Selected Topics in Mathematical Physics
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Module - 11
Lecture - 31
Green Function for (Del Square + K Square); Non relativistic Scattering (Part III)

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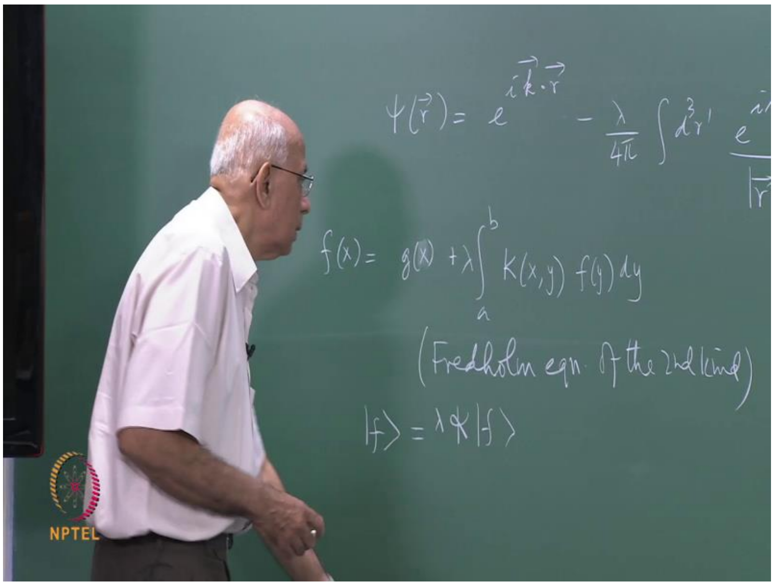

Can psi of r was equal to the incident wave e to the i k dot r minus putting in the green function lambda over 4 pi d 3 r prime e to the i k mod r minus r prime divided by mod r minus r prime. And then a potential U of r prime central potential times psi of r prime it is if, that was the integral equation first scattering lambda is the strength of the potential, and I recall to you that this guy here is related h cross squared times the potential V of r. The potential that appears in the Schrodinger equation is lambda times V of r.

Now, this is an in homogeneous equation, this is the in homogeneous terms and it is integral equation it is called a fredholm equation of the second kind. So, general equation of the following kind, and I will say few more words about it for simplicity let us look at a scalar variable just a single variable. So, if you have an equation of the form f of x equal to some given function g of x that is this in this case in the present case. So, some given function plus an integral K of x comma, some kernel function K of x comma y times f of y d y over some range some given range a to b, this is called a fredholm equation of the second kind, and in homogeneous fredholm equation of the second kind.

Now, we can we will say lot more about this at least a little more about this as we go a long, but it is exactly of that form the kernel here. Instead of y I have got r prime the kernel here is some constant time, it will the i k mod r minus r prime over all minus r prime times U of r prime, and the unknown functions is psi of r prime instead of f here, a given function is this thing here. Now, they are equation were if this is 0, for example this is 0 on this side, then it says the unknown function integrated weighted with this kernel is equal to some given function, if become minus sign whatever that is called a fredholm equation of the first kind, whereas this is in equation of the second kind.

By I want emphasize that is not an Eigen value equation, because of the present of this term here had this not been present, then it is an Eigen value equation. And let us for comparison say put in the lambda here explicitly, then it is an Eigen value equation. Because if this g is not there in abstract form, this thing looks like f equal to lambda some operator on f and integral operator. Then of course you can see that this lambda place the role of 1 over the Eigen value here, that is a fredholm equation which is homogeneous.

And it is can Eigen value problem, but in this case we have an not an Eigen value problem, we have something which is got in inhomogeneous terms present here in this side, and the significant of this will become clear as we go along. Now, let us come back here, and ask what is this going to be as intarticle, because remember we want to identify what the value of the scattering amplitude is where what is this scattering amplitude given by in terms of the wave function. So, for that we need to extract the large r behavior of this kernel here.

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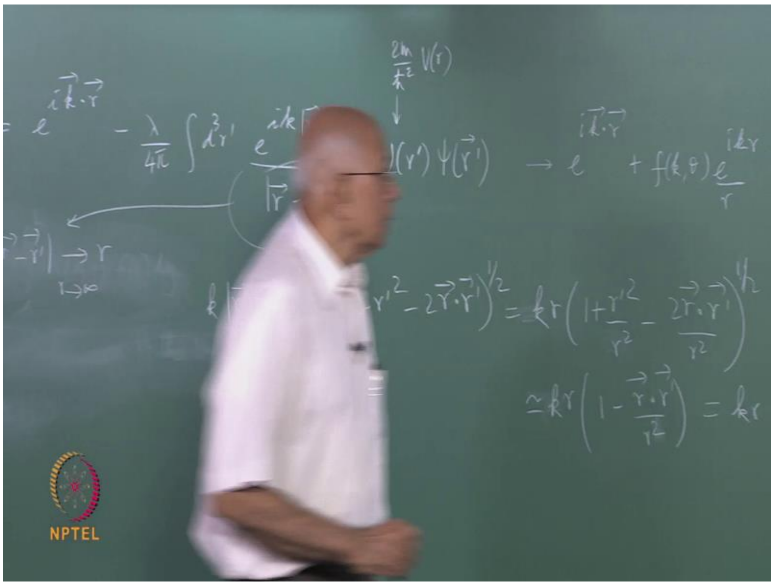

And of course, you can immediately see that mod r minus r prime, the leading terms is r for r becoming very large r tending to infinity in this factor. So, there is going be 1 over r which is what we expect, and we would like this whole things to become e to the power i k dot r plus f of k comma theta. It will be i k r over r that is are asymptotic form, and I want to extract this coefficient f by taking the exact equation here, and extracting what the large r behaviors is. You cannot do the same thing here, because this is the phase factor.

