this is completely rigorous we will just explicitly found their exponential.

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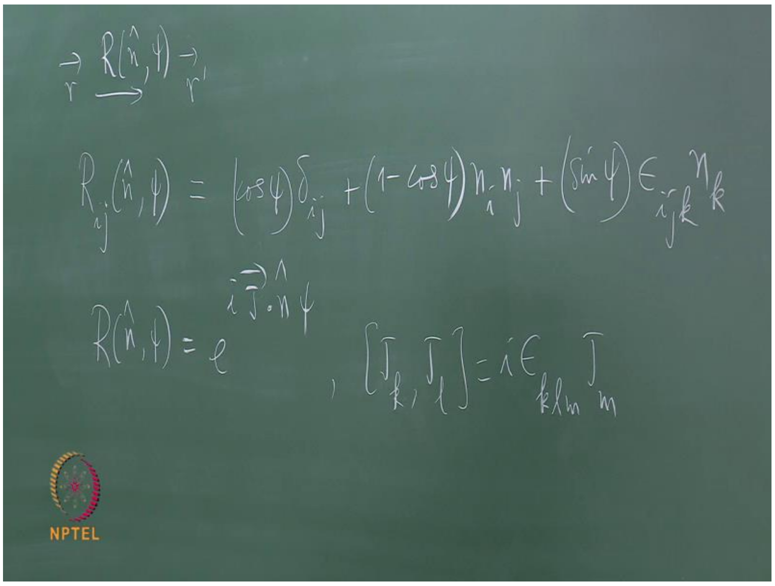

This matrix n psi could be written down explicitly we could write down the actual formula close formula for the finite vector, and it was just a cosine psi times delta i j plus 1 minus cosine psi times n i n j plus sin psi times epsilon i j k n k. So, from this the formula for the transformation of any vector followed immediately. Now, the key point here is the fact that the 3 generators of rotation do not commute with each other.

So, the main point to notice is that this rotation is generated in general R of n psi was the exponential of this quantity here J dot n psi, where J had was set of 3 vectors and they satisfied the commutation relations J k, J l equal to i epsilon k l m J m where all the indices run over the values 1, 2 and 3. Now, we also notice that these matrices J 1, J 2, J 3 when we wrote them down as 3 by 3 matrices, they happen to involve something like the sigma matrices in between one of the sigma matrices.

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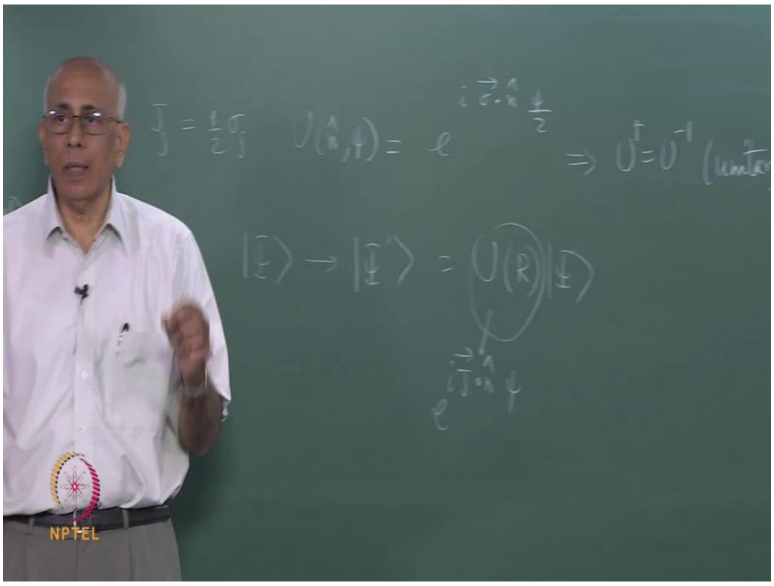

And indeed it terms out that if I said J j equal to 1 half sigma j, when because of the properties of the poly matrices, these matrices here satisfy exactly the same algebra as was satisfied by the 3 by 3 representation of the generators. This is not an accident that fact is that in 3 dimensions rotation and 3 dimensional spaces is characterized by the Lie algebra of the infinite as generator J 1, J 2 and J 3. And as long as you have 3 objects which satisfy that computation relation, you have a representation of the rotation group of the group of orthogonal matrices etcetera, etcetera.

