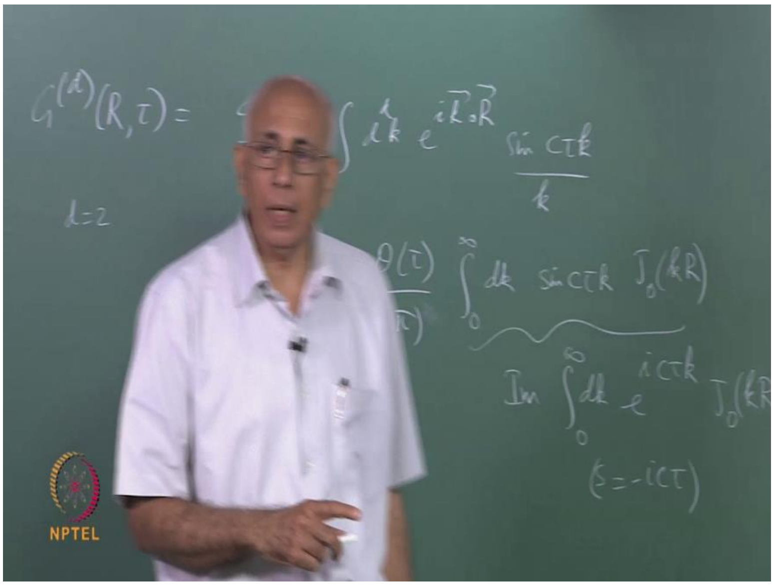
Selected Topics in Mathematical Physics

Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Module - 12
Lecture - 33
The Wave Equation (part II)

(Refer Slide Time: 00:24)

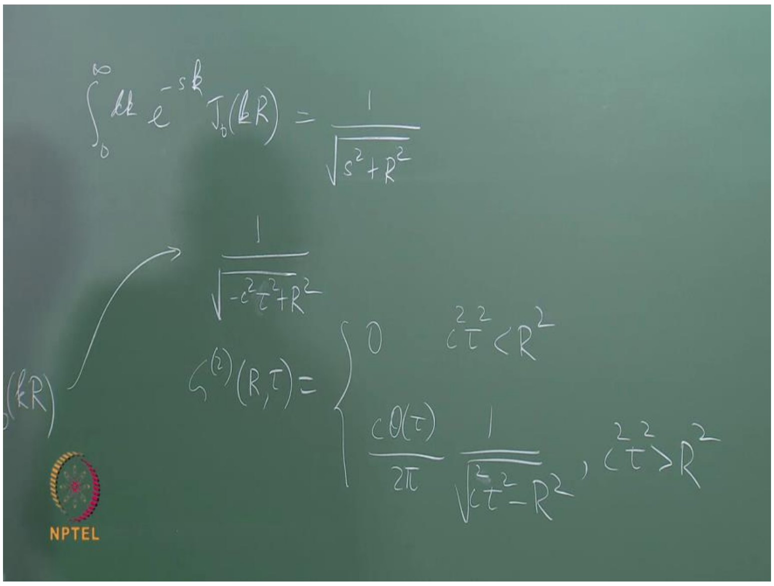


So, we had got to a stage where we analysing the green function for the wave equation the ordinary wave equation, and we found the formula for the fundamental green function in d dimensions, which you write something like G. G dimensions of R and tau was if I recall write it is there is the c theta of tau, and then I 2 pi to power d, and then an integral d d k in k space, it will be i k dot R sin C tau k over k this was the general expression.

After we did the integration over the frequency over omega will left to the integral over the wave vector k. And it is this expression wave which you have to analyse and we did this for d equal to 1 I believe, and d equal to 2 also I mentioned it quickly just run over what happen for d equal to 2, which was plane polar coordinates with the x axis along the direction of the vector R, and then this becomes k d k, so that get is killed. And then we have G 2 of R and tau equal to C theta of tau divided by 2 pi the whole squared, and then 0 to infinity d k, because that the k gets killed sin C tau k.

And then we have an integral 0 to 2 pi it will be i k or cos pi which was J 0 of k R, apart from a factor 2 pi so that goes away and then you have is J 0 of k R this function. And of course, one way to do this integral is to write this as the imaginary part of 0 to infinity d k e to the i C tau k J 0 of k R, and to regard this as the Laplace transform to s equal to minus i C tau of this function of the J 0. And then we already got earlier we had a formula for the Laplace transform of this of J 0.

(Refer Slide Time: 02:42)

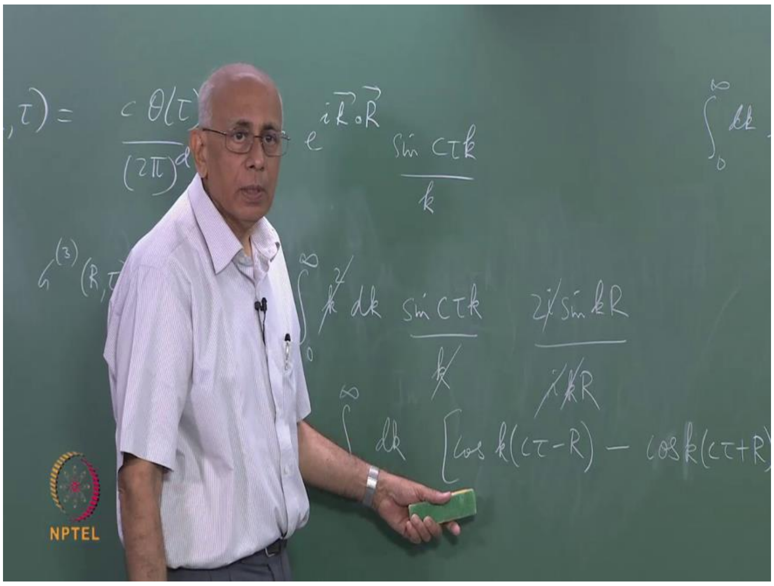


So, we had 0 to infinity d k e to the minus s k J 0 of k R is equal to 1 over the square root of s square plus R square. We can do that in many ways you can expand this in power series and do it term by term, because you know the Laplace transform of a power, and then recombine terms and it becomes this algebraic expressions. So, this thing here will give you 1 over square root of minus C square tau square plus R square in take out the i take imaginary part etcetera.

And you end up with the result that G 2 of R and tau equal to 0. If C square tau square is less than R square, because it is a causal green function and it is equal to apart from these factors that this other factors sitting around. So, you have C theta of tau over 2 pi 1 over square root of C square tau square minus R square greater than R square, that is the cause of green function. And it is slowly decaying function goes like 1 over t for very large values of t for any fixed spacial point, so that was a 2 dimensional green function.

And the surprise here is that you start with delta function pulse at t equal to 0 tau equal to 0 at the origin and capital R equal to 0, and it passes by u your point, wherever you are located the feel point as a slowly decaying pulse after a retarded time. Now, what happens in 3 dimensions that is the one of maximum interest to us, so in 3 dimensions slightly different thing rather different thing happens, let us see what goes on.

(Refer Slide Time: 04:54)

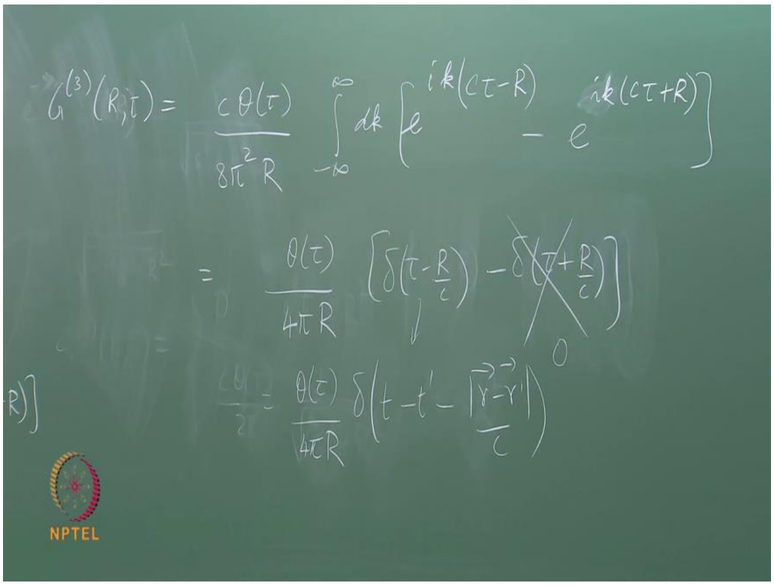


In 3 d, G 3 of R and tau is C theta of tau over 2 pi whole cube, and then 0 to infinity this is a k square d k of their sin tau k. And then this e to the i k dot R the obvious thing to do is to choose spherical polar coordinates with the actual direction with the C direction along the vector R, and then this is just e to the i k R cos theta. So, the pi integral will give you the 2 pi factor that is sit here 2 pi squared, and then you have to do an integral minus 1 to 1 d of cos theta in a very familiar bines of familiar in a word e to the i k R cos theta, this is going e to the i k R minus e to the minus i k R that is 2 i sin k R divided by i k R.

So, this whole thing becomes 2 sin k R divided by i k R in this fashion, I am not happy about the i (s) this is 2 i of there and the i cancels yeah the k cancels as well. So, there was already a sin c tau k over k, and then there is an extra k here. So, all these k square factors cancels, k factors cancels, the i cancels and you left with just this I thing here. So, this is equal to C tau theta divided by I am just trying to check that we have all are factors, I am trying to make sure will be have all our factors this is a 2 pi to the d. So, it was 2 pi q one of them when to pi integration, and the other one gave us 2 sin k R, so there is 2 sitting there and let see.

So, this is 4 pi square R just of make sure that all our factors come out, yes an integral from 0 to infinity by the way, so let us d k and then 2 sin a sin b that is equal to cos k times C tau minus R minus cosine of k times C tau plus R, that is the integral. And the cosine of course in even function, so I might as well write this as minus infinity to infinity, and put a factor 2. So, this is equal to 1 half minus infinity to infinity of this guy. Now, adding the sin will not do anything, because that is an odd function.

(Refer Slide Time: 08:42)

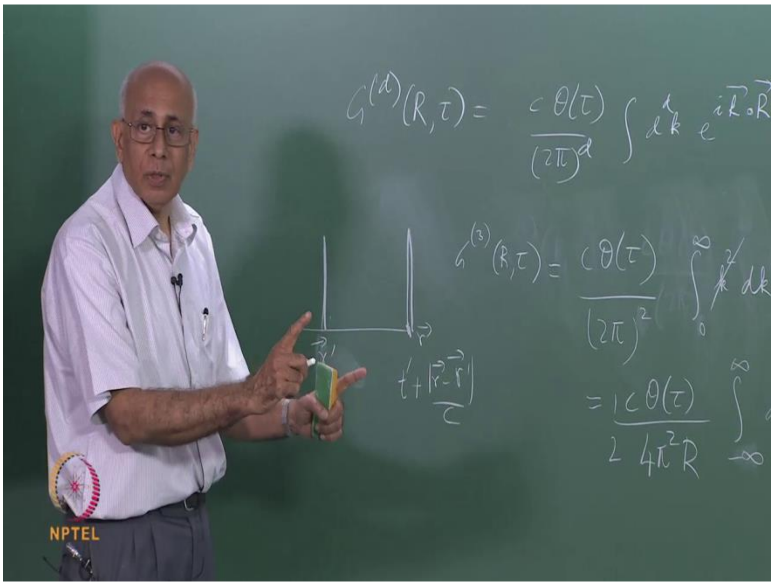


So, if I do that in both cases this immediately gives us the result, because it says G 3 of R and tau is equal to I am worried about a minus sin was it a minus here to start with right from the beginning had a minus sign, was it a minus is check whether it was. We could go back and do the whole thing, but I believe there was a minus sign sitting here. There were so many minus signs that I am ((Refer Time: 09:20)) is this All right cos a cos b plus sin a sin b and this is minus sin b, so that part is should have been a minus somewhere here is that was a minus sign yes, was that a minus sign right from the beginning.

We start relevant we can fix those factor later, but I want to make sure that we get all our things right, I just want to make sure that I just want to make sure that we get a little right, but up to a minus sign we have to worry about this little later. But, let us see where it takes us this is equal to C theta of tau divided by 8 pi square R, and then this integral can also be written as minus infinity to infinity d k e to the power i k times C tau minus R, because the sin part of it is 0 by symmetry minus e to the minus i e to the i k C tau plus R and this fashion. And these are delta functions as you know these are delta functions, so this thing is equal to C theta of tau over apart from a 2 pi factor.

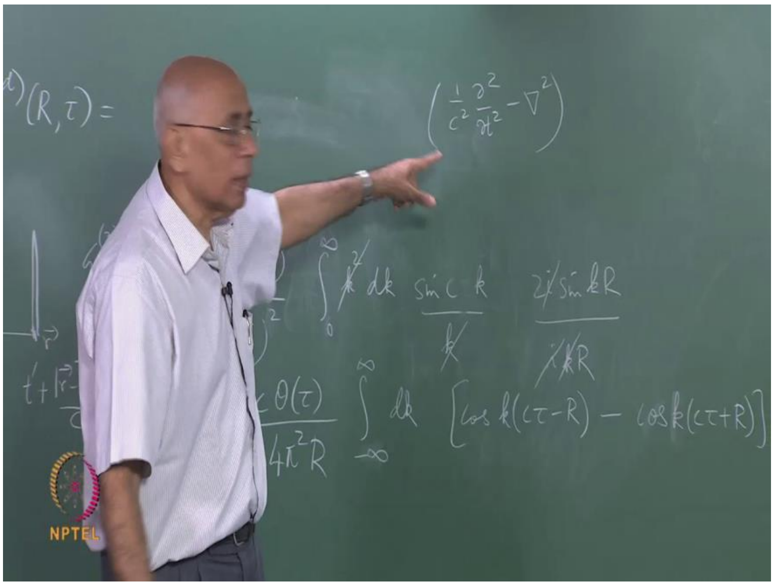
So, this becomes 4 pi R, and this is a delta of C tau minus R minus of delta of C tau plus R and this fashion. And I can pull down factor C, we can take out of factor C this 1 over the C times the same guy, so that goes away and it is tau minus R over C minus delta of tau plus R over C, but you see both tau and R are positive. So, this thing never false this thing is 0 essentially 0, we left to the green function causal green function which is a pure delta function at this is delta r t minus t time minus mod r minus r mod over C.

(Refer Slide Time: 12:20)



So, the initial pulse at capital R equal to 0 there was a pulse here at R equal to 0, and wherever you are at any point at time, so if this thing must emitted at time t prime, it reaches at time t prime plus mod r minus r prime over C this is the point r prime, and this is the point r. And it is again delta function pulse. So, you have this magic that what starts of as a delta function you means a delta function, modulated of course by this factor here, so this is equal to theta tau over 4 pi R. The reason I am un happy with this thing here the sign which we have to fix is that, if C equal to infinity as C tends to infinity, yes it is perfect it is All right as C tends to infinity notice what happens?

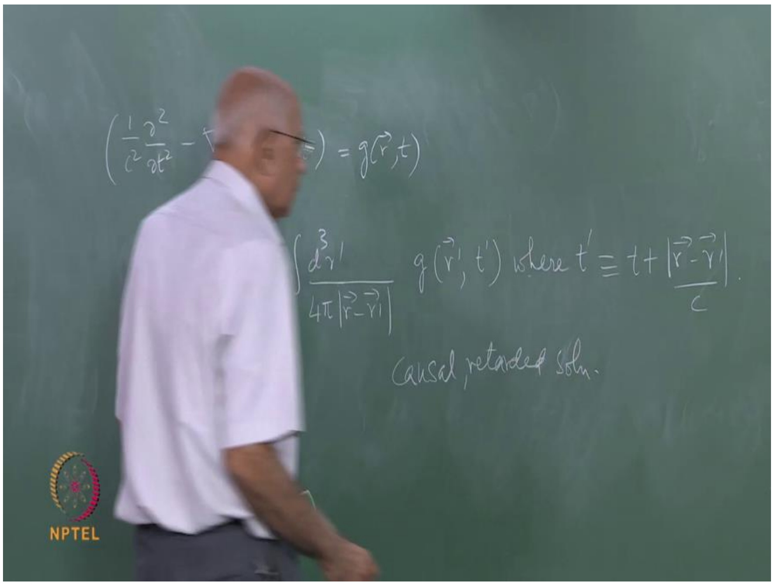
(Refer Slide Time: 13:23)



The operator you are talking about is 1 over c square d 2 over d t 2 minus del square, as c tends to infinity this disappears and essentially you have minus del square, but remember that c tends to infinity you have a pure delta function of t minus t prime which was there in the initial equation for G as well. So, you have precisely minus 1 over 4 pi R as the green function for the operator del square, we have the operator minus del square here. So, you got a plus 1 over 4 pi R, so signs are.

This fact that the green function for the wave operator the causal green function goes over into the green function for the Laplace an is again not always guarantee, but in 3 dimensions it happens. But the important thing is that unlike the 2 dimensional case or the 1 dimensional case, you again have a delta function signal un distorted a pure delta function once again at a retarded time. Now of course, this implies that if you write down solutions of the wave equations, so let us go right back write down solutions of the wave equation.

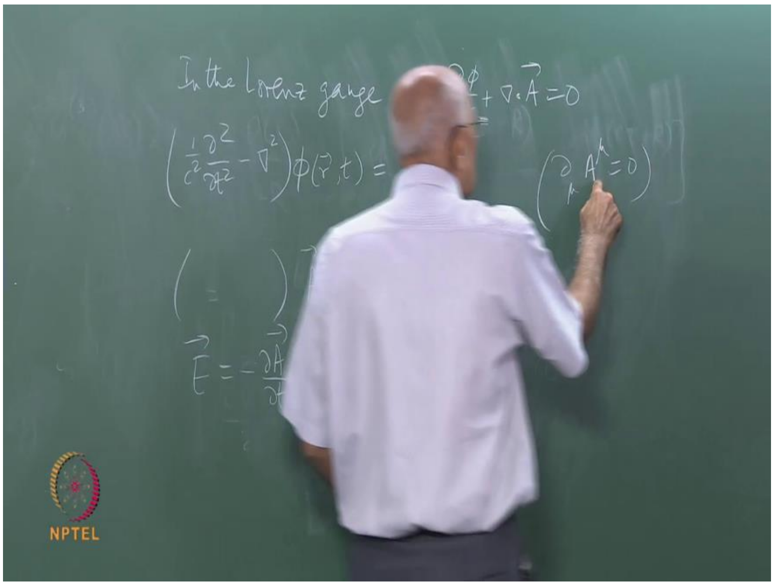
(Refer Slide Time: 14:43)



So, if you gave me the equation 1 over c square d 2 over d t 2 minus del square acting on a function of r and t equal to given function of r and t, when the particular integral corresponding to the causal solution here would be f of r and t equal to integral d 3 r prime in 3 dimensions this guy here, divided by 4 pi mod r minus r prime. And then there is an integral over t, but the point is t must be set equal to t prime plus mod r minus r prime over c, because of the delta function. So, essentially you get g of r prime t prime, where t prime stands for t plus mod r minus r prime over c, this is the retarded solution.

So, the source at an earlier time at t prime such that t prime is t plus this, so if you looking for the solution of time t, then you must have started at an earlier time t equal to t prime minus mod r minus r prime over c. And that is what plays a role in the signal at time t, which is exactly what we expect of the retarded solution, so this a causal retarded solution. And we are familiar with this in the context of say electro dynamics or something like that where, if you are work in the Lorentz’s gauge we know that in the Lorentz’s gauge both the scalar potential as well as the vector potential, obey the wave equation. And then you can write down a formal solutions to it.

(Refer Slide Time: 16:54)



In terms of for the retarded solutions and that is it very, very straight forward matter now, because if you recall the equation satisfied by the scalar potential is 1 over c square d 2 over d t 2 minus del square phi of r and t equal to the charge density epsilon naught in this fashion in the Lorentz's gauge where we have 1 over c square d phi over d t plus del dot A equal to 0 in that gauge. In this gauge, you have the wave equation with the source term being the scalar being the charge density for the scalar potential, and the same thing acts on the vector potential A r comma t equal to area have mu naught j r comma t.

If that write this for each Cartesian component of A, and then you have a set of wave equation and a formal solutions to this equations which are causal and retarded are given by this in this side, where g is respectively either rho over epsilon naught or mu naught g each Cartesian component. And there of course, after that you have the task still of finding the electric and magnetic fields, so you still need to write down after solve for phi and A you still have to find E, which is minus delta A over delta t minus grade phi as you know, and B of course is curl A.

So, principle once you have the retarded solution, you can compute now what the e and b are at any space time point given the sources rho and j. Now, you also know that the Lorentz's gauge happens be Lorentz's in variant, namely if you are in the Lorentz's gauge in one frame of reference you are in the Lorentz's gauge in all frames of reference automatically. Because, this condition here is really a 4 dimensional divergent, so this is really del mu A mu equal to 0, where this is the 4 vector potential, whose time component is 5 over c and space component are 3 vectors A.

So, this is a one way of deriving this retarded solution explicitly, it is still an exercise which you must undertake to find to put this in and then find the curl find the curl and divergent on the derivative etcetera not all together trivial, because remember sitting inside here there is an odd dependence, which will get differentiated. So, you have to a little bit careful, and that is how you get the correct retarded solutions, so that is an exercise in electro magnetism which standard exercise yeah.

Student: ((refer Time: 19:47))

Pardon me.

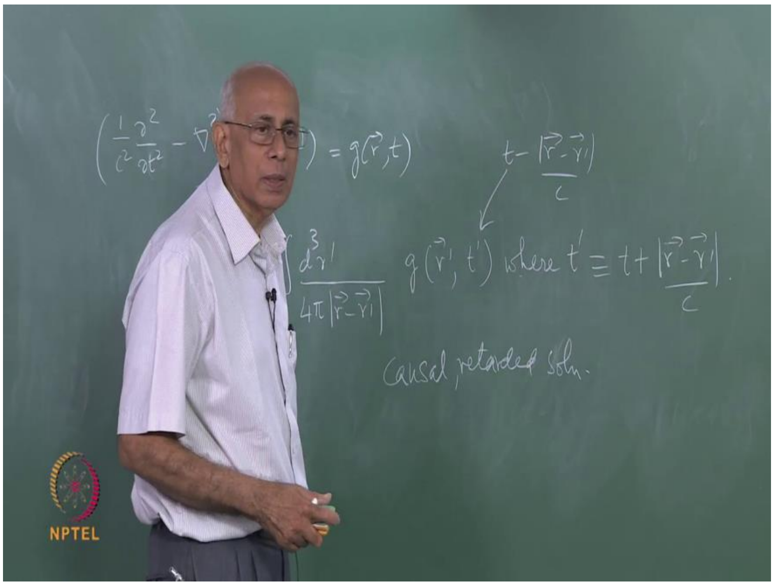
Student: This 1 by r dependence on the delta function as their function.

Yes.

Student: So what is that ((refer Time: 19:56))

No 1 by r is always there, because there is a del square sitting there. So, this is really reflection of that, but the fact is in 3 dimensions the t is replaced just replaced by t prime plus mod r minus r prime over c exactly, that is the crucial point that this solution is over a specific time out here.

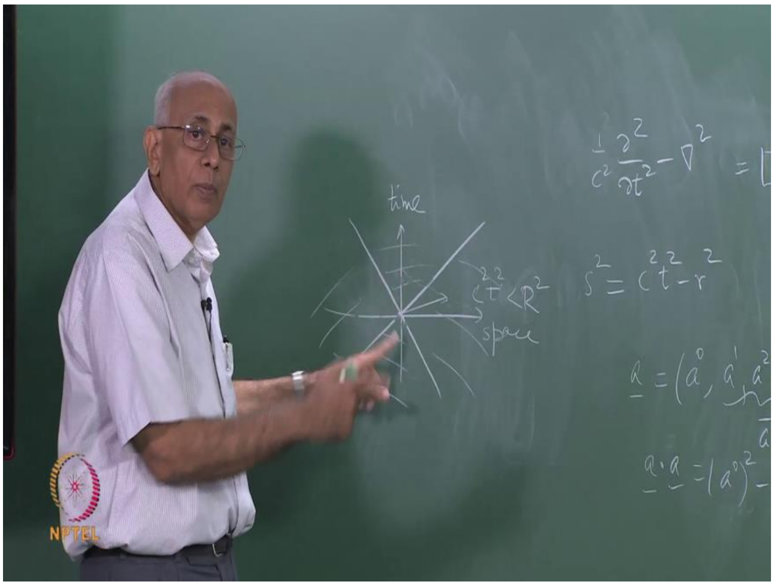
(Refer Slide Time: 20:25)



This t prime stands for t minus mod r minus r prime over c, and then this 1 over r will always be there whenever you have a Laplace in it is 3 dimensions you expect as you know this kind, because you expect the effect to decreases or increases is whole thing the inverse square northing. So, that is really what this reflects, but the crucial point is this that it is un distorted un like 2 dimensions or 1 dimension, where the delta function impulse does not come to any other point as a delta function impulse.

Now, the question is what happens in higher dimensions, I can mathematically look at it and ask what do I do forgot the green wave function in an arbitrary dimensionality d greater than or equal to 4 for instance, while that requires a little bit of effort. But does not interesting way of doing this problem, and that requires relativistic in variance that is if you for the moment say that c is the speed of light in vacuum and the wave operator corresponds to light propagation for instance.

(Refer Slide Time: 21:34)



When this operator as you know del square this is really the so called box operator in 4 dimensions, which is the 4 dimensional ((Refer Time: 21:47)) or the analog of the Laplacian 3 dimensions, it is realistically in variant. In other words, this equation if and g are scalars is in variant under Lorentz transformations not just under rotations completely. When we can say a lot more, because it turns out then that the proper quantity to look out is this invariant quantity c square t square minus r square is interval between the origin and space time, and any point with space time coordinates t and r t and components of r.

So, if you call this quantity s square equal to this, then this green function can be written down by looking at the fact that is really s square that appears everywhere. You saw that in 2 dimension we had 1 over square root of this s square essentially. And we also know that for s square less than 0 this green function must be 0 by causality, because I said generally answer being 0 for c square tau square less than r square. So, in terms of the light cone which probably familiar with, so if you plot a space access here and the time access here, when the 45 degree line this is how light would propagate in this direction a flash of beam of light it is word line is this in here.

Everything here is the future for this person everything here is the past, and these are elsewhere and else when, because there space like separated. No causal signal can have happen can communicate between this point and this or this or this, no causal signal can occur. So, for instance if this person throws a ball at this person and he catches it the speed of these ball is less than the speed of light, so this thing will have a slope which is lower than this and go along a path like this, and this person will catch it after gets a little older.

So, this thing here by the way if I pretend that there are 2 more direction here for the space this is really a cone, this is the famous light cone. And what we are saying is that this region corresponds to c square tau square greater than R square, and here it is less than R square, and here it is greater than R square inside that is the region where the green function is not 0 and 2 points which are separated by time like interval, can communicate with each other causal 2 points space like separation cannot communicate with each other.

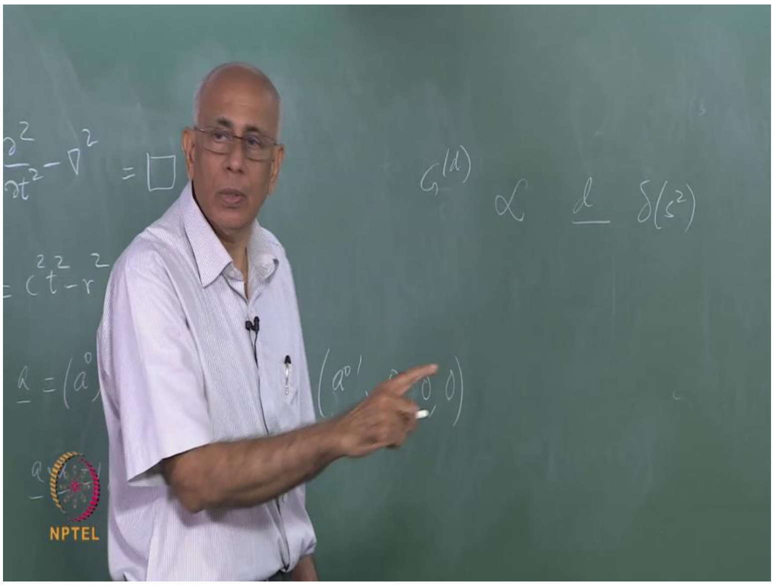
So, these green function we are taking about are identically 0 in these regions outside these like cones inside it got some value out here. One can use this relativistic invariance to actually solve this problem, because when you have time like separation, and you have vector which is time like. So, you have a 4 dimensional vector, when you call it a this is equal to some a 0, a 1, a 2, a 3 and these 3 are both will normally combined into this in a. Then the dot product of this vector with itself is a 0 square minus a vector square it is quantity here, when it is positive you say it is a time like vector, when it is negative it is a space like vector, where it is 0 you say it is a light like vector, because for a light c t equal to r.

Now, turns out that when you have a time like vector, you can always make a Lorentz's transformation to another frame of reference another inertial frame such that in the transform frame, you have only the component and all these component are 0. In other words you can make the interval purely time like no space component at all that is always possible, you can always go to another frame of reference. Where this vector will look like this will go to some a prime, which is equal to a naught prime and 0, 0, 0 and always do this.

And this a practical a limit will come back and tell you there is an you actually do this all the time and you look at material particles, you go to what is called a rest frame of this particle. In the rest frame the liner momentum is 0, and there is only an energy rest energy of these particles. On other hand, when the vector is space like, when the sign of this quantity is negative then it means a vector like this, it is space like this point here that space like interval.

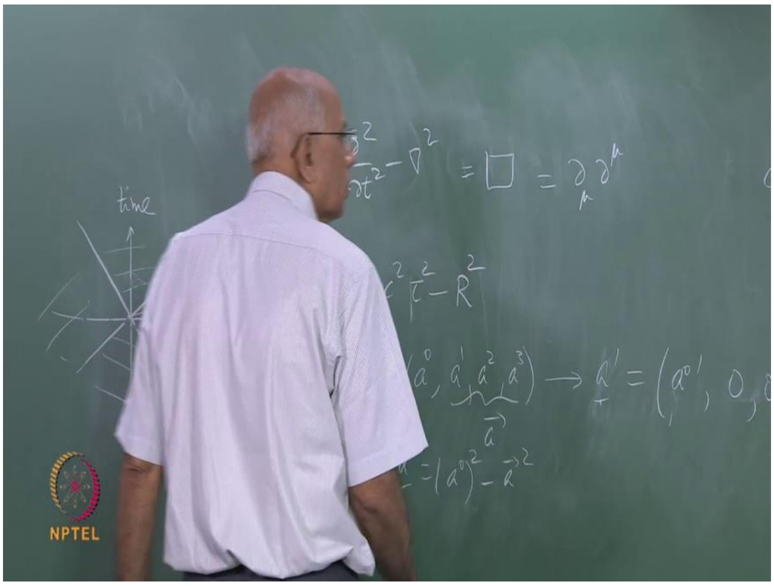
In such a case, you can always goes to a Lorentz's frame such that you have simultaneity, such that this separation has 0 separation in the time coordinate, but it is purely space like you can always do this for space like particles, space like intervals. Now, our green functions are 0 outside and non 0 inside, and you can always go to a frame by this interval s square root of s square, if you like can you can we made to have only a time component. Then the matter becomes very simple and you end up with the following result.

(Refer Slide Time: 27:03)



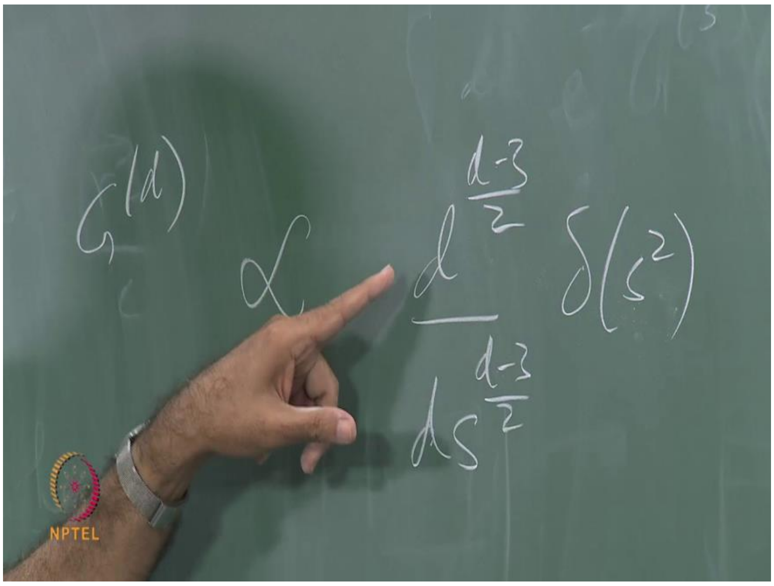
You end up with a result that G in d dimensions really a function of this variable s square, if you like this thing here becomes proportional to just write the result now. A derivative of a delta function of s square, at s square equal to 0 that is c square t square tau square minus capital R square, if you like so...

(Refer Slide Time: 27:31)



Let us stick to that notation c square tau square minus R square. So, it is the derivative of this delta function this is a scalar quantity. So, it is a single 1 dimensional delta function.

(Refer Slide Time: 27:44)

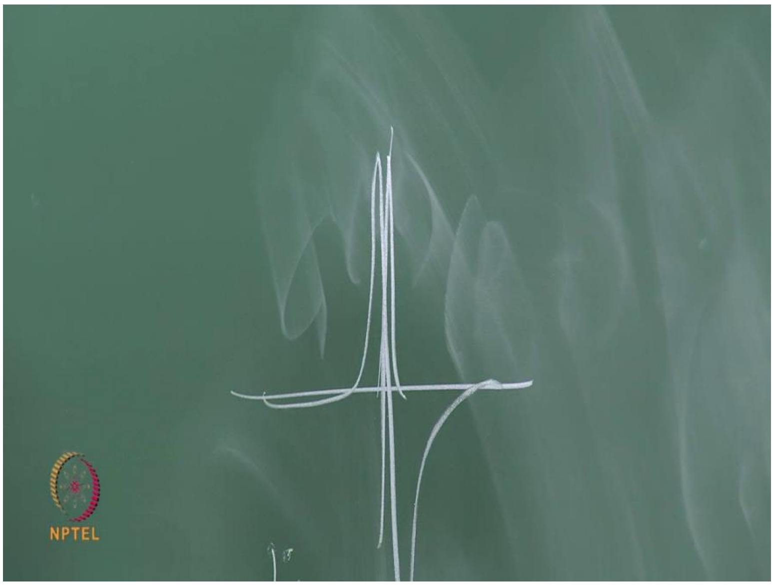


But, the derivative is very strange, it is d s to the power of d minus 3 over 2, it is what is called a fractional derivative of the delta function. Now, how do you define this, well you define how do you define a delta function, you define it by multiplying by it some function and then integrating in given a rule for this integration. Similarly, derivative of delta function, you suppose to do integration by parts so that the derivatives act on the function the test function.

And a fractional derivative is defined by it is a Fourier transform, and turns out that well obvious from here that, when you can giving meaning to this for all values of d both even and odd, when d is an odd number, when d is an odd integer odd number of dimensions. Then d minus 3 is an even number divides 2 derivative, so you get a derivative you get an actual derivative of some order some integer order of the delta function.

So, this would be a pulse once again, and notice that in d equal to 3 it is the delta function itself that is it, that is why in d equal to 3 you got a pure delta function as the answer. In d equal to 5, you get the first derivative of a delta function. Now, what is the first derivative of a delta function look like, it is more singular than the delta function.

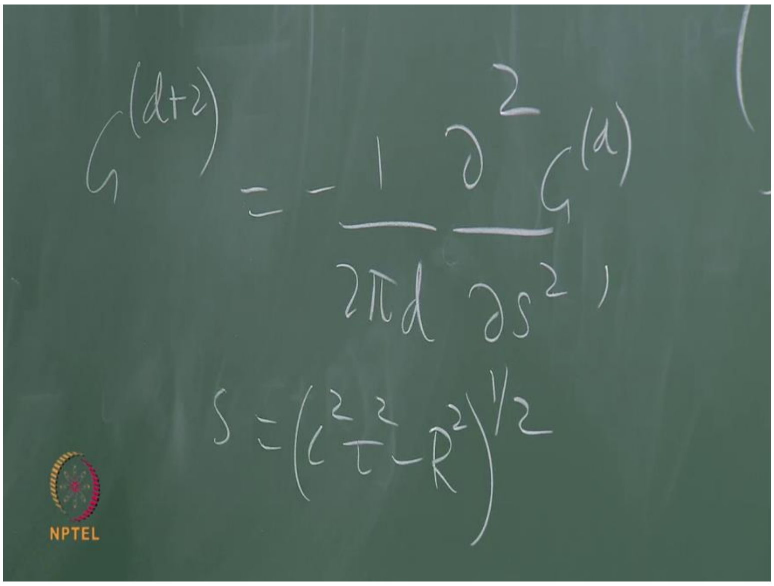
(Refer Slide Time: 29:03)



I am just very roughly realistically you can see, that if this is your delta function. So, imagine approximating it by a very steep function in the origin, if I differentiate such a function you want a getting something like this. It is going to be very singular in the limit more singular than the delta function, roughly speaking delta prime of x is like delta of x over minus x. The second derivative is even more singular and so on. So, for all odd values of d for higher dimensionalities, you can see that you get singular objects they are pulses, but they are not the original delta function pulses they are like derivative of this delta function.

On other hand if it is an even value, d is an even value like for instant d equal to 4, then you get half derivative of this fellow here, what this mean is that really an integral of some kind. It is an integral transform you can write this delta function fractional derivative as an integral transform, and then it is an extended function. We saw already that d equal to 2 we had an extended function 1 over square root of this s squared, and that is what happen in all the even number of dimensions. So, this single formula can be derived using the relativistic in variance of this problem, and it tells you in a very compact wave all the del green function is in all cases here.

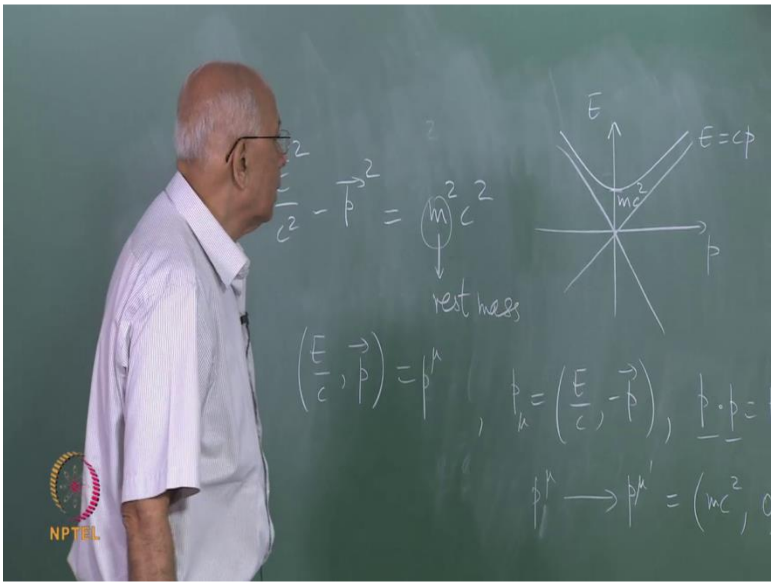
(Refer Slide Time: 30:27)



There is another relation which you can establish is also very interesting as say G at dimension d plus 2 is minus 1 over 2 pi d, a second derivative of G of d. So, that s, so if you give me the green function in dimension d, I generate for you the green function in dimension d plus 2 etcetera. These relation can be proved using this invariance relativistic invariance, I want go further into this expect point out the mathematically you can solve this problem for every dimensionality.

And this relativistic in variance are clever trick to find this green function. Now, let us turn to slightly more complicated problem then the wave equation, and this is the operator which you get in quantum mechanics, when you do relativistic quantum mechanics of a particle. And I am not sure if you already familiar with characteristic quantum mechanics the Klein Gordon equation for scalar particle. Well, since not all of you may be familiar with it, let me do this in a extremely simple way.

(Refer Slide Time: 31:57)



And at least motivate where this equation comes from, we know that if you have a relativistic particle, then the energy and momentum of a free particle are related by the following equation E squared over c squared minus the liner momentum squared is equal to m squared c squared. Well, this fellow here is called the rest mass of the particle that is the energy momentum relation for a free particle relativistic free particle. Now, this quantity E over c comma P is a 4 vector a 4 dimensional vector, and you can see now that what we got here is the analog of the s squared, you square the time component and you subtract the square of the spacial components, and you get the dot product of the vector with itself.

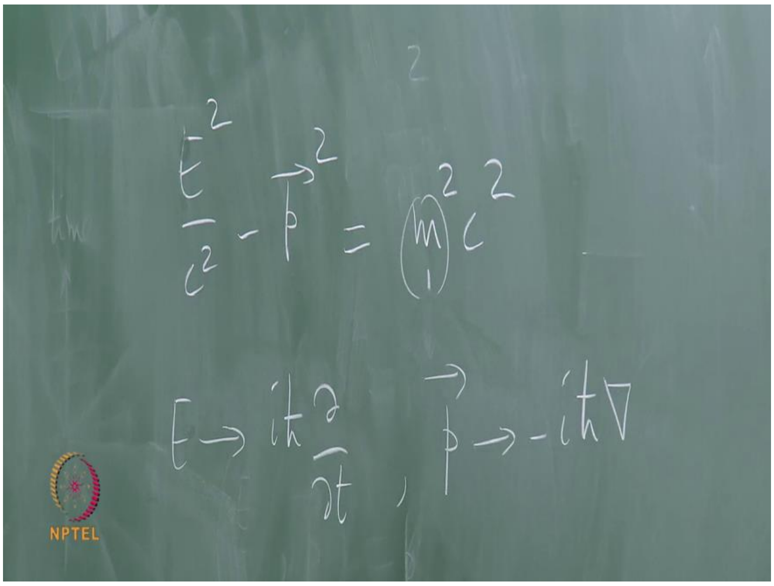
So, this implies that p mu is E over c minus p and p dot p is p mu p mu sum over this mu value 0, 1, 2, 3, this is precisely E squared over c squared minus p vector squared. And that is equal to this quantity for real m real masses positive masses is greater than 0. So, this means that a physical particle of non 0 rest mass positive rest mass, it is energy momentum 4 vectors is a time like 4 vector, which means you can always go to a frame of reference, in which it is components are such that the space components are all 0, and there is only a time component.

So, you can always go to a frame of reference in which this particle the p mu goes to prime equal to whatever it is value is when the momentum is 0, and you can see that E squared is m squared c 4, so e is m c squared. So, we have m c squared and 0, 0, 0 that is what you called the rest frame of a particle, because it is at rest but it has a non 0 energy the rest energy. And when it is moving it is moves on a hyperboloid, because if I plot say the x component of the momentum p versus E here, and the y and E come out you need more dimensions for it.

Then you can see that this line is E equal to c p this case, and this is E equal to minus c p on the other side that is the light cone, and what is a physical particles relation look like this thing here. If I plot E versus p it look like this, it is really a hyperboloid of revolution, and this quantity here is precisely m c squared that is the rest energy. But when it is moving it is on the hyperboloid it is called the mass shell, it is stays on the hyperboloid as long as it is a free particle. If you put a potential then it changes the energy momentum relation like change, but this thing here is for a physical particle here light would travel along this or that on the light cone.

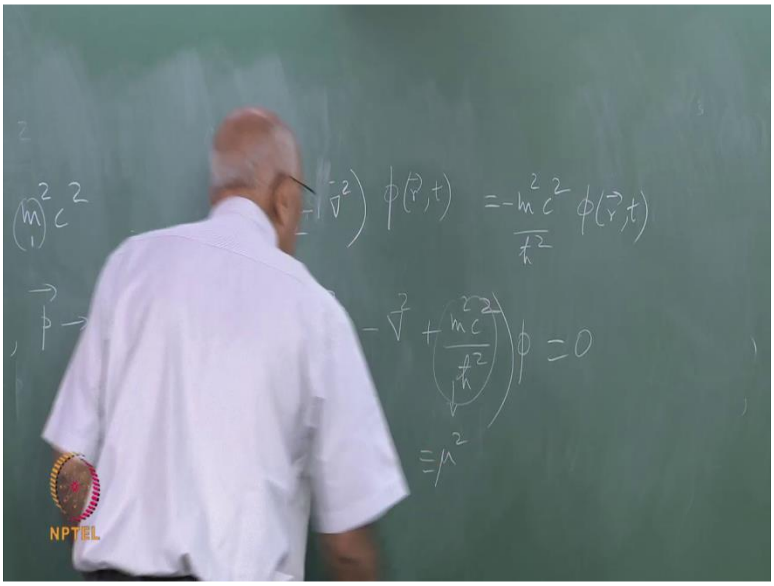
And then no physical particles, which corresponds to vector here or here and this place, anyway to get back to this when you introduce quantum mechanics, what you do essentially is to change to wave function you say look I do not have things like momentum and position and so on for these particles. I have some wave function which describes this particle, and then on this wave function these quantities energy momentum etcetera, replaced by operators they act on this wave function, and depending on what basis you choose the representation of this operator is different depending on the representation, but you know that this is essentially the Hamiltonian operator.

(Refer Slide Time: 36:05)



And we know that crudely speaking the correspondence is E replaced by this, when it acts on wave functions, and p is replaced by minus i h cross gradient, when it acts on position phase wave function that is the prescription. Now, put that in here and see what happens?

(Refer Slide Time: 36:30)

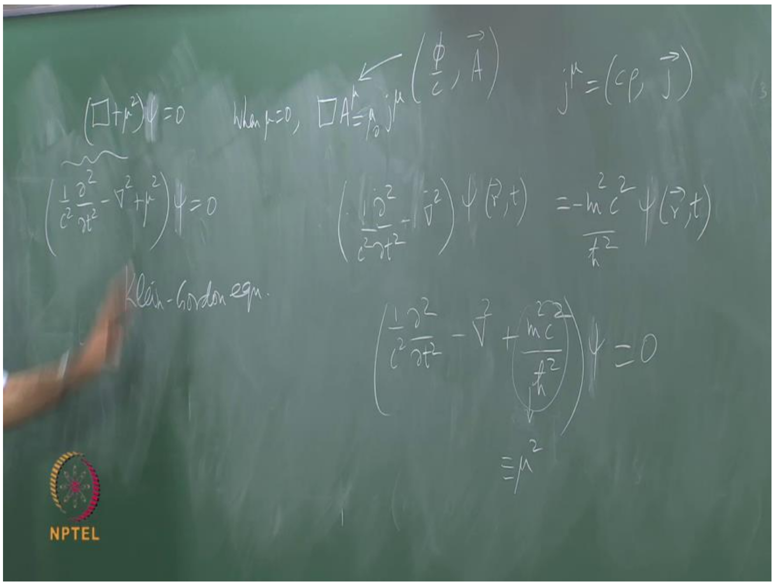


We says that if you have a particle, whose wave function is represented by phi of r comma t, then this thing here is minus h cross squared d 2 over d t 2 over c squared, because I put in E squared acting on that wave function I and up to the second derivative.

And this is minus p squared, so this is equal to plus h cross squared del squared acting on this is equal to m squared c squared at phi r comma t. So, that must be the relativistic wave equation obeyed by a particle, whose wave function is phi of r comma t in the case when it has no internal degrees of freedom. So, no spin or any complication like that, but let us rewrite this equation, and what you get you get 1 over c squared d 2 over d t 2 I divide through by h cross squared.

So, let us get rid of this, and then I change the signs. So, there is a minus here and minus here, so it is minus del squared plus m squared c squared over h cross squared phi equal to 0. Now, what is this quantity here we can readily identify this quantity h over m c is the Compton momentum, h cross over m c is the like the Compton wave length. So, this fellow says that out here this has dimensions of 1 over length squared. So, does this 1 over length squared, and this is 1 over Compton wave length squared. So, let us give this some name, let us called this mu or something like that mu squared.

(Refer Slide Time: 38:30)



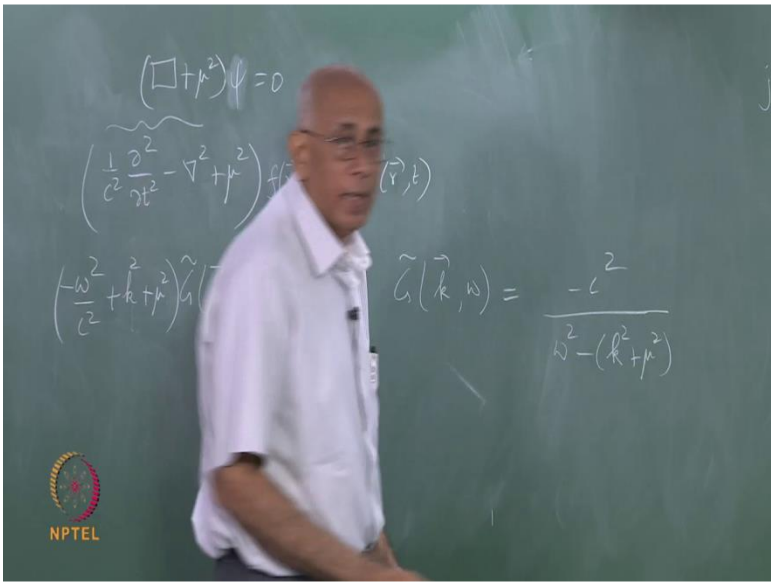
And then you have this equation, which is 1 over c squared d 2 over d t 2 minus del squared plus mu squared and phi equal to 0 this equation is called the Klein Gordon equation. It goes one beyond the wave equation, and gives you an extra time out here. Now, of course this portion alone is the box operator, so really what you have is box plus mu squared phi equal to 0. Now, in the case of a photon in the case of electro magnetism in the Lorentz's gauge, when you have del mu a mu equal to 0 then the rest mass of photon is 0 out here, and sure than enough we know that for photons.

When mu equal to 0 you have box, and then A mu equal to 0 this is different mu by the way ((Refer Time: 39:28)) for variance you have to use this, but this is the 4 vector potential. I am again to use this phi I should have use some other function here, let us call this wave function psi, because there is more traditional psi and use phi, because phi is very often use for a scalar field with no spin, so that psi is used for spinners and so on and so anyway. So, this is the scalar potential and electro magnetism and A if you write the potential.

And together Maxwell's equation sat this phi in the absence of sources, what happens if you put in sources? Suppose, you have just for that we know what we are talking about, you take c times rho, and current density and together these constitute of 4 dimensional vector called the 4 current density. The transform and the Lorentz's transformation like the 4 vector, now what is this equation become when you have this sources, when they are non 0. Obviously, that is the left hand side is the 4 vectors, right hand side must be a 4 vector and must be depend on the sources the only one is j mu, so it is clear this is equal to j mu.

Just we show that we have the same normalization there it is equal to mu not j mu, in those units that is the in homogeneous equation, this is the homogeneous equation in the absence of sources pure radiation field for example will satisfy this equal to 0 in the Lorentz's gauge. And we describe electromagnetic waves, but you have the problem of solving the Klein Gordon equation as well. So, let us see what happens in the more general case?

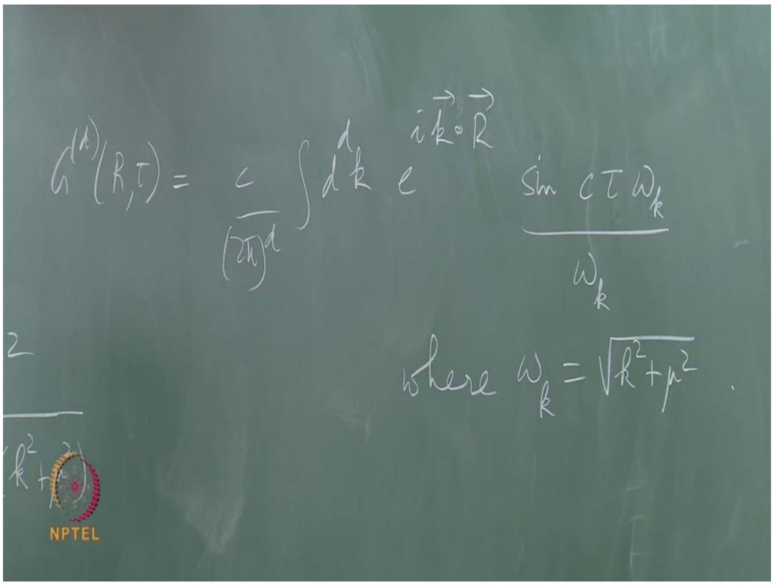
(Refer Slide Time: 41:39)



Suppose you have again some function of r and t equal to given function of r and t, I will like to write down the green function for this operator once again, what would you do? You do exactly the same thing as before take a Fourier transform it space and time, and then you get g tilde of k vector and omega let us do this in 3 dimension, we can do this in any number of dimensions let us write it down in 3 dimensions, what is going to happen? It is more less the same as before except that this fellow here would satisfy the following equation, if I do Fourier transform which respect to time, I pick up a minus omega squared over c squared this fellow here it is a minus k squared, so thus become a plus k squared plus mu squared on this.

And that must be equal to 1 the Fourier transform of a delta function. So, this will imply, but in this case G tilde of k and omega equal to minus c squared over omega squared minus k squared plus mu squared, while you are here have k squared, now you got k squared plus mu squared here. So, we play the same game as before we do the integration etcetera, and then you left with picking up poles. So, you will end up with wherever k appeared you going have square root of k squared plus mu squared appeared.

(Refer Slide Time: 43:10)



So, this will mean that this G (R) 3 dimensional for example, you can actually write it down in d dimensional why not. So, G d of R and tau in this case is equal to apart from this factor c squared whatever it comes out c, an integral d d k usual to 2 pi to the d factor here e to the i k dot R this will be sitting there. And then you get a sin c not c yeah sin c tau not k, but you get omega sub k divided by omega sub k that is this. So, that is all that will happen, when you pick up this poles instead of the poles at plus or minus c k, it occurs at square root of k squared plus omega squared and mu squared on both sides.

So, we have to do this integral which of course is considerably more complicated than doing the integral with just k here. You had k then this become little relatively easy, especially the fact that factors of k cancelled again k to the d minus 1 here in 2 and 3 dimensions etcetera we made life very easy, but now we also struck with this you have to do this. Well, you can do it the various kinds of bassel functions the Neumann functions and so on, but the reason I brought this up was because once you do the wave operator you can also do a Klein Gordon operator in this case.

And formally write the solution down although it is not so trivial to writing down here, you can again write ask for the causal green function write this and write the expression now explicitly. So, the earlier case just the wave operator would apply for instance to light, but this would apply or in this language to 0 rest mass particles, but the more general case where you have a non zero mu squared also come within this family of hyperbolic equations, and you can solve it pretty much the same way in this case.

There are many other interesting things that happen, which I do not want get into here. For instance this quantity d 3 k is not relativistically invariant, because there is a length contraction. So, this will contract the 4 dimensional volume element is invariant and Lorentz’s transformation, but this quantity is not, however d 3 k divided by omega k is Lorentz’s invariant once again etcetera. So, you can use this fact you can again very powerful statement, just as I showed you that general d dimensional thing could be written down using Lorentz’s invariance of this operator, you can do similar things.

By the way this is also scalar operator and Lorentz’s transformation this is quantity here is invariant. So, one can use this fact and writing down the green functions not much interested, actually writing this down explicitly except if you want to solve the problem explicitly, but all the properties that you need of this green function can be read out from the integral representation itself.