Selected Topics in Mathematical Physics
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Module - 12
Lecture - 32
The Wave Equation (Part I)

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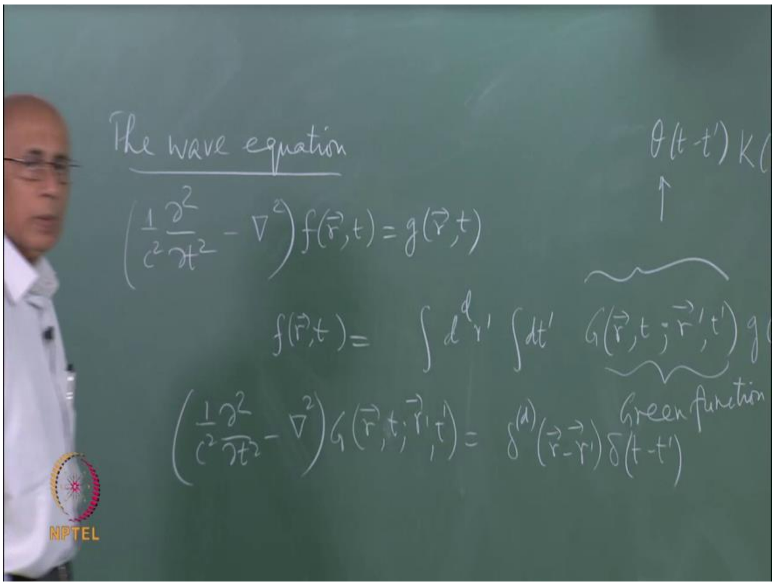

Today we will discuss the green function for the wave equation, which will be last of the equations of mathematical physics for which we find the fundamental green function solution. So, the wave equation as you know could be written first scalar fields vector fields and so on and so forth will start with the scalar field, because we just want to find green function. And the equation I have in mind is of the form 1 over c square partial derivative with respective to time minus del square time some function of r and t equal to a given function of r and t. So, this access source for this disturbance and the idea is to compute this disturbance here therefore for r and t.

Now, this problem unlike the earlier problems is mathematically hyperbolic differential equation, because what happens here is the minus sign here. And the space and time in disease variables have exactly the same order of derivative both are second order derivatives. And in the classification of differential partial differential equations, if the second order this is a hyperbolic equation, it has it is own implications in particular signals can propagate. So, there are propagating with were solutions for this such an equation.

As usual our interest is an finding the fundamental green function for this operator namely, what is the inverse of this operator here, it sort about boundary conditions the natural boundary conditions for instance. Now, what we like to argue physical grounds is that the green function we want to select is one that is going to ensure, that before we switch on g there is no f at all. In other words, before we switch on the source of this way there is no disturbance f, there is no result and disturbance this is called the principle of causality, you want make sure that they cause precedes the effect always.

So, the general solution that we have in mind is going to be of the form f of r comma t equal to an integral. And now we might as well do this in an arbitrary number of dimensions d special dimensions, so d d r prime, and then an integral d t prime the special times of green function of r and t given r prime t prime, and then the source at r prime t prime. So, that is the general form of this solution that we are interested in, and this green function here of this wave operators inverse of this operator must satisfy must reflect the fact that we want a causal solution. So, this quantity we expect should disappear, if t prime is greater than t, so it is exist only as known as t is greater than t prime.

So, in general I would like to look for us solution which has this built in to it. So, I would like to have a step function in t, this vanishes for t less than t prime multiplied by some function of these variables. And as usual, since this operators in variant under translation and t, in variant as under r goes to r minus r prime, this thing here is going to satisfy a delta function which as exactly the same properties. So, I expect that the solution, it look like r minus r prime comma t minus t prime. So, I expect to find this K, this propagator K such that it is function of just the differences r minus r prime t minus t prime, we need to see how their comes out.

Now, as usual the equation satisfied by g is precisely the same as this original differential equation with delta functions on the right hand side. So, g satisfies 1 over C square minus del square G of r t r prime t prime equal to a d dimensional delta function fine, and then t minus t prime on this side, both in space and time. Now, as usual as always we look for a solution such that G goes to 0 as r minus r prime goes to infinity. So, I want something that dies down at special infinity, as view done in all the earlier cases this is the natural boundary condition. We also want something, where G equal to 0 and since it is a second order differential equation equal to 0 for t less than t prime.

So, you want causality and this puts in the requirement of causality, so without look for a green function with satisfies this boundary condition and these conditions initial conditions for t minus t prime equal to 0, for less than 90 you wanted to be equal to 0 identically. So, now took at a long story short as always this thing here is a function only of r minus r prime and t minus t prime, because a boundary conditions are the differential operator itself is and the inhomogeneous terms also such a function.

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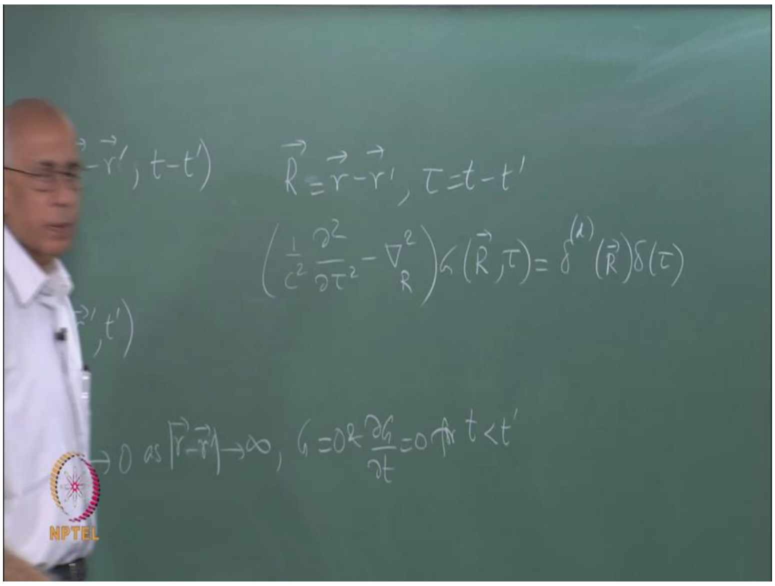

So, as always will set R minus r equal to r minus r prime, and let us put tau equal to t minus t prime, then this equation could be written as 1 over C square d 2 over d tau 2 minus del with respect to capital R square G of R tau equal to delta function d dimensional of this vector R delta tau this fashion, and you want to solve this equation. Now, the obvious thing to do is to do a Fourier transform both this respect to space as well as with respect to time, do double Fourier transform. And this will ensure that this differential equation becomes an algebraic equation for the transform double transform of G. So, let us write that down it is.

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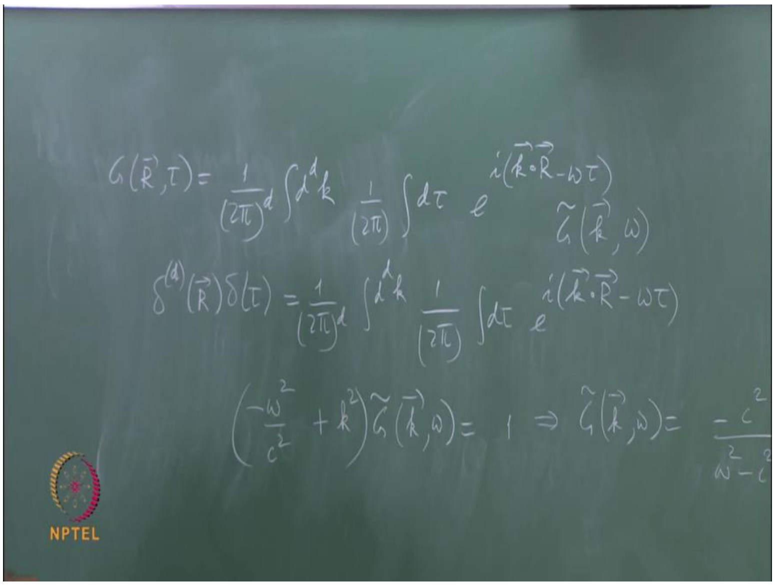

So, as always let us put G of R tau equal to 1 over 2 pi to the part d, d d in case space, and then 1 over to 2 pi for the time variable the tau, either the part i k dot R minus omega tau, and then G tilde of k and omega. Notice that the Fourier transfer convention, where I have a function of coordinate is an integral over k, it will the power i k x to the plus sign. And then inverse will have a minus sign, for the time I invert it that is convenient, because this sort of thing represents some way moving in the k direction as t increases. So, that is a minor matter, but the 2 pi convention I keep exactly as it is to avoid confusion, and then what?

Well, the delta function delta d of R delta of tau is equal to 1 over 2 pi to the power d integral d d k 1 over 2 pi integral d tau e to the power i k dot R minus omega tau, that is a representation of the delta function with one as a Fourier transform of the delta function. I put thousand into the wave equation, and then equate co efficient of e to the i k dot R minus omega tau is a obvious. And you can see what is called happen? This derivative with respect to tau, each time you differentiate with respect to tau, you produce minus i omega in the denote in the bring down minus i omega. So, when you square it you get minus i squared omega square minus omega square.

So, this going to be a minus omega square over c square, and then the del R square is just going to produce a minus k square, and then minus del R square becomes a plus k square. So, this only give me a plus k square G tilde up of k and omega is equal to 1 on the right hand side. So, this immediately implies the G tilde up of k and omega is equal to. Now, let us write this as multiply through by c square, so it is c square over omega square minus c square k square put a minus sign that is it. This make sure I have an made a sign mistake, so it is ok.

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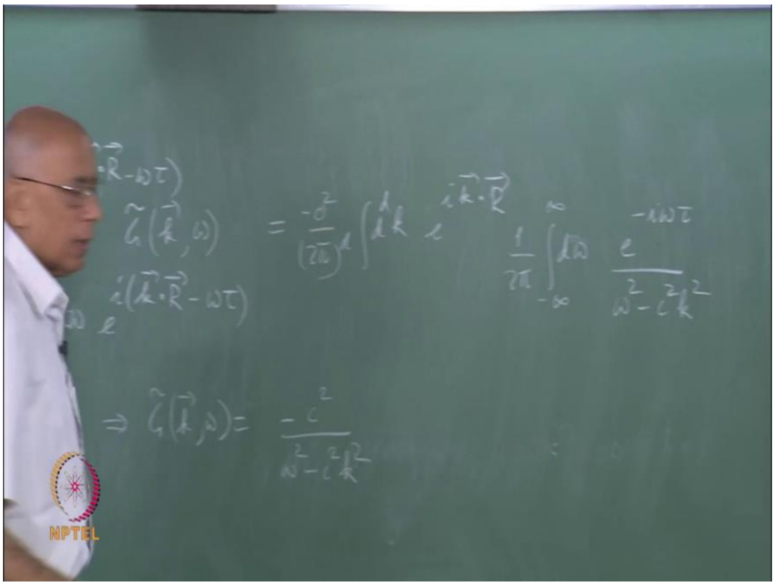

So, this green function is now solvable this is equal to1 over 2 pi to the d d d k, it is the i k dot R let us take that out this kai here, and then 1 over to 2 pi integral d the d omega. This is this is a d omega, I am integrating over omega here, here too d omega the transform variable. So, it is that multiplied by this is minus infinity to infinity e to the power minus i omega tau divided by omega square minus c square k square. And I believe there is a minus c square outside, so minus c square times that, that is the integral we have to do. Now, as always as it stands the integral makes no sense at all it is divergent, because you can see that it has poles on the real axis at plus or minus c k, where k is the magnitude of the vector k.

We make sense out of it by making an i epsilon prescription as always, and that i epsilon prescription should be such that the conditions you want on this green function be satisfied. Now, this is an omega integral the conjugate of the time variable. So, therefore the conditions that you place on it with respect to causality that is the condition you would like to invoke, you like to make sure that these green function vanishes identically for tau less than 0, because remember tau is t minus t prime. So, now what is going to happen?

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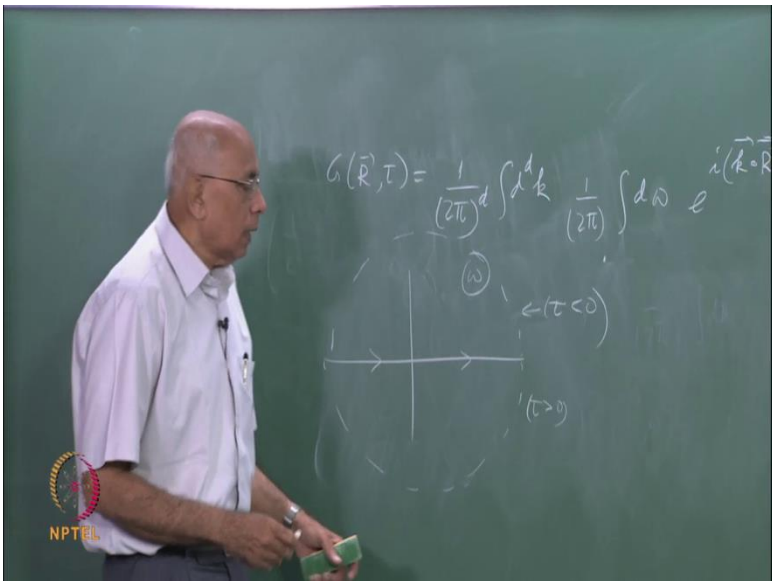

Well, in the omega plane the control of integration is like that, and you have 2 poles at plus or minus c k, I could move them up or down, but I want to ensure that this contour is such that this integral is such that the value of this G is identically 0 for tau less than 0. Now, what is that mean? Well, if tau is less than 0, this chi is negative this minus sign goes away. And then if you close this omega in which half plane should you close it, you close it in the upper half plane, because then that is with this i is going to produce an extra minus sign.

So, it is clear that for tau less than 0 you must close in the upper half plane, and for tau greater than 0 you compile to close it in the lower half plane, but we want the answer to be 0 for tau less than 0, this means there should be no poles in the upper half plane at all. You need to close the account to in this fashion just as a mnemonic for tau less than 0, and you gone to close it like this in this fashion for tau greater than 0. And whatever you do 2 these poles it should be such that this integral is 0, and of course it 0 if it encloses no singularity at all, which means both the poles have to be displaced in a lower half plane.

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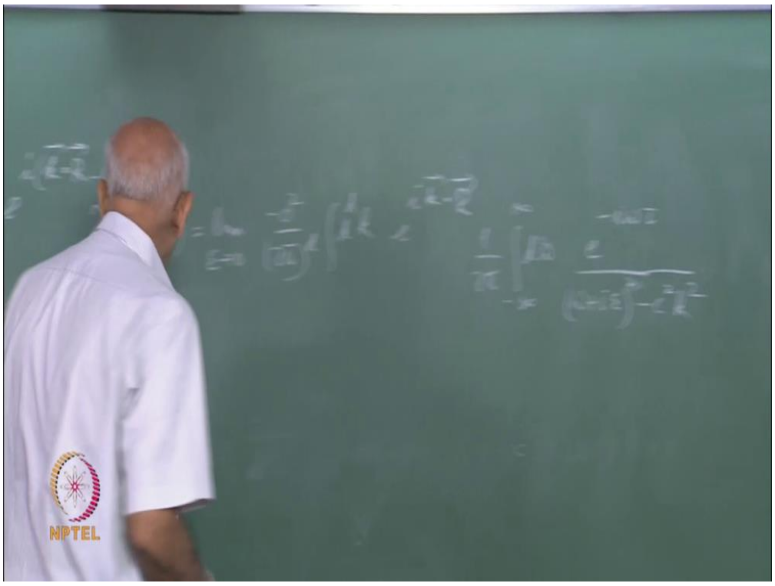

So, the correct way to define this green function is to define at as equal to a limit epsilon goes to 0 from a buff, of something where the poles are displaced in the lower half plane. In other words you have omega plus i epsilon whole square minus c square k square.

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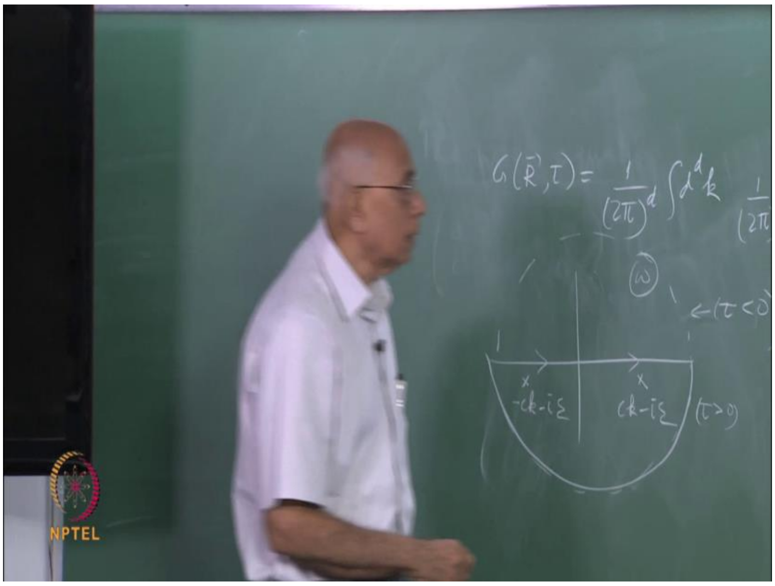

When one pole is here at k minus i epsilon c k minus i epsilon, and the other pole is here symmetrically at minus c k minus i epsilon. And then when I close the contour in this manner here, I am going to pick up a nonzero value, but there is nothing up here. So, for tau less than 0 it is identically 0 is not it.

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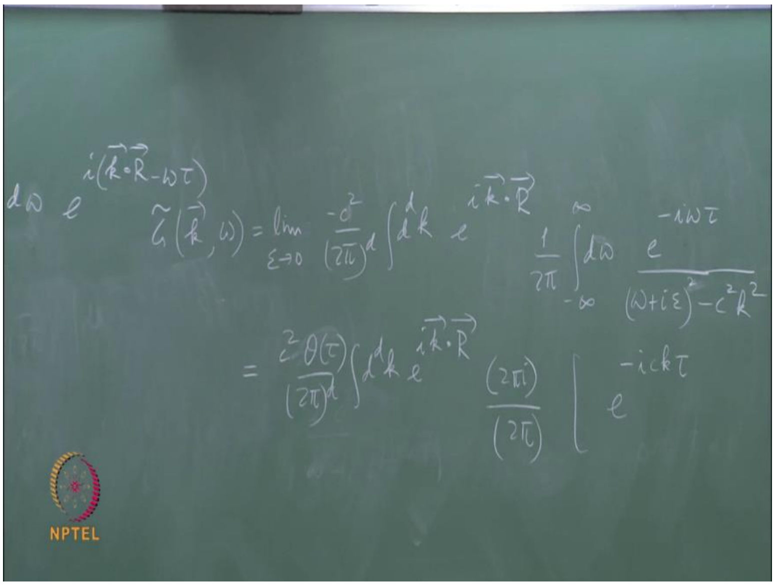

So, it promptly tells as this is equal to minus c square over 2 pi to the power d integral d d k e to the i k dot R. And then, so let us now put that in let us put that in as minus c square fate of tau, because we just showed that when tau is negative this answer is identically 0. So, let put that step function n, and then let us do see what this does, when you close it in this fashion, when I close it here and gone to pick up both these contributions in a minus 2 pi i because it is been traversed each of these contours, each of these poles is being encircled in the negative sense. So, there is an extra minus sign that goes away, and then there is a 2 pi i over 2 pi 2 pi is will cancel out gradually.

And then e to the power minus i omega is c k, so omega c k tau that is this term divided by you have essentially omega minus c k and omega plus c k, and you multiply by omega minus c k and then put omega equal to c k. So, this gives you 2 c k out here plus e to the power i c k tau divided by minus 2 c k, and that is it. So, let us collect these guys together, you have a 2 c k and then a minus sign of their, and that c one of them cancels here, is it 2 here they end up with the k here. And the 2 pi is go away and there is an i here, that is it taken all factor in to account, but this fellow is minus 2 i sign c k. So, let us put that in the i goes away.

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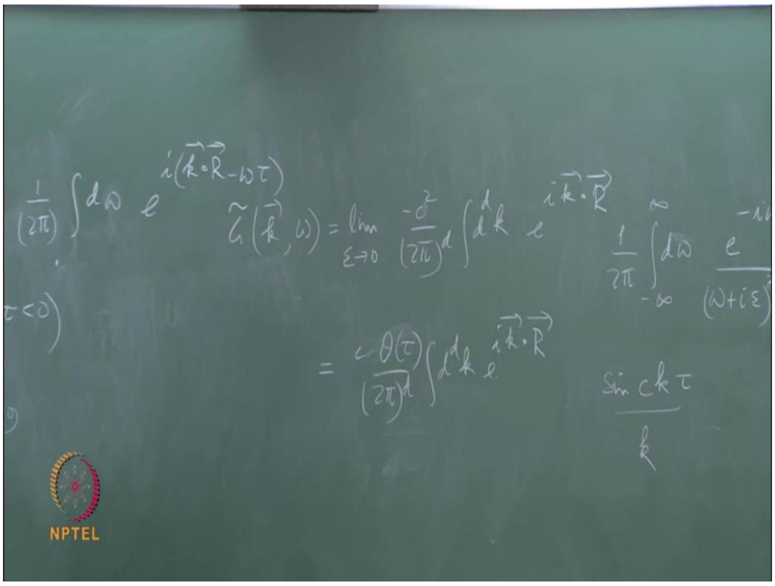

So, 2 sin c k tau and the 2 goes away, let us put this here so we do not forget this factor that is it, so that is the answer. We done the omega integral and you got this state of function out, which takes care of causality.

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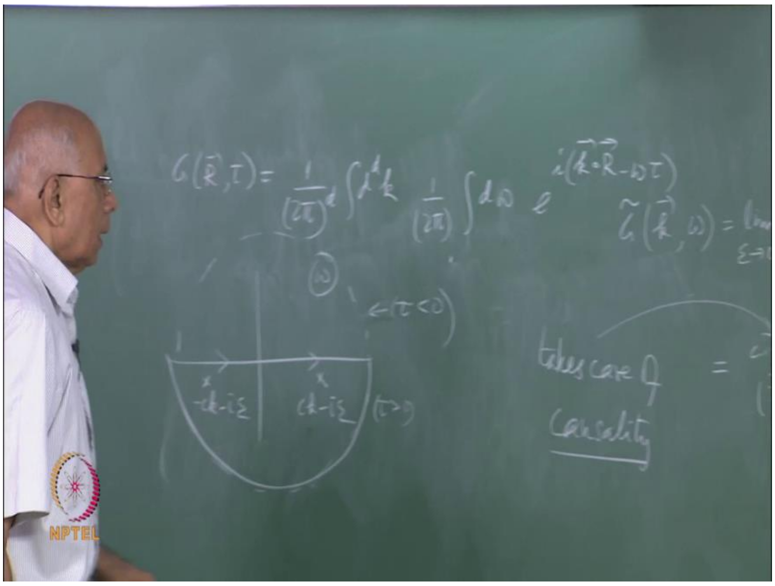

So, this guy here takes care, and you left to the job of doing this integral. Now, of course you have to tell me in how many dimensions you are working, because that will tell you what this face phase factor is in what this only look like. So, there is an explicit dimension dependents out here, but it is causal is green function is causal.

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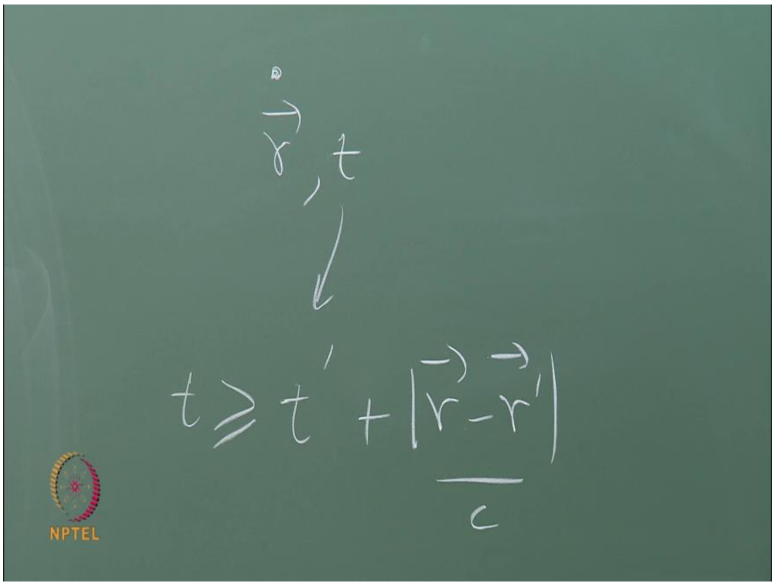

Actually we do not have to get something more out of this, because if you start with the point r prime at t equal to sometime t prime, there is an source which axed the point r prime at time t prime. For you to detect the effect at r at time t is clear enough time must elapse, t must be sufficiently larger than t prime such that this is less signal has trying to propagate from one place to the other. So, what you going to do is, actually see it only for t greater than or equal to t prime plus mod r minus r prime over c, where c is the speed with which this propagates.

You going to have not only t greater than t prime, but t must be greater than this number greater than or equal to this number or else you do not see the you would not see the signal propagates faster than c. And that by let us causality once again, we have and said that c is a speed of light in vacuum, of course it applies to that but this is true in general. All we are saying is that there is causal retarded response this response, such that t is greater than equal to t prime plus that extra piece is called retarded response.

It takes time finite time for a signal to propagate with the speed d or c whatever, so that should come out. We got this data function out fail easily by desire small prescription, but we need to also see that is automatically going to immerge or not. So, let us see, now let us work things out specifically different dimensionality is to see what is go happen, and then there is a little bit of a surprise. Then it briefly what will happen is?

The way this signal propagates and what is this green function after all, it is the signal corresponding to unit impulse at the prime at the point or prime or if you like in capital R at the origin a time tau equal to 0. So, it is the response to this unit impulse, this delta function and that point. Now, it will tell out that in spaces of even dimensionality, when these even you have a very different kind of propagation, than when you have d horn that emerges it comes entirely from this factor. We will see how this magic happens in a minute.

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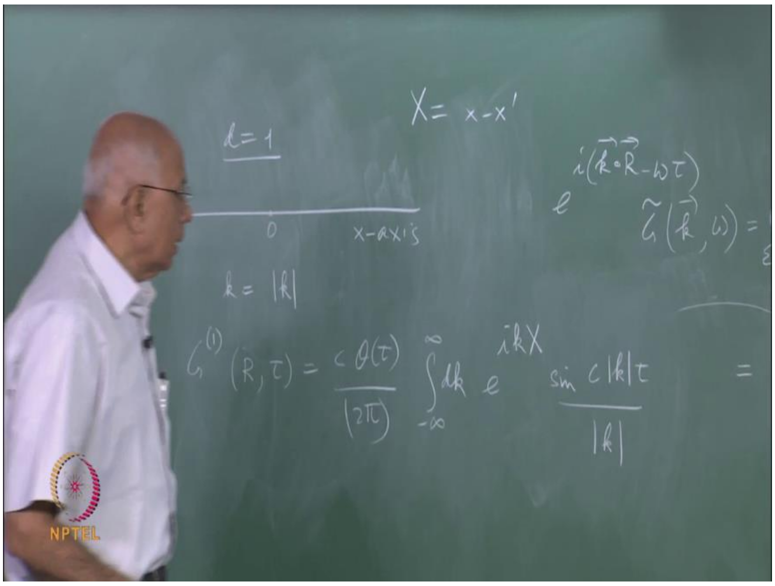

So, let us do it in one dimension first, remember that in one dimension what I call k. So, you have just the x axis in there is the origin here, and what I call k should really be call k is really equal to modulus of k, because in one dimension you just have one coordinate conjugate, and but I called magnitude of vector is just the modulus. So, in one dimension we have G, and now let me put 1 here to show that it is in one dimension of R and tau. By now it should be fairly obvious to you from previous experience that the answers going to be a function only of the magnitude of this vector R, because the whole thing is rotationally invariant.

So, not only it is scalar, but it is also going to be depend only on the magnitude of R. So, might as well write that down, but let me leave it as it is for a moment, you may write it like this G of R and tau is equal to c times theta of tau divided by 2 pi, it is in one dimension. And then there is a just d k, and this once minus infinity to infinity that is the volume enable to one dimensional space e will be i k X, let me call capital X equal to x minus x prime, the x component of capital R.

So, capital X sign c mod k tau over mod k, you should be careful make it magnitude, but of course this is an even function of k. So, again forget about the mod again remove this, it is a same thing. And I need to do this integral, as you know the singularity at k equal to 0 is avoided by the fact that you have this sign here, this goes to 1. And then this kai is going to oscillate some fraction, now you want the green function to be real that was sure. So, if you write this as cos k X plus i sin k X the cosine part as you can see disappears minute, the cosine part remains and the sign part disappears.

So, the imaginary part disappears as it should the green function it is a real know real form of function. So, you permit me to write this as this thing here as cos k X by symmetry, this is running from minus infinity to infinity, cos is an even function that is in even function, so it contributes the sin vanishes that is why. Let us write this as cos k X sin c k tau cos k X divided by k. So, let us put a 2 here and a 2 here, and this thing here I write as some of two signs.

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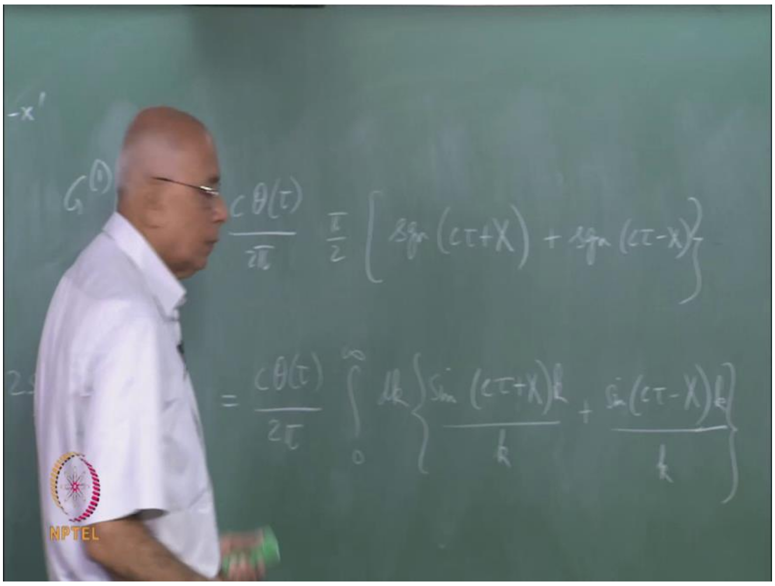

So, this is equal to c theta of tau over 4 pi and integral minus infinity to infinity d k, before I do that this is a symmetric function. So, let me just write it as an integral from 0 to infinity with the factor of 2. So, this 2 goes away, and then I have this 0 to infinity. So, this is back to a 2 pi, and this is 0 to infinity d k 2 sin a cos b, that is of course sin a plus b plus sin a minus b, so this is sin c tau plus X k over k plus sin c tau minus X k over k. Now, what us these give you? It is pi over 2 provided that quantities positive, and ditto here for this fellow also.

So, G 1 of in this case not r, but X this is just X and tau equal to c times theta of tau over 2 pi, and then pi over 2 and the sign of c tau plus X, the sign of that quantity plus the sign of c tau minus X in this fashion times a pi over 2 and that is it. So, let us kill the pi, and we get c tau over 4 times this kai here. Now, what is this look like as a function of capital X tau is remember positive, because this thing sees to it that for tau less than 0 the whole thing vanishes. So, tau is some possible number, and then you want the sign of this.

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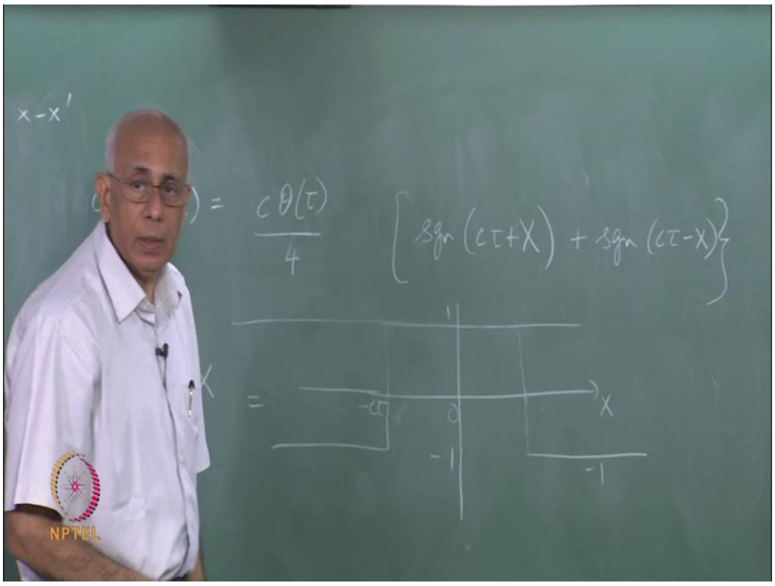

So, that is equal to we plot this as a function X for tau positive some fix tau, then this number is plus 1 as long as X is bigger than greater than minus c tau. So, you have minus theta here and then it is this fellow in this passion, and it is equal to minus 1 the rest of the way, so this is minus theta, this is 1 and this is minus 1, and what about sin of c tau minus X?

Student: ((Refer Time: 27:11))

When X is bigger than c tau this going to be minus 1, so beyond this point it is going to be minus 1 this sign. And below that point, it is going to be plus 1 in this fashion and extremely I am happy, because this is suggesting now the answer is when you super post these 2 guys that the answer lies, that is right that is exactly what you want? This is precisely what you want exactly? So, the answer therefore it is 0 beyond these 2 points these 2 fellows add up, and it is equal to 1 plus 1 in between. So, there is factor 2, which cuts of between this point and that point.

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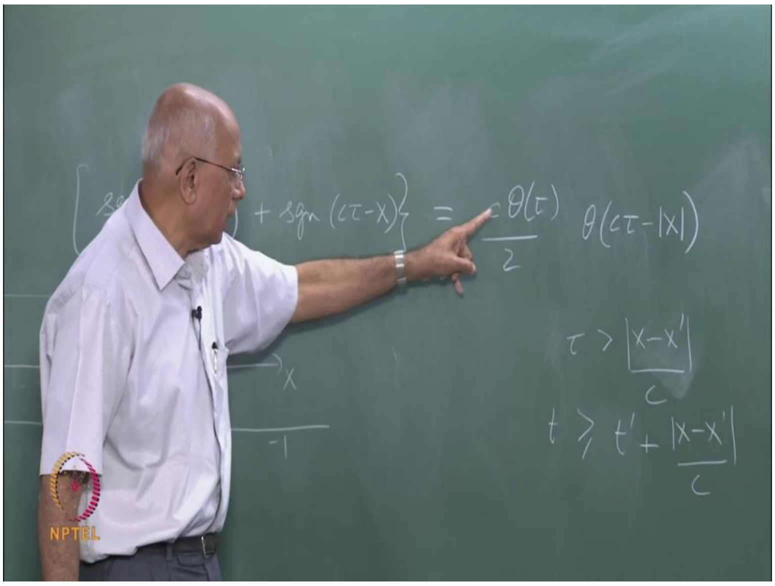

So, we can rewrite this perfect, so the factor 2 and you say c theta of tau divided by 2, and then it is equal to a theta of function c tau minus mod X, X must lie between minus c tau and plus c tau. Otherwise there is no signal that is exactly what you want, because you cannot reach this point here at all and have any signal, because it takes finite time for it to go, so it is clear. This will tell you that c tau is greater than X minus X prime modulus, there does not equal to mod X minus X prime, which is exactly what you want? Because it says that you will not be able to reach there say any quality in t, as you can see this is C.

So, t minus t prime greater than mod X minus X prime over C or t is greater than equal to t prime plus this that is exactly what you want? By causality plus the fact that you have retarded response, and it is come out automatically as you can see, when over to this integral, but by it in that integral by it in this thing here is this nice feature, that it respects the fact that you have a finite velocity of propagation, and that is emerged automatically. Notice another thing and need you to check the physical dimensions of this quantities, this has dimensions of a velocity this are all dimensional guys. So, I leave you to check the g indeed has dimensionality of velocity in this case, notice that what this implies is that the response case for ever.

You started with the delta function that one infinity strong signal, but that is now spread out. So, for all time such that t is greater than t prime plus mod X minus X prime over c, this chi here is a constant. This green function is a constant that is the peculiarity of one dimension.

So, in this in a sense what happens is that if there is it is infinity bright spark on the x axis, and t equal to 0 and you are sitting here. It will reach you after time t which is given by the distance divided by c, and after that there will be a steady illumination. Due to this infinitely strong source the beginning, this is a peculiarity of one dimension do not violating energy conservation; you not do any of those things and get to have.

Student: ((Refer Time: 31:11)) here you seen a wave guide.

There is no way there is no why do why do you say we that in a wave guide.

Student: There is no power loss we see if we.

There is no power loss because of the boundary conditions in the wave and so on and so forth, but what I am saying is in one dimension. And one dimension alone what is starts of is a delta function spike does not pass you as a delta function spike. So, it is not as if there is a pulse there and that pulse is traveling and passes you as a pulse, it does not do that at all. It just gets smooth doubt becomes constant that is a peculiarity of one dimension, but now we got see what happens in two dimensions in that so little harder to do. Let us you get 2 and of course the most important case is d equal to 3. And then just for mathematical curiosity we would like to see what happens in d greater than 3, 4, 5, 6 dimensions etcetera.

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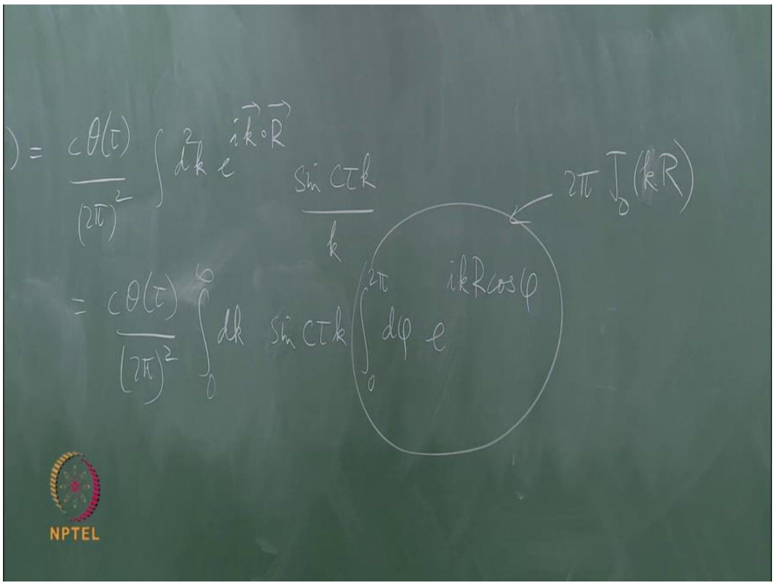

So, let us look at d equal to 2, and what was the green function G 2 of R and tau was equal to there was a c, you have to the minor the factors now, there was automatically a theta of tau and then divided by 2 pi. And then there was a d 2 k e to the i k dot R this fashion. And then sin C tau k over k, where k is the magnitude of the vector k in this case, that is the two dimensional case pad me.

Student: ((Refer Time: 32:53))

2 pi whole square 2 pi square good. Now, of course in this case, this case there is again a function only of the magnitude of this R, this saying here the obvious thing to do is to go to polar coordinates plane polar coordinates. And choose the x axis along the direction of the vector R. So, you going to get this equal to C theta of tau over 2 pi whole square integral 0 to infinity d k. And then there is a factor k which cancels again this guy times 0 2 pi azimuthal angle d phi. Let us write the sin C k tau sin C tau k, and then this 0 to 2 pi d phi e to the power i k R cos phi that is the integral, but this is not an elementary integral become do this in many ways.

You could expand this in a power series, you get functions powers of cos phi from 0 to 2 pi. And you can convert that you can do those integrals write it in terms of various gamma functions factorial and so on. The result is this quantity here is 2 pi times J 0 of k R, the J 0 is the Bessel function of first kind of order 0. After you do the algebra this is what emerges, and then you know tau me at because you still have some problems to face, just as we faced an infrared diversions, when we talked about that green function for the replace an operator.

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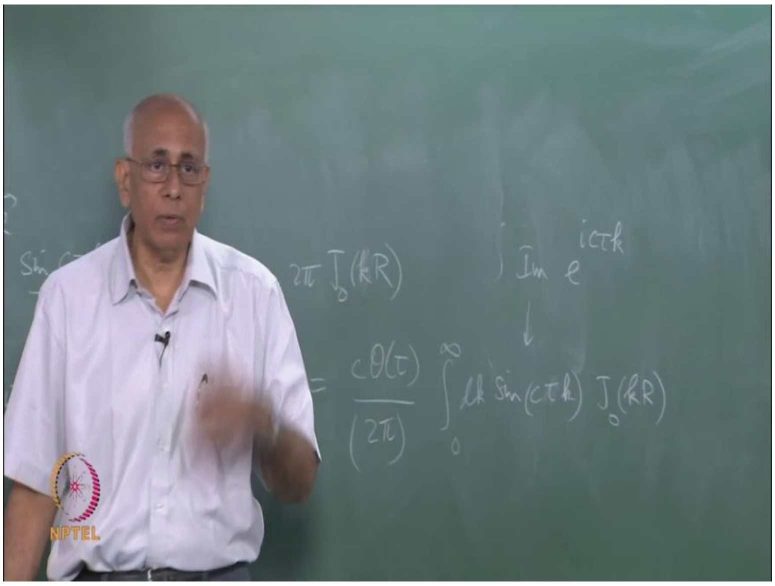

Similarly, d equal to 2 is very strange dimensionality will face a problem, but can overcome it. So, this is C times theta of tau divided by 2 pi one of the 2 pi goes away, and then you have an integral 0 to infinity d k sin C tau k J 0 of k R, in this fashion. Now, again rewrite this as the imaginary part of e to the i c tau k time J 0, you look up tables you will going to this integral, but the quick is swear you do this is a small trick, I can write this as the imaginary part of e to the i c tau k.

And then preteen that this is the analytic continuation to the point c tau k c tau of the Laplace transform of this quantity here 0 to infinity, if I write e to the minus s k then it is a Laplace transform, but now if I put s equal to i c tau minus i c tau, and I have got the analytic continuation of the Laplace transform. So, that is one way of doing this integral defining a giving at meaning and so on, because these are all tricky integrals you can see they not absolutely convergent.

Because this function here at infinity goes down like 1 over square root of k R k and that is not enough to make it converge, but then it change a sign this changes sign that changes sign, and then of course you have a convergent integral like a directly integral. So, took at a long story start I will just write the answer down, and then you will come back and look at it little more in detail and interpret it turns out this quantity here.

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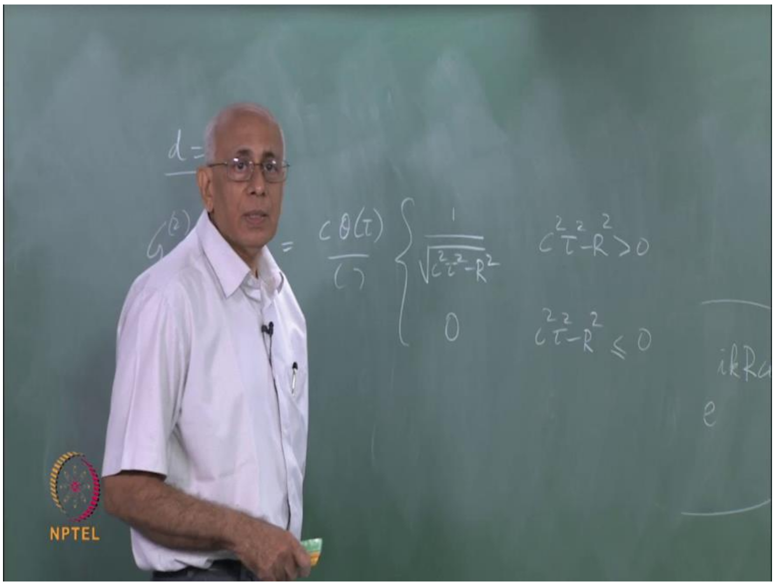

Apart from say there is constant, so there is a c theta of tau etcetera, and then there is some numerical factor here. And then there is a 1 over square root of c square tau square minus R square. So, there is this 1 over square root factor. So, then it dies down as you can see as t increases any point at any given points space point in this fashion, but this ensures again that the response is retarded, because this say c tau is greater than R which is t greater than t prime, t must be greater than equal to t prime plus mod R over c.

So, you have a diminishing effect here, now it is actually decreasing you start of at a large value, but then this single gradually decreases as time increases, if you are at some fix point this flash here passes through and is a the retarded time, but then it goes on persisting. So, there is an afterglow if you like and then goes on forever, but it is intensity it is amplitude decreases like one over time, this case that too is a peculiarity of two dimensions, but you can see it is very, very different from what the original delta function pulse was.

So, you started with delta function pulse at the origin, and then where ever you are this thing does not reach you as a delta function at all, but it reaches you as 1 over square root this kind of signal here in to. The important question now is, what happens in d equal to 3, what does it look like? We will see next time that will d equal to 3 and d equal 3 is the only dimensionality where it happens, this delta function pulse reaches you as a delta function pulse.

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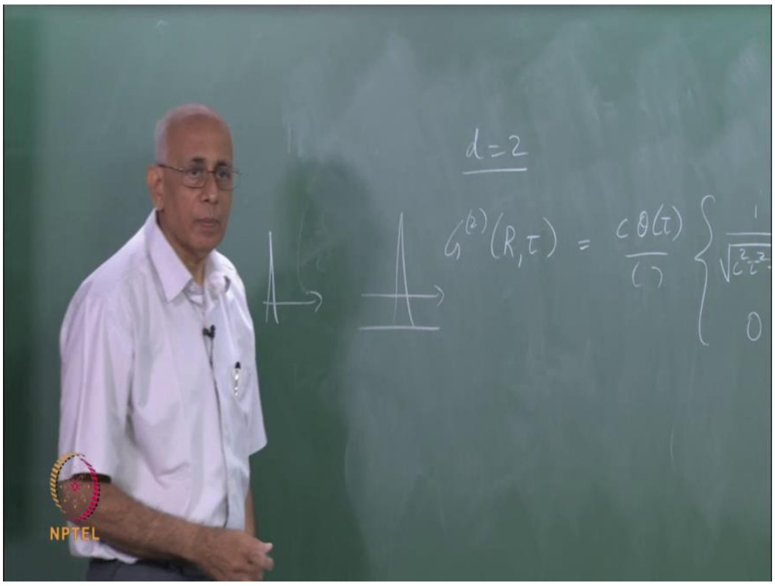

So, you have this searching which started off at this point a travels and where ever you are it passes you as a delta function pulse process point, which has deep implications, because this means you can communicate. It means you can start and stop a signal, and you receive this also as a start and stop sharply it is defined signal without distortion. We only talking about vacuum and we are talking about this simplest wave equation, but many other factors which could changes that, but the fact is this is possible only in three dimensions.