So, you have an actually representation of this group of the rotation group in 3 dimensions, which can have any dimensionality depending on what dimensionality these things have explicitly when you write them down as matrices. So, that is the important thing this here where this J is not necessarily a 3 by 3 matrix satisfying this algebra, but any set of 3 objects which satisfies this Lie algebra here is a representation of the rotation group. In fact, you already know that if you have for instance get vector psi in quantum mechanics, and now you change your coordinate system describing let us say the size and state vector of a particle, and you change your coordinate system by rotating it through an angle psi about some axis n.

Then in general this psi would go to some other ked vector psi prime in the same space which would be related to this size and is nothing changes physically, the norm of psi does not change etcetera under this rotation. It would be related to this original psi by a unitary transformation. So, there would be some transformation U, some transformation operator U which would be a function of this rotation that would date on the coordinates, and that would act on this psi.

So, I write U of R loosely in order to tell u that it would depend on the parameters n and psi of course, but the representation of this U does not have to be 3 by 3 matrix, because after all it is acting on a ked vector and that could be a ked vector in an infinite dimensional Hilbert space. In which case if for instance this psi is represented by an infinite dimensions column vector. Then this U would have to be an infinite dimension matrix again, but what is for sure is that this U is guaranteed to be e to the i j dot n psi I am sorry to use this psi is a small angle here this small psi, where this j comprises 3 objectives J 1, J 2, J 3 satisfying that Lie algebra.

And that the dimensionality of this J, if you write it as a matrices would have to be compatible with whatever is out here. If this is an infinite dimension column vector then this J would have to be an infinite dimension matrix as well, so that it could act the exponential of it could act on that. So, here what is happened is that the coordinate the rotation of the coordinate system has induced a transformation in Hilbert space given by the operator U the unitary operator U. So, that is spirit if I write J I j is half sigma j then it matter of it is the matter of trivial verification that those 3 matrices satisfy exactly this algebra. Therefore, you expect that there exist a 2 by 2 matrix representation of the group S O 3 in 3 dimensional spaces.

Now, what is that representation explicitly? Well here it is all you have to do is to write

U of n psi equal to e to the power i times sigma dot n psi over 2 and that is it, because these matrices sigma over 2 they actually satisfy the Lie algebra, and that is like of the general form where these J (s) are represented by half the sigma matrices here, why did the I put the U here, because you notice that because the sigma are hemission you notice that U is a unitary operator. It is easy to see that this will immediately imply that U diver equal to U inverse, U is unitary. So, this here is a unitary representation of the rotation group in 3 dimensional space, but it is a 2 by 2 representation.

Now, you could ask if it is a 2 by 2, 2 by 2 matrices then how do I implements it in 3 dimensional space. I have points in 3 dimensional space specified by 3 coordinates, and it is convenient when I use the representation of 3 by 3 orthogonal matrices, what I do is to write the coordinates one below the other x 1, x 2, x 3 and then R acts on the left and gives me the new coordinates. Now, what kind of thing object will U act on it has to be 2 by 2 object either a column vector with two numbers or it has to be a 2 by 2 matrix. These are the only possibility is it can be a column matrix with two numbers, because you actually need 3 pieces of information to specify a point in 3 dimensional space. So, it is got to be it is the 2 by 2 matrices and what is that matrices?

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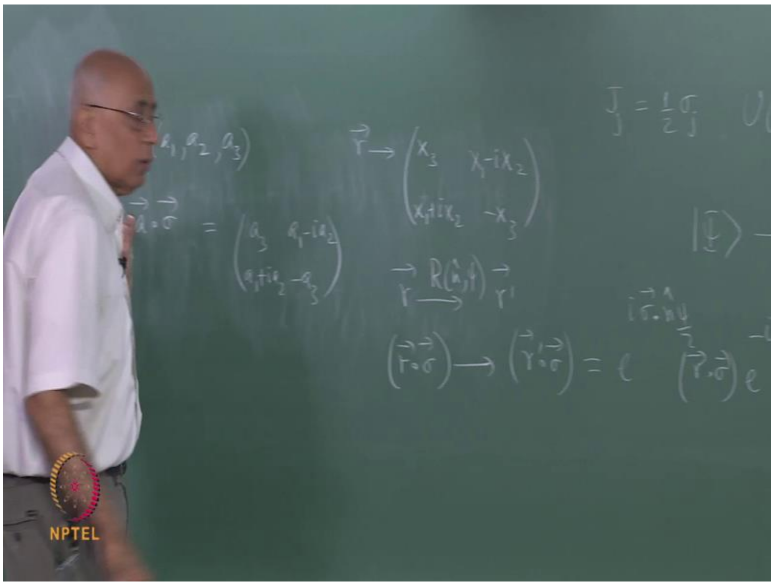

