Selected Topics in Mathematical Physics
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Module - 13
Lecture - 34
The Rotation group, and all that (Part - I)

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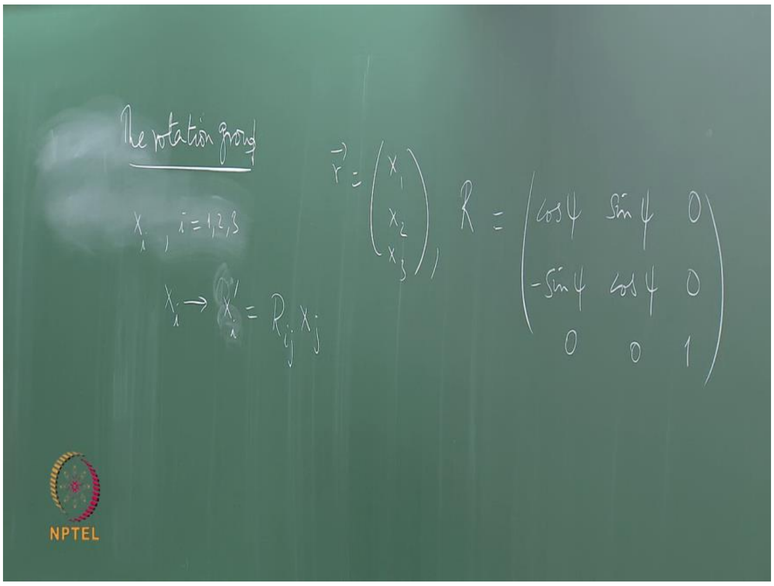

So, today we will change stream, and today, and tomorrow we will talk about slightly different subject dealing with all these days namely partial differential equations, today we will talk a little bit about lie groups, and about the rotation groups specifically the rotation group it is connection to the unitary group, and the little bit of the properties topological properties of the parameter spaces of these groups. So, this will be just about enough material for today, and tomorrow, and let us start off with material which we already know now about the rotation group we could just as well consider the rotation group in n dimensions.

So, very frequently I will talk about n dimensional rotations in n dimensional Euclidean space or occasionally specialised to three dimensional Euclidean space, and point out there are some very major differences between rotations, and even number of dimensions, and odd number of dimensions the very important physically now the notation i use will be that of Cartesian tenses which are presume everybody knows the index notation with summation convention. So, in the series like i j k etcetera will run from one to n in the n dimensional case. So, one to three in the three dimensional case. So, let us look at the simplest case the three dimensional rotation one frequently talks about we have coordinates x i i equal to one two three, and one knows that under a rotation of the coordinate axis about the origin keeping the origin fixed one knows that distances do not change distances between points do not change at all, and that this rotation is a linear transformation in the sense that x i goes to some x i prime in the new coordinate system which is a linear combination of the coordinates in the old coordinate system, and it is a homogeneous transformation, because the origin does not change.

So, the point zero remains the point zero the origin remains itself, and it is a linear transformation, because it is linear combination each new component is the linear combination of the old component here now the first property we know now let me give a example of this thing here we can write this as matrix, if I write position vector r in the form x 1 x 2 x 3, then this matrix r is a three by three matrix with elements where the matrix itself is orthogonal as we will see within a minute, because distances are preserved, and a rotations, and it is got unit determinant if it is a proper rotation let us see what is this look like an example for instance of what r is like would be if you rotate in the x y plane through some angle psi leaving z axis untouched, then this lonely affect x, and y coordinates we can immediately write down from elementary trigonometry what this matrix looks like it looks like cos psi sine psi zero minus sine psi cos psi zero zero zero one.

So, that is the matrix which had determines what the new coordinates look like what linear combination they are of the old coordinates here, and since the x 3 axis are untouched all points on it remain by the way originally you had one here zeros here ensuring that only x, and y coordinates get exchanged with each other get mixed up with each other that is one example, but more generally more general way of writing rotation is to parameterize this rotation in some suitable form this rotation we fixed the rotation axis to be z axis, and we set that the rotation through an angle psi in the x y plane. So, you could do this in general, and say there some direction in space unit vector n about which you are going to rotate the coordinate system through some angle psi. So, we will use this notation, and I will write r of n psi to denote the face that this the axis of roatation, and this is the amount by which the rotation occurs. Now of course, this is an add fact of three dimensions, because rotation about an axis the concept of rotating about an axis is peculiar to three dimensions the reason is we rotate in three dimensions if you are in a plane evolve this a plane Euclidean plane, then what you doing is really rotating about the origin there is no third dimension at all.

So, there is no access of rotation as such if you rotated in four dimension for example, and the coordinates are x 1 x 2 x 3, and x four you could rotate in the x 1 x 2 plane leaving the x 3 x four sub space untouched, and of course, there is no question of unique access about which you make a rotation it is a whole subspace that left untouched. So, the idea that you rotate about an axis which is a common way we always imagine rotations is two, and three dimensions for a very peculiar reason in three dimensions with dimensionality n equal to three, and a number of mutually orthogonal independent planes that you can find one set of planes independent set of planes would be in three dimensions x y y z, and z x planes, well it really means you choose two coordinates, and that is a plane two coordinates fix a plane.