So, we have to retain a little more than this, and notice that r minus r prime is actually equal to r square plus r prime square minus 2 r dot r prime to the power half that is the meaning of mod r minus r prime. And if I take out an r outside this is r times 1 plus r prime square over r square minus 2 r dot r prime over r square to the power half. And now, let us do a binomial expansion the idea being that you looking at a feel point far away from the region were the potential is significant know 0. So, you have origin we have a potential around it is spread out in space, but we looking at asymptotically very far away in a region were the potentials effects is essentially negligible.

And that situation r prime is certainly smaller than r much smaller than r throughout the range of this integration. And you can write this as r times by the binomial theorem, the leading term is going to come from here the leading correction, because there is an r prime square over r square here, but this is a first order r prime out here. First order term in r prime over r, because there is an r here and then r squared here, so whole things goes like a 1 over r.

And that is equal to 1 minus r dot r prime over r square the half cancel against the 2 here and that is it leading term, but therefore k times this k times this k times this equal to k times this. So, that is equal to k r the first term minus k times r dot r prime over r 1 factor of r cancels and you get, but r over r, r vector over r magnitude is just the unit vector in the direction of r. So, this is equal to k r minus unit vector along r dot r k times that.

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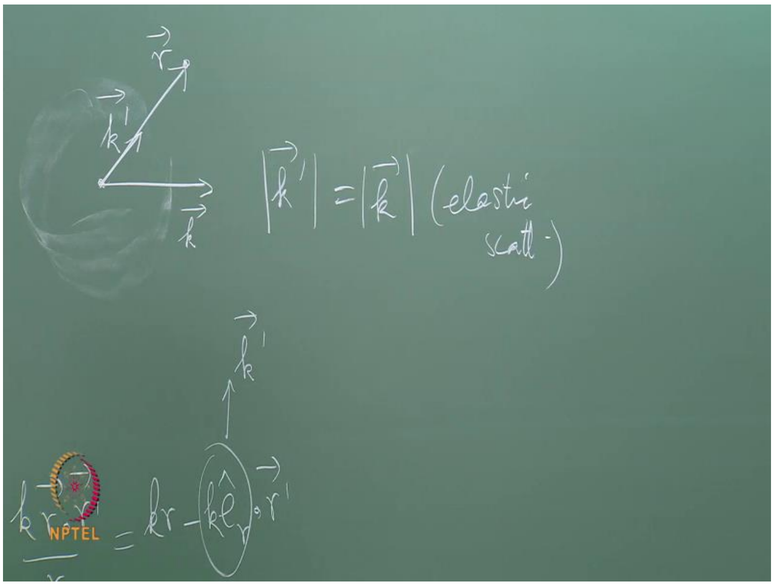

Now, for a little bit of geometry, if you had the origin here the origin of coordinates, and there this potential all around this verse the direction of the incident wave the polar axis here. Now, we are looking at the wave function at some point r in space some arbitrary point. So, you are here this direction here that is the point r at which I want to look at the wave function for very large values of the magnitude of r, and some arbitrary direction that is where may detected is.

Obviously, that is also the direction of the scattered wave k prime, remember that they initial momentum time was h cross k, and the final momentum depends on the direction we are looking at this flex in all directions all. So, this is the direction along which the feel point long which you looking for the wave function, that is the directions of k prime here. So, clearly since the magnitude of k prime is equal to the magnitude of k, because it is elastic scattering for this elastic scattering this implies the k time in r is in fact k prime. Because, it is the magnitude sitting here, and the radial direction in a radial directions, that is the direction of the scattered wave that particular scattered wave the wave vector.

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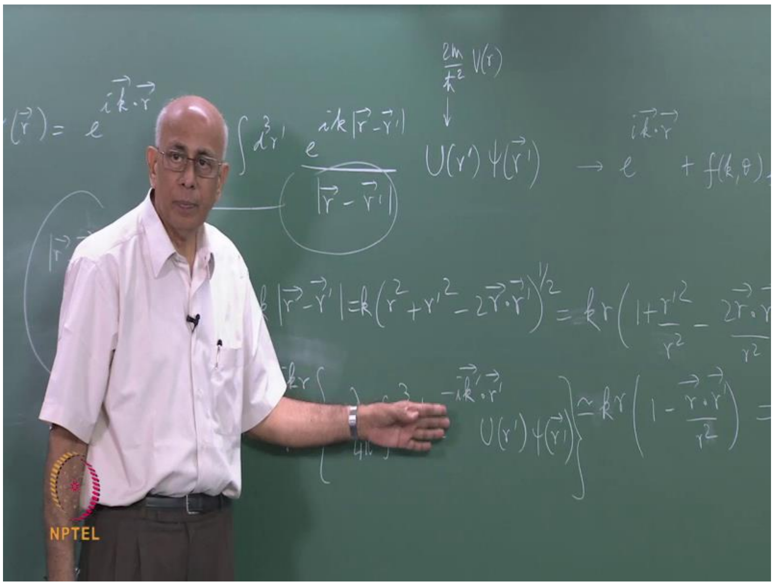

So, you could go back here, and write this whole things out as approximately equal to the e will i k dot r minus a plus e to the i k r over r, the k are comes from here in the exponent here. And then the r comes out form this factor there is 1 over r, so e to the i k r over r multiplied by minus lambda over 4 pi, and integral over d 3 r prime. We cannot get way form that, because r prime could be small you have to integrate of all values of r prime e to the power minus i k prime dot r prime that is this term here, times U of r prime and then a psi of r prime, and that this closed.

So, for large values of k r large com parity unity this is the factor that multiplies e to the i k r overall, and there is an integration over r prime. So, this whole thing is the function of just one quantity the vector k prime, and what is the vector k prime?

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