Well, you already know that given any 3 dimensional vector a made up of this order triple a 3, you could construct the matrix a dot sigma the 2 by 2 matrix a dot sigma which is equal to since sigma is they is diagonal in the usual representation, this is a 3, this is minus a 3 and then this is a 1 minus i a 2 a 1 plus a 2. Specifying this matrix is equivalent to specifying a 1, a 2, and a 3 and vice versa. So, this could be taken as a representation of a vector in 3 dimensional space 3 components. In which case the coordinate are would just look like instead of r, the representation for this r would be x 3 minus x 3 x 1 minus i x 2 x 1 plus i x 2.

So, instead of r I replace it by r dot sigma which is this 2 by 2 matrix, and then ask what does a rotation due to it. Now, since you got a matrix which is more like an operator than a column vector. It is clear that the way the rotation mode operates is the following just as r goes on the rotation r inside to r prime; the quantity r dot sigma goes under the same notation now to r prime dot sigma where r prime has components x 1 prime, x 2 prime, x 3 prime it is the another 2 by 2 matrix that is related to the original 2 by 2 matrix by e to the power i sigma dot n psi over 2 r dot sigma e to the power minus i sigma dot n psi over 2.

So, it is U times this original matrix times U inverse in the right hand side, and it is easy to check that indeed you would get the finite rotation formula for the components of r, it is exactly this same thing. Because, if you wrote this out this one here is this matrix here, and this is with the prime here, and then you compare you discover it just the finite rotation formula. For that you need to able to write this explicitly down, the exponential of this 2 by 2 matrix. Now, we already wrote the exponential of 3 by 3 matrix, the r matrix yesterday the finite rotation formula did that.

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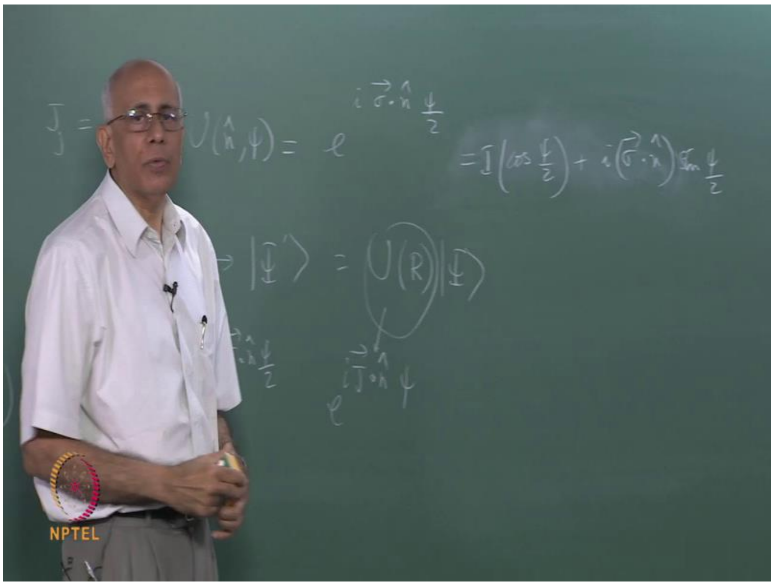

On the other hand this thing here is easily written down by this little generalization of Euler’s formula. It is cos psi over 2 times the unit matrix plus i sigma dot n sin psi by 2, so that is easy to verify. It has to be something like that, because any 2 by 2 matrix can be written as a linear combination of the unit matrix and the 3 poly matrices. So, you could use that fact and compare coefficients or you could explicitly find the exponential here using the fact that the square of every sigma matrix is the identity matrix 2 distinct sigma matrix and I commute with each other.

So, sigma 1, sigma 2 plus sigma 2 sigma1 is 0 and so on, and then this formula pops out very easily. And all you have to do is to put that matrix here r dot sigma is sitting here, and this is the same thing with psi goes to minus psi all the complex conjugate put that in here, and that is it when you verify that you get exactly the finite rotation formula. So, what does this mean?