So, the number of planes that you could find is n times n minus one over two, and this is happens to be equal to n for n equal to three. So, the accident that you have three planes mutually orthogonal planes coincides with the number of axis that you have is peculiar to three dimensions. If you have two dimensions you have one plane, and no third axis at all, if you had you have two sorry you have two axis the x 1 x 2 axis, and you have one plane on other hand, if you had four dimensions you have six planes you have the one two one three one four two three two four, and three four planes, but you have only four axis the one two three four axis. So, you see this accident is what enables you to say i rotate about z axis one should technically say this is the rotation in the x y plane, and so on. Of course, these mutual orthogonal planes need not be perpendicular to the coordinate axis you can choose any set of mutually orthogonal planes, but the numbers fixed in n times n minus one over two.

So, I will use this language loosely occasionally, but one should remember that it is really rotation defined in a plane always yeah well, if I have n coordinates a plane is determined by any two of them, and these two you can choose n times of n c two ways. So, you have that many planes, but, then you have n coordinates. So, the number of axis that you have in Cartesian coordinate system is n, but the number of planes is n c two, we just not same as n except when n equal to three. So, the question of there can I associate always an axis with the rotation is a tricky one it is clear that in even dimensional space is mean be the case at all in two dimension there is no third axis that on. So, you cannot associate an axis with the rotation, and that about the origin, and in four dimensions you have two extra axis which may not be touch by a rotation at all.

So, the question associate always a unique axis with the rotation is not always the case is not always possible, we will see how in the case of odd dimensional space you can always do this, but in even dimensional space this does not have necessarily have to be case you could have rotation about an axis in your four dimensional space, but all I am saying is all rotations do not necessarily have to have unique axis associated with it I mean four dimension I could rotate about the x 1 axis, I am say here is you know rotation.

So, it will do something together the coordinates leaving x 1 un changed, but the fact is that you do not always have to do right ,there are rotations where you cannot associate a unique axis rotation all right. Now the next question that arises is speaking of three dimensions, then we say little more about these rotations the answer is even in general these rotations form a group, because thus combination of two rotations a succession of two rotation is again a rotation as you can easily verify every rotation has an inverse which you brings you to the identity rotations which means no trans transformation at all the identity transformation, and that is all you need. So, there is under matrix multiplication of these matrices you can produce the result of a succession of rotation. So, it is clear that there is a correspondence between the set of abstract rotations in n dimensional space, and a set of matrices, and n dimensional space the matrices would be (( )) matrices that is one we have representing the rotation, and the matrices would be orthogonal orthogonality is immediately established is not difficult at all, because under our rotation i want x i x i to be equal to x i prime x i prime i want this to happen, and now let us write what this implies well x i prime is equal to r i j x j, and the other x i prime is some r i k x k if you like.

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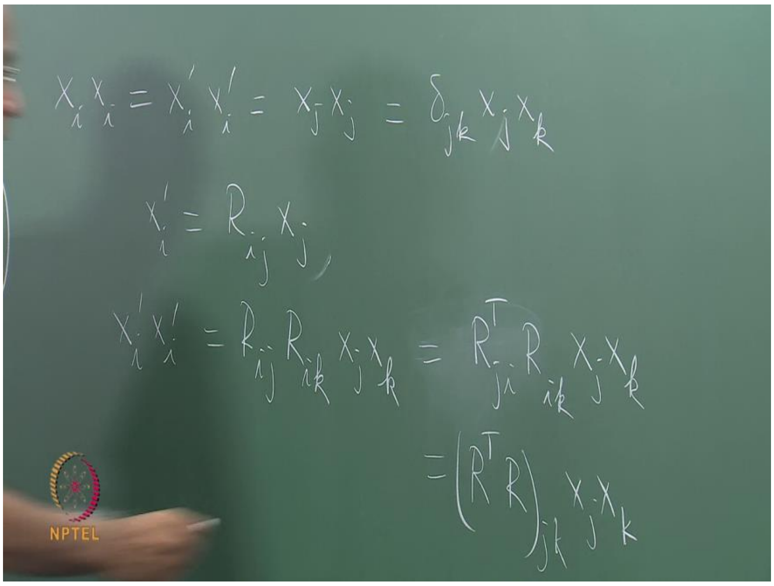

So, this means that x i prime x i prime this stands for r prime squared, because the i is summed over. So, it is the some of the squares of the coordinates here this is equal to r i j r i k let say x j x k. So, j k, and i all of them appear twice therefore, they are dummy in the series they have to be summed over, and this could be written also as equal to r i j r transpose, let us write it other way this is equal to r transpose j i r i k x j x k which is equal to r transpose r j k element x j x k on the other hand that this thing here should be equal to x j x j, because r prime squared equal to r squared, but I could also write this as equal to delta j k x j x k that delta j k is chronic delta.