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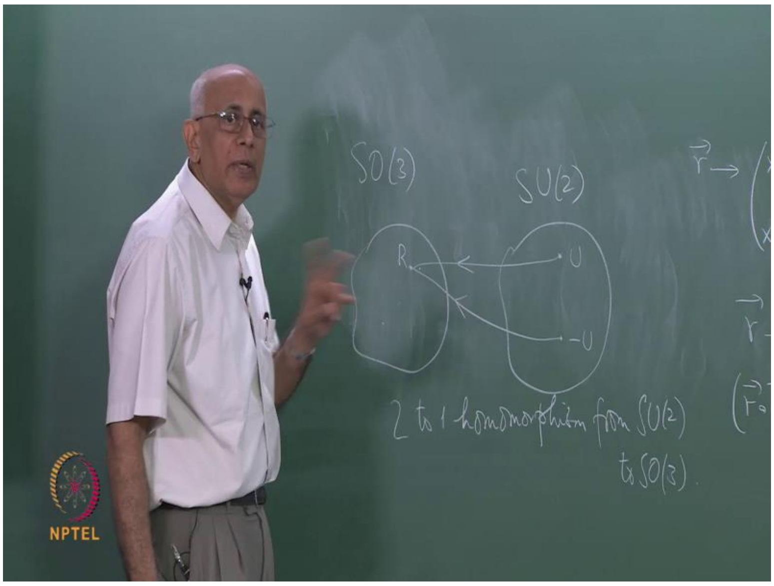

This means that there is a connection between every 3 by 3 orthogonal matrix S O (3), every 3 by 3 orthogonal matrix with unit determinant every proper rotation. There is a connection between every such matrix and the corresponding matrix U in this case with the same parameters n and psi. So, it is a direct relation between these 2, the question is what sort of mapping is this between one and the other. So, you have the set of matrices S O (3) these are all elements of this points in this space, each point is the unimodular orthogonal matrix 3 by 3 matrix describing a rotation.

And on this side you have a collection of matrices of the form of U is a unitary matrices, and the 2 dimensional, so one write it is as U (2) and the determinant is also 1 as you can easily check and it is S U (2). This is the special unitary group of 2 by 2 matrices with possibly complex entries in general, and there is the connection between this and that. The question is is it a 1 to 1 connection, for every matrix here do you have 1 matrix here or vice versa. The answer is you do not it is not a 1 to 1 connection, because of this fact that if I change U this is U r dot sigma U dagger, but if I change U to minus U, U dagger also become minus itself, but r prime dot sigma does not change.

So, it is immediately clear that if you give me a particular R here, and there is a matrix U here minus that matrix gives me exactly the same physical rotation. And therefore, there is 2 to 1 connection between them, this is a called a 2 to 1 homomorphism from S U (2) to S O (3) both these fellow map on to the same point. S O 3 of course unique a little more to as sort this, S O 3 of course is parameterized by 3 real parameters, the components of the unit vector n two of which are independent, because there is the some of the square is equal to 1 and the angles psi, so there are 3 real parameters in this space.

Every element here is parameterized by 3 real parameters that should be the same here otherwise there is no question of mapping from one to the other. So, we need to know, what is the most general form of a 2 by 2 matrix that is unitary and has determinant equal to 1? Now, ((Refer Time: 32:53)) general 2 by 2 matrix in general with complex entries it has 8 independent parameters, because there are 4 elements in each of them is complex in general. So, there are 8 independent parameters, let us count the number of parameters in S U (2).

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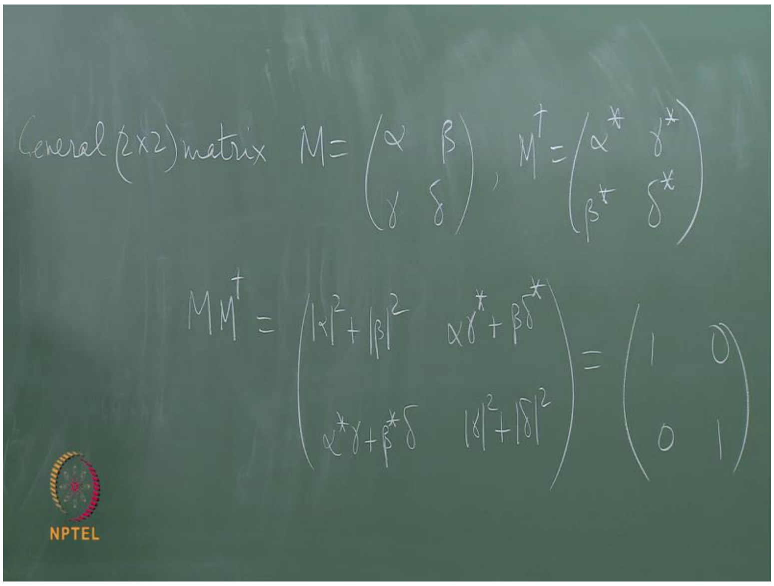

So, let us write a general 2 by 2 matrix M let us write this in the form alpha beta gamma and delta some complex number alpha beta gamma delta, and then impose the condition that this be unitary and let us see what happens? So, M dagger in general is alpha star delta star gamma star and beta star complex conjugate transposed in this form. And then it is construct M M dagger, this is equal to mod alpha square plus mode beta square that is this term. And then alpha gamma star plus beta delta star and then out here you have alpha star gamma plus beta star delta. And then mode gamma square plus mod delta square and you want this to be unitary, so you want this to be the identity matrix and then be solve for these elements and see what happens.

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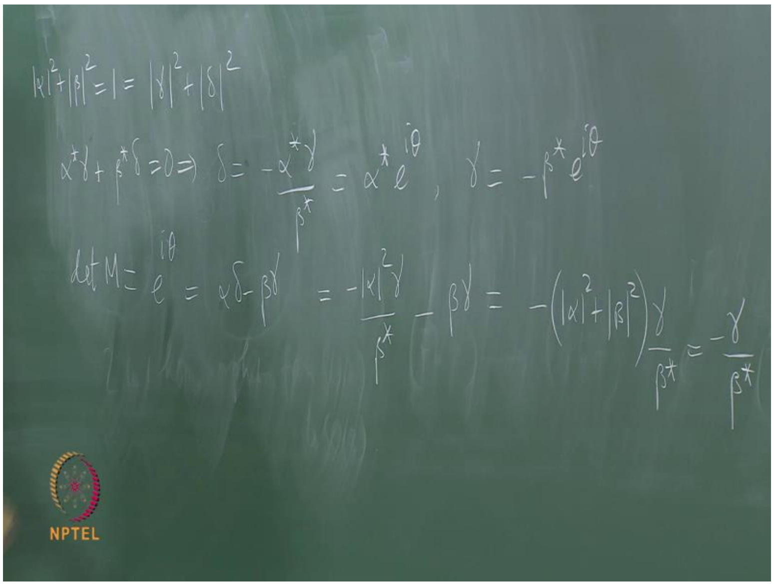

So, we right away have mod alpha square plus mod beta square equal to 1 which is also equal to mod gamma square plus mod delta square, so you got 2 conditions 2 are the parameters are gone eliminated. And then the half diagonal element alpha star gamma plus beta star delta equal to 0, let us eliminate delta for example delta equal to minus alpha star gamma over beta star. The other one is just a complex conjugate of these conditions it does not give you anything new, but we also want to find out what the determinant conditions says?

So, you want to put in the determinant and see what that does, and what is determinant of M equal to? It is a unitary matrix that is the matrix analog of a complex number with unit modulus, and what is the most general form of a complex number with unit modulus e to the i some real number. So, it says determinant of M must be of the form e to the i theta, the theta is any arbitrary real number, because it is a unitary matrix. This of course immediately says this is equal to alpha delta minus beta gamma equal to e to the i theta, and you can once again eliminate one of the variables favors, so what shall we do?

So, let see do this in a little more efficient way I like to do this in a little more efficient way. So, if I put this in, so this is equal to alpha delta, so that is minus mod alpha square gamma over beta star that is this term here I put in this value of delta here minus beta gamma that fellow which is equal to minus mod alpha square plus mode beta square gamma over beta star is equal to minus gamma over beta star, because mod alpha square plus mod beta square is 1 all have done is to simplifies this in use that condition here.

So, if put that in minus gamma over beta star this says delta is equal to that should equal to e to the i theta, so it is says alpha star e to the i delta is equal to that. And similarly, gamma is equal to minus beta star e to the i that just this conditions here. So, it is says the most general 2 by 2 matrix which is unitary must have the following form.

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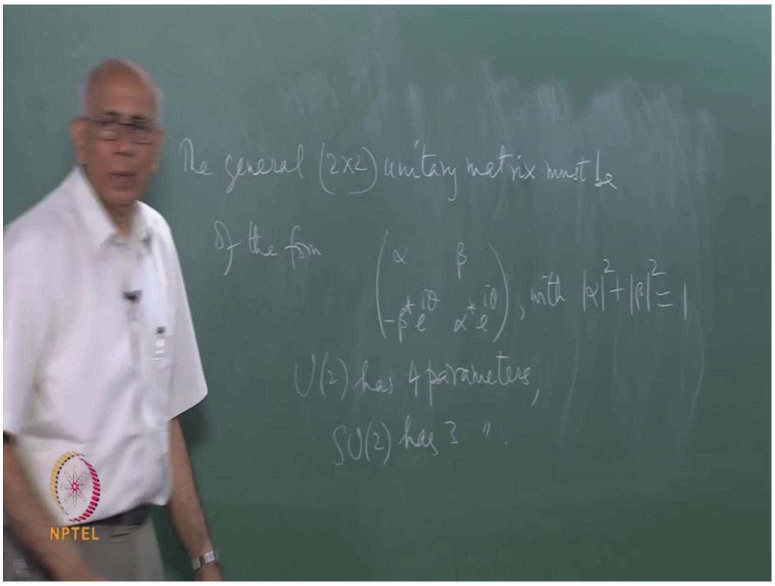

So, the general 2 by 2 unitary matrix must be of the form is write that down. The some complex number alpha, then there is a complex number beta and then a minus beta star e o the i theta, and then delta what was that alpha star e to the i theta with a condition mod alpha square plus mod beta square equal to 1 that is the most general form of a 2 by 2 unitary matrix, and determinant equal to e to the i theta, how many parameters are there in this how many parameters are there in this?

Student: 2 3

Why do you say that?

Student: ((Refer Time: 39:34))

There how many real parameters are there? Will you count parameters you have to count how many numbers have do you have specified independently? When you specify complex number you have to specify both it is real and imaginary parts which you can do independently. So, that is really being counted as 2. So, what is the total number of parameters in this?

Student: 3

There is 3, because between alpha and beta you have 4 parameters, but there is one real condition between them, so there are 3 independent numbers here, and there is a theta. So, this is a 4 parameters group that cannot match the rotation group in 3 dimensions, because that is a 3 parameters group, but the moment you make the determinant equal to 1, then theta must be 0, because you saying the determinant is e to the i theta must be equal to 1. So, this means that S U (2), so this is U (2) is the group U (2) has 4 parameters, S U (2) has 3 parameters the homomorphism that we talking about the homomorphism is between S U (2) and S O (3), and e to the power i sigma dot n over psi over 2 is a unitary matrix with determinant 1 as you can easily check. And what is the most general form of this matrix in S U (2)?

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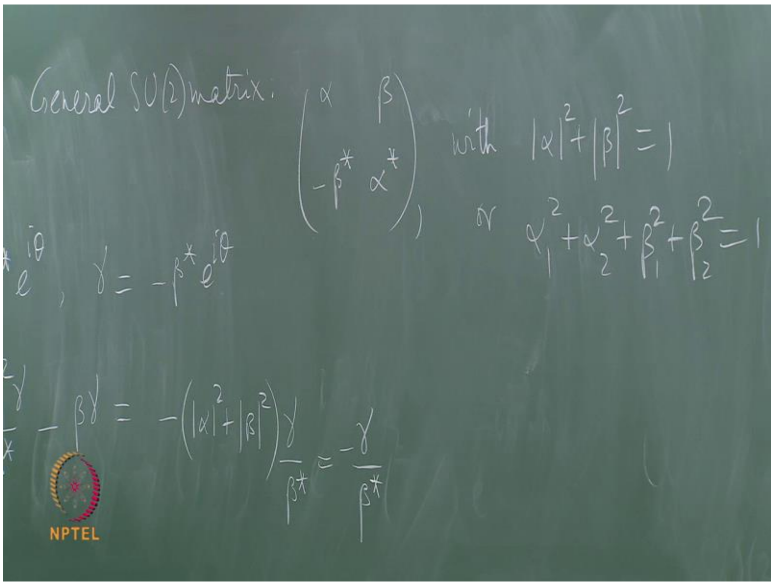

What is the most general form of in S U (2) matrix? Put theta equal to 0 that is it. So, general S U (2) matrix is alpha beta minus beta star alpha star satisfying the condition with mod alpha square plus mod beta square equal to 1 or if I write it in terms of real parts and imaginary parts. And the client now is that you have 2 matrices in this space in this group of matrices S u (2) corresponding to each physical rotation in 3 dimensional space.