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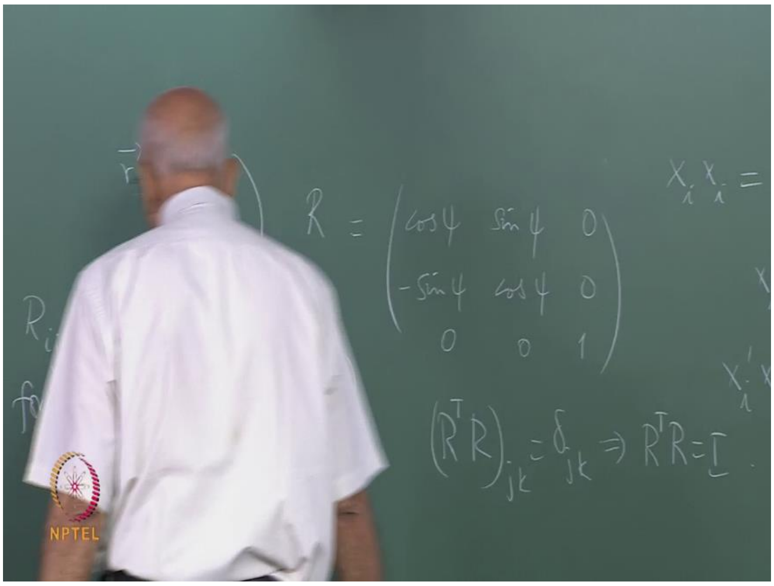

So, this means that r transpose r j k must be equal to delta j k for all j, and k, which immediately implies that r transpose r j k th element equal to delta j k, but the chronical delta is just the matrix element of the unit matrix. So, this implies that as a matrix r transpose r equal to the identity matrix the matrix whose inverse is equal to it is transpose is an orthogonal matrix as you know. So, this means that rotations are orthogonal matrices, and this as we arrived at the conclusion by simply imposing the condition that the distance from the origin to any point should be unchanged under the rotation about the origin that is all we used now for finite dimensional matrices r transpose r equal to i also means that all r transpose equal to i, because left inverse equal to the right inverse in such cases.

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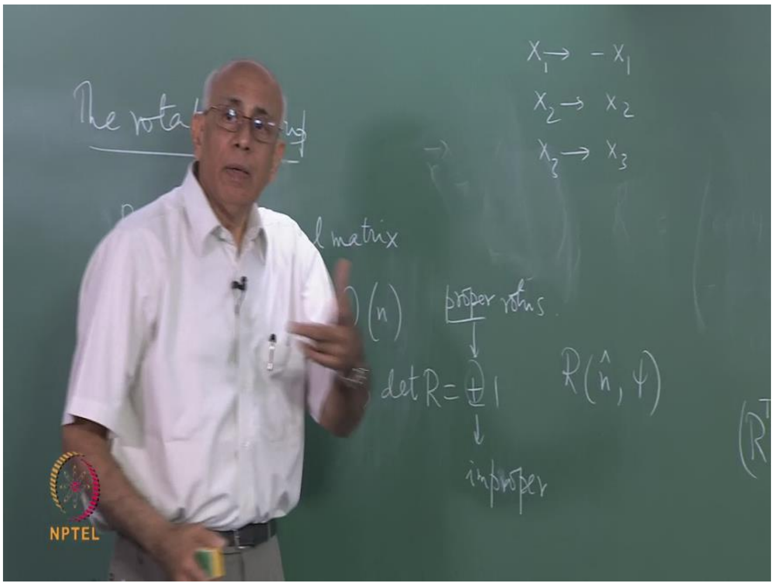

So, very easy to establish that, and what does this imply is r is an orthogonal matrix by the way I did not use the fact that you are in three dimensions for this is true in general in n dimensions, it is immediately true an orthogonal matrices n by n orthogonal matrices form a group the set of orthogonal matrices form a group, and it is denoted an element of the orthogonal group which is denoted by o n that is the group of n by n orthogonal matrices, and we have restricted ourselves to a real matrices matrices with real entries real elements I have put that in explicitly otherwise you should put n comma r here to show it is real. Now this set of matrices an interesting property it is a group, but that group it is not very, very trivial group it has some structure to it, and that you can see from that fact that if you take the determinant of this both sides of r transpose of r equal to i you immedietly discover that determinant r squared equal to plus one that is the determinant of the unit matrix, and the determinant of r transpose. Of course, as same as the determinant of r, because the matrix determinant does not change if you interchange rows, and columns.

So, this implies that determinant r equal to plus one or minus one two possibilities, and the both included here in this now matrices with plus one determinant we call proper rotations, because the identity transformation itself has plus one, and this means that all those matrices with determinant plus one this thing here corresponds to proper rotations we can actually be construct it continuously by moving continuously from the identity. So, you start with original coordinate system, and you want to get proper rotation coordinates look like this you can do. So, by a succession of infinitesimal transformations each of which has determinant plus one. So, this property of being connected to the identity is what defines a proper rotations here on the other hand there are matrices whose determinant as minus one, and the discontinuous transformations there is no way of reaching a determinant minus one by continuously starting with the identity succession of transformation continuously like has to be a discreet transformation somewhere it would still be orthogonal, but it would not have determinant plus one.

Now in example three dimensions for instance would this. So, if x goes to x 1 goes to minus x 1, but x 2 goes to x 2, and x 3 goes to x 3 this corresponds to an r it is still in the rotation group, but it correspond to an r whose matrices minus one in the first diagonal one in the other two diagonal, and zero everywhere else, and the determinant is minus one. Now of course, you will this is not really a rotation in technical sense of the term why it is not a rotation it is a reflection really it is a reflection about the x 2 x 3 plane the x 1 axis goes to minus x 1 right. So, reflection of this kind is still in the rotation group in the orthogonal group, but it is an improper transformation. So, having made this reflection.

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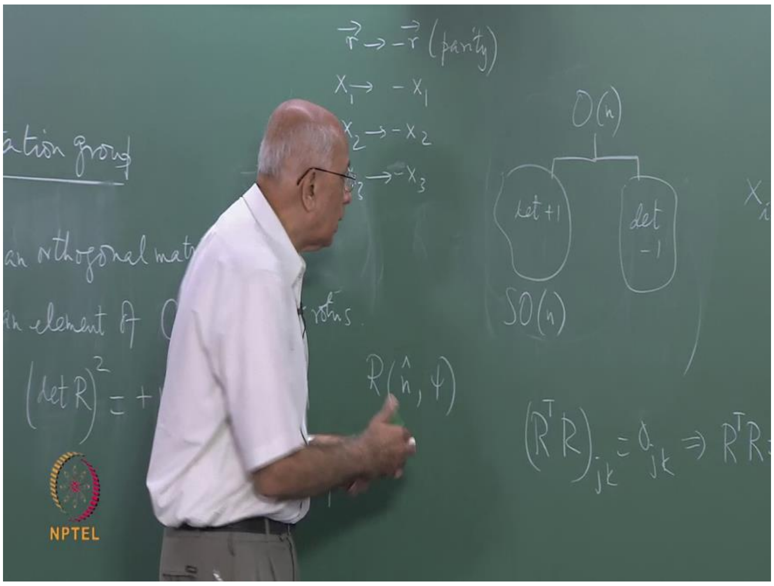

If you now rotate the coordinate system by a proper rotation the result is still a determinant minus one transformation, because the determinant of a product of a matrices is the product of determinac right. So, you would construct all in the in the in an abstract sense here is the set o n, but it really has two disjoint pieces there is one piece you should write it really like this, there is one piece which is determinant plus one, and another piece which is determinant minus one, and o n has both the set of matrices is disconnected the one set has the determinant minus one one set which has determinant plus one, but the both together belong to the rotation group the orthogonal group.

However, that will it cause here, now where do you get elements of this from this consist of all proper rotations things, which you can continuously build up from the identity transformation this thing here consist of all those matrices which you can get by taking a discontinuous transformation like reflection or parity you could also have had this had this is the parity transformation r goes to minus r; that is got determinant minus one as well. So, reflection or parity transformation or any discontinuous transformation which has determinant minus one, and you can multiply every element of this set here with one of these transformations, and you get an element of this part of this group what these things form a group among themselves, and this is called the special orthogonal group. So, this is s o n this s stand for determinant plus one or special a uni modular portion, and product of any two of them will still have determinant plus one. So, they form a group, and there is an identity element in there. So, they form a group on the other hand this set does not form a group that is obvious, because there is no identity element here minus the identity matrix was there, but not the identity matrix to start with or another way I am seeing it is, if I multiply two of these transformations one following the other the determinant becomes plus one. So, it is no longer in this set , but the parameter space of o n therefore, is disconnected there is a piece which is proper that forms a subgroup, and that is the special orthogonal group, and there is a piece which is got only minus one matrices which are proper transformations multiplied by one of these discontinuous transformations, you have to be little careful here in an odd dimensional space the parity transformation is not a improper transformation you know that is very, very immediately obvious.

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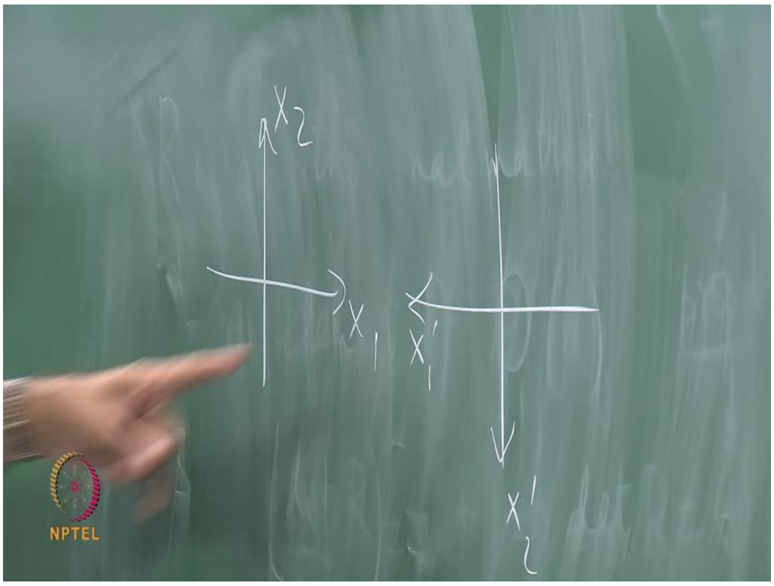

If you had just two coordinates x 1 x 2. So, you have x 1 x 2 in what in this by parity we just do this would become x 1 prime x 2 prime. So, x 1 goes to minus x 1 prime x 1 prime is minus x 1, and x 2 prime is minus x 2. So, it looks like this, but this is a proper transformation, because the determinant is got a diagonal matrix with minus one, and two two elements, and this regard plus one as the determinant on the other hand you can see, but this is attain from this by simply rotating about the origin through pi. So, the parity transformation is a proper transformation in an even dimensional space, because the determinant is plus one, but if I reflect about this axis or this axis the one of the coordinate changes sing, and then it becomes discontinuous transformations, but in three dimensions which is most common one you are interested in parity is an improper transformation all right. Now having done this, let us ask what is the basic property ah what what do we do about the fact that these are group, and how is these group generated. So, I speak the idea is the following at like my target finally, is to able to write down this matrix as a three by three matrix, I would like say you give me an arbitrary access in space in the original coordinate system unit vector.

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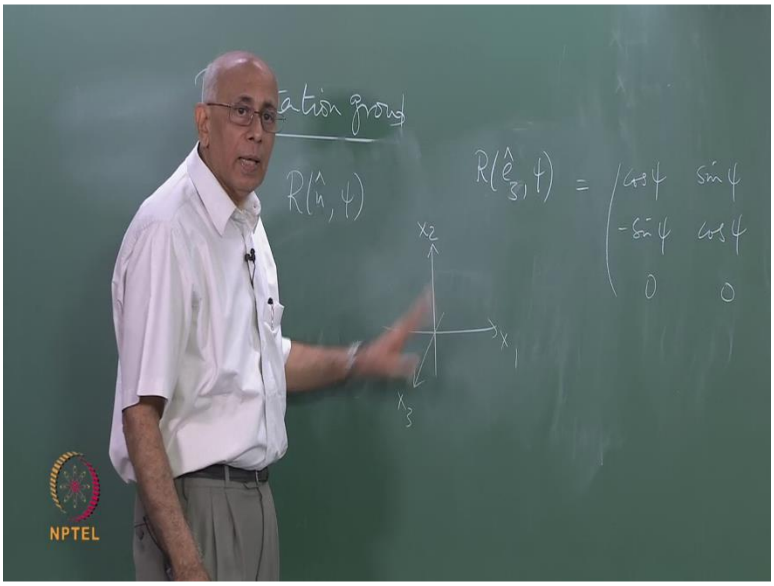

And I rotate through an angle psi for any way from zero to two pi, and I want write the matrix now three by three matrix just as I wrote it down in the case where we rotated in x in the in x 1 x 2 plane. So, that is the target, and the way to do this is to observe the following it is a little algebraic, and the trick is following if you looked at r of inverse of e three (( )) direction psi this was a rotation in the x 1 x 2 plane this matrix we wrote down by elementary analysis this is psi zero, and I have been saying that this rotation should be obtainable by a succession of transformation if for instance draw picture, and i say this is x 1 this is x 2, and then x 1 cross x 2 the x 3 axis which is coming out of the plane of the both x 3 this part, and I make a rotation in this plane, then it is clear to get a finite angle, I can rotate by a succession of infinitesimal rotations.

So, let us see what an infinitesimal rotation cause to first order. So, the same thing tell me r of e three delta psi this is first order infinitesimal this thing here is equal to first, and I the unit matrix, where i replace all these by one to first order cos psi is one, and I take out the unit matrix, and then plus a matrix which is basically zero there is delta psi comes out, and then a one zero or minus one zero zero zero zero zero, I have replace sine psi by sine delta psi which is delta psi itself to first order, and I pulled out the delta psi constant, and that is it that is the matrix. Now for reason you should clear in a minute these quantities here will play the role called what I call the generators of these rotations repeated action by these matrices will lead to the rotations, and in physics we could leave it as it is, but in physics very often especially in quantum mechanics it will turn out that these generators of rotations corresponds to physical quantities namely the components of the angular momentum, and that you would like in physics, and quantum mechanics could be represented by remission operator, because we would like to have it is Eigen values real. So, make sure that you do not make a mistake there.

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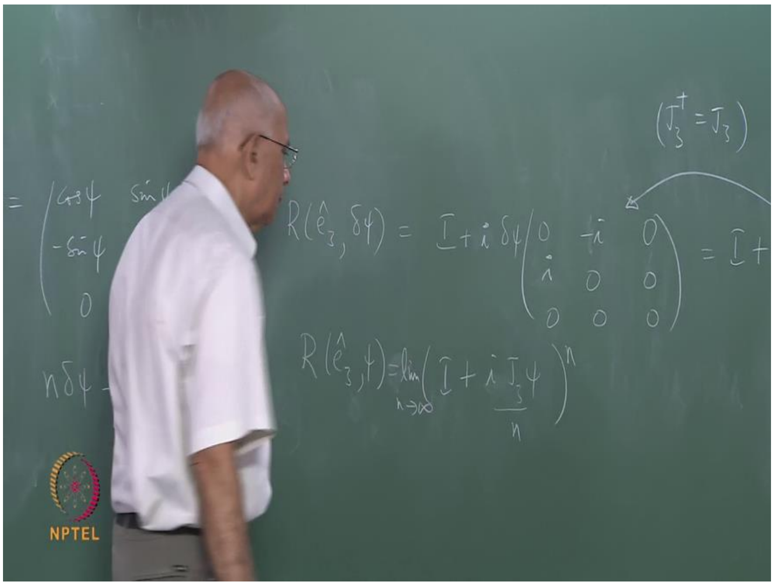

Let us pull out a factor i in the definition here, and there of course, this is minus i this is i, and zero that is what an infinitesimal rotation in the x y plane looks like let us give this matrix a little name since which about the three axis. So, let us write this as i one plus i j 3 delta psi j 3 is trans for this matrix, and notice its omission i pulled out this i to ensure that this matrix is omission. So, j 3 die there now what we need to sure ensure i got to go back, and check this out little bit, and see where the trick is we need to ensure that the repeated action of this indeed leads to this finite rotation in other words I am simply saying that if you had the rotation these were the new axis x 1 prime, and x 2 prime here at this angle is psi, and I am building it up with succession of infinitesimal delta p s is with same axis about the same axis when what I am trying to do is to say that I have n such rotations succession of n rotations each by an angle delta psi, such that in the limit, when n goes to infinity delta psi goes to zero this quantity goes to psi.

So, I build up this finite rotation where succession of infinitesimal rotations when it immediately implies that r of e three psi should be equal to should be equal to repeatedly acting with this matrix. So, it would be it would o i plus i j 3 delta psi least to the power n one after the other , but now usely speaking, if I replace delta psi by psi over n, and I need to take the limit n tends to infinity when it is not hard to show that this quantity you pretend that this is really a number other than a operator, but it works just as well as you can verify for matrices this track goes to e to the power i j 3 psi if i had one plus x over n to the power n the limit n has goes to infinity e to the power x definition.

So, it is turns out even for operators this is true, and this is e to the i j 3 psi notice out hard for you to do this, because j 3 is a very interesting matrix it is a commission matrix. So, it is easy to compute square cube and so on, and explicitly show that this thing here reduces to this matrix. So, let us a little exercise show that this matrix which is i plus i j 3 psi plus all thing squared cube and so on will. In fact, to reduce to this you need the infinite series representation for cos psi, and sine psi, that is all unique, and then it will equal to this right.

So, first lesson it is looks like if you generate a rotation finite rotation by this if generator of an infinitesimal rotation, then if you exponentiate it you end up to the finite rotation that is the general rotation. So, these matrices like j 3 multiplied by some parameter when you exponentiate it with an i, because you want it to be an emission you end up to the finite element the finite matrix. Now turns out that these matrices matrices like j 3 will obey a certain algebraic relation called a lie algebra, and this group itself an example of a lie group namely it is a group whose elements are parameterise by certain parameters in this case psi for instance, and these parameters are kernel analytic fashion in the sense of for a complex variable theory, but more relevantly they appear generally as exponentials.

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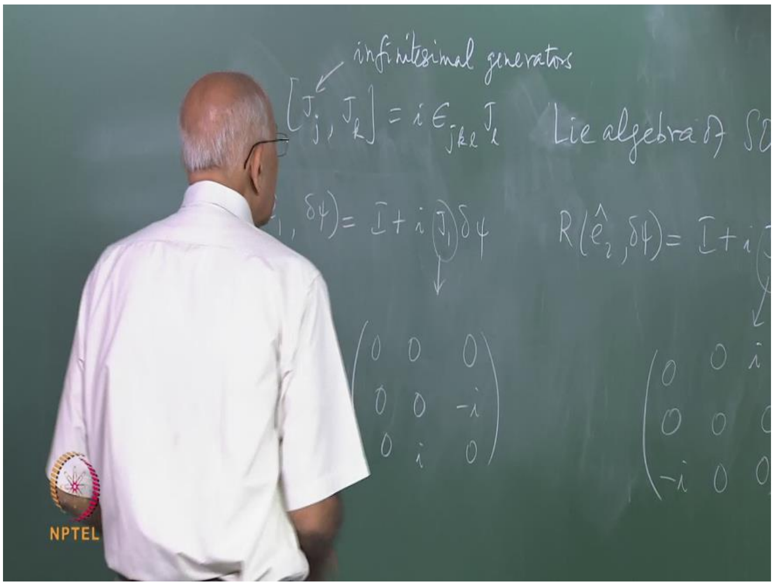

And as you know the exponential of z is an (( )) function lies analytic function that is that is a lesson, but now let us see how we can push this a little further you could ask what happens in three dimensions, if I rotate about the one axis through an angle psi what would this look like where I need to write down the infinitesimal vary variation version of this. So, let us write delta psi here, and this thing here is one plus i j 1 delta psi in an obvious notation, and the question is what this j 1 equal to well it is clear that all we have done here is instead of z axis we have chosen the x axis. So, in cyclic permutation we can write down what it is going to do the original about the z axis, the x, and y coordinates got mixed up now in you rotate about the x axis the y, and z coordinates are going to mixed up.

So, it is obvious that this quantity here is going to be just like copying what have written down there is going to be zero zero zero zero zero minus i zero i zero obvious this is what it is do not look like. So, where ever you had x, and y now you got y, and z that is it, and what is this fellow going to be what is r of e two delta psi going to be this is going to be i plus i j 2 delta psi, and what is this matrix going to be now it is one, and three axis that gets mixed up, and the two remains completely untouched right. So, you going to have where as zero zero zero zero zero here, and these four elements are not going to be zero the diagonal one’s are still going to be zero, but the off diagonal one’s what is going to be here, and what is going to be here which one r i which one is minus r i the z will play the role of the original x, because you got y, and cyclic permutation it is z x right. So, it is clear. So, this is going to be minus one, and that going to be a plus one.

So, that is j two, and this thing is now simply coming we got all the three generators. So, if the group the number of planes is three, and are three generators corresponding to infinitesimal rotations in each of these planes, but interesting thing is these matrices j 1 j 2 j 3 they form what is called a lie algebra in other words they form a linear vector space among themselves such that the commutator of any two of them is again an element of the same set of set of elements same class.

So, in this case it is not hard to show that the commutator j i let us see its not i j k equal to i epsilon j k l j l. So, commutator of any two of these is the third of these matrices in cyclic order j 1 j 2 commutator gives you i j 3 and so on, and epsilon j k l is the completely anti symmentric symbol in three dimensions it is equal to plus one if j k l are in the even permutation of the natural order one two three minus one, if it is our permutation, and zero if any two indices are equal. So, epsilon j k l has as it stands twenty seven components most of them are zero twenty one are zero six of them are non zero three of them are plus one three of them are minus one this is called the lie algebra of s o three it is denoted by s o three the convention is to use small letters to denote the lie algebra of generators infinitesimal generators corresponding to a lie group the group o three itself is a lie group we talk about s o three, because we are talking about rotation form from the identity continuously, and the generators of those rotations.

So, these guys are the infinitesimal generators that is a bit of a mismomer it is technically the generators of infinitesimal transformations there is nothing infinitesimal about j 1 j 2 j 3 themselves well just call them generators they form a lie algebra these quantities here a constants in front they are called the structure constants of the lie algebra we will not worry about that right now except to see that this defines for you in a sense the rotation group in three dimensions, because it means that if you give me all the properties of these j's j's from this algebraic relation, then by exponentiation i can actually find the general element of this group, and everything that follows from that. So, it is all going to set in this the algebra here (()). Now of course, you could ask what about our target of finding this quantity r for a general axis what those look like and. So, on that has a bit of a problem as it stands, because what happens is that the general rotation is specified by an axis n.

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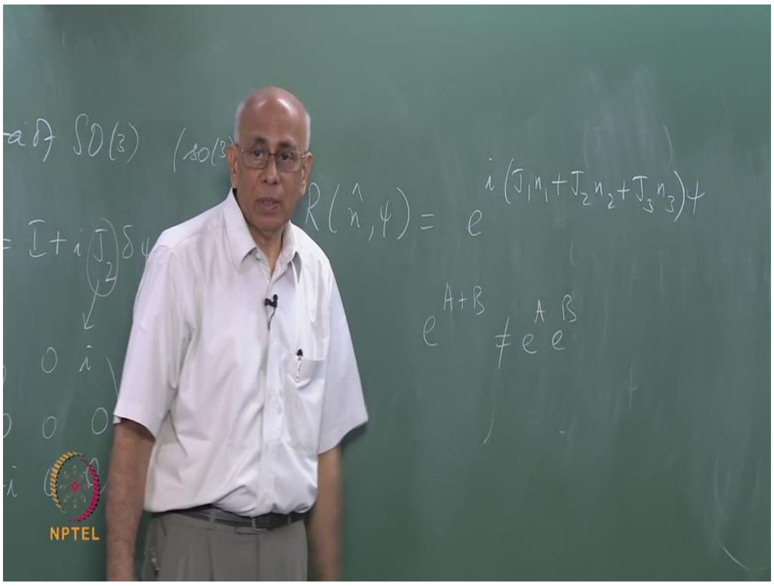

And the element corresponding to that is this n psi this is equal to e to the power i j 1 n 1 plus j 2 n two plus j 3 n 3. So, you first find an infinitesimal transformation about the new axis which is generated by not j 1 j 2 j 3, but j 1 n one plus j 2 n two plus j 3 n three, because one is some arbitrary direction, and then you must exponentiate it, and multiply the whole thing by psi. So, you need this quantity by our prescription, but you need to exponentiate the generator i times of generators times the parameter, but the difficulty is that the j's do not commute with each other, and whenever they commute with each other tan two operators a, and b do not commute with each other e to the a plus b is not equal to e to the a times e to the b it is immediately clear, because e to the a e to the b on this side would have all powers of a on the left hand powers of a b on the right, but of course, if you score a plus b it is a squared plus b squared plus a b plus b a since the do not commute.

So, this possesses the serious problem, and this problem here is at root of all that all the physics all the phenomena that occur in three dimensions, because it essentially says that rotations about two different axis do not commute with each other on the other hand if you are in a plane i rotate by an angel psi, and subsequently by an angle psi prime the result is an rotation by an angle psi plus psi prime, since there is both of them are in same plane, and you could have added them in either order . So, the rotation group in two dimension which consist which is represented by two by two orthogonal matrices in a plane this group s o two is an Abelian group.

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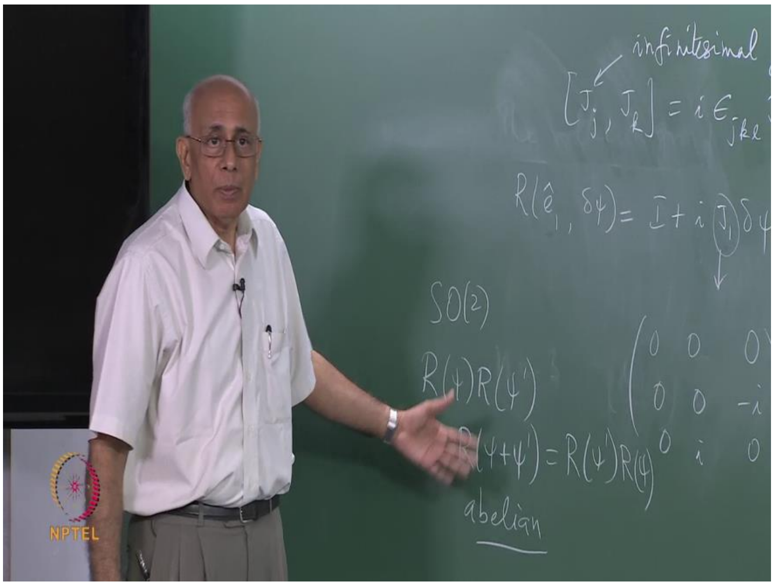

Because it is a commutator group different elements will commute with each other in either order. So, if you rotate since there is only one point, which is can fix the origin i do not there is no axis here let me just write r of psi in this case r of psi r of psi prime is equal to r of psi plus psi prime this is also equal to r of psi prime r of psi, and the group is said to be Commutator or Abelian, and that is make a huge simplification that is not true in three dimensions, because this commutator is not zero.

So, because of this, and the fact that you have the non trivial algebra finding this exponential is not as straight forward as it could have been of these operators is commute (( )) matter would be trivial , but we really need to deal with this separately the fact that the round commute the identities to do this, but in this particular case it is not very hard to do this as you will see in a minute, and I write the answer down, but you will see that in general this will be a serious problem, and if you go to n dimensions, then the problem is made working much worse, because you have many more generators you got n times n minus one over two generators, and the general transformation can be quiet non trivial to write down even this will not be very simple algebraic expression you can write this down as follows.

So, let us see let us write this j 1 n one plus j 2 n 2 plus j 3 n 3 explicitly, and see what happens. So, the first point is, and this I will establish regorsely the fact that I called them j 1, j 2, j 3, and three of them suggest you should suggest, but may be the components j 1 j 2 j 3 themselves are like a vector the matrix value vector in the sense that each of those quantities three by three matrices, but maybe it is a vector transforms like a vector under rotations. Now how do you define a vector in three dimension set under rotations how do you define a vector it is a set of three quantities an ordered triple of quantities which transform under a rotation exactly a same way that the coordinates themselves transform.

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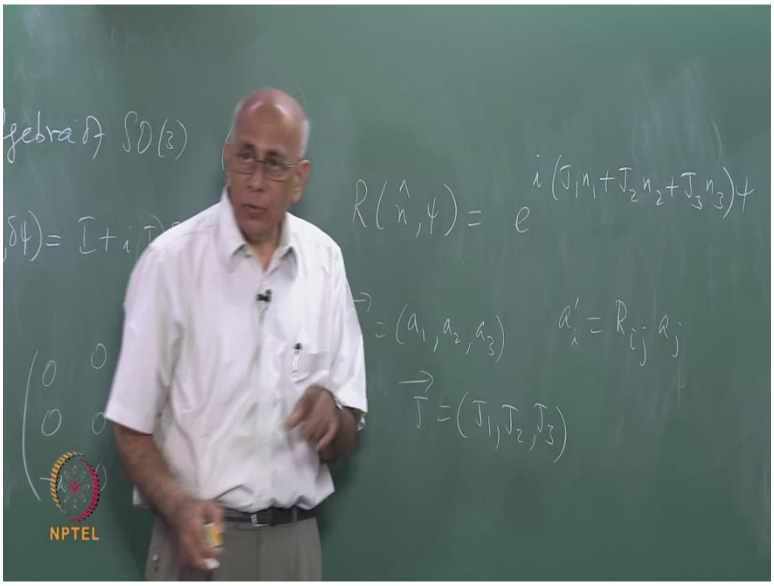

So, vector in three dimensions for example, would be a quantity a, and I would write this as a one a two a three i call this a a vector, if under a rotation a goes to a prime, such that a i prime equal to r i j a j if the coordinates transform like x i prime is r i j x j, and these quantities three quantities transform in the same way, then I call it a vector if does not change at all I call it a scalar, if yeah yeah all constant we do not care in general that would be functions of coordinates we do not care, that is my definition of a vector the elementary high school definition of course, is vectors quantity with magnitude, and direction, but this is unsatisfactory as it definition, because direction with respect to what direction with respect to what, then you said with respect to a coordinate frame who decive this frame.

So, it implies immediately that this statement when you write an equation like f equal to m times a Newton’s law something like that you write it as a vector nobody specifies what coordinate system is. So, it is obvious it implies that this law must be valid in all coordinate system outside in class with respect to each other, and does not matter now when will I lobby invariant when it expressed in terms of quantities whose transformation laws is prescribed to you. So, laws are forming invariant. So, why did you express them in terms of form co in terms of covariant objects whose transformation laws already encoded in the object itself that is what you get the that is the whole idea scalars vectors tensors and so on any way we are talking here about vectors under three dimensional rotations that is sort of transformations. And then of course, this is a definition of a vector, now this is not an accident it will wrote it in this form. So, turns out that indeed, then can write a vector j equal to j 1, j 2, j 3, and moreover these quantities transform under rotations exactly as coordinate coordinates themselves to. So, we going to write down a finite rotation transformation formula for a vector, and then I will show explicitly that the j themselves do the same thing, and therefore I will anticipate that result, and say that this is a vector I will use this notation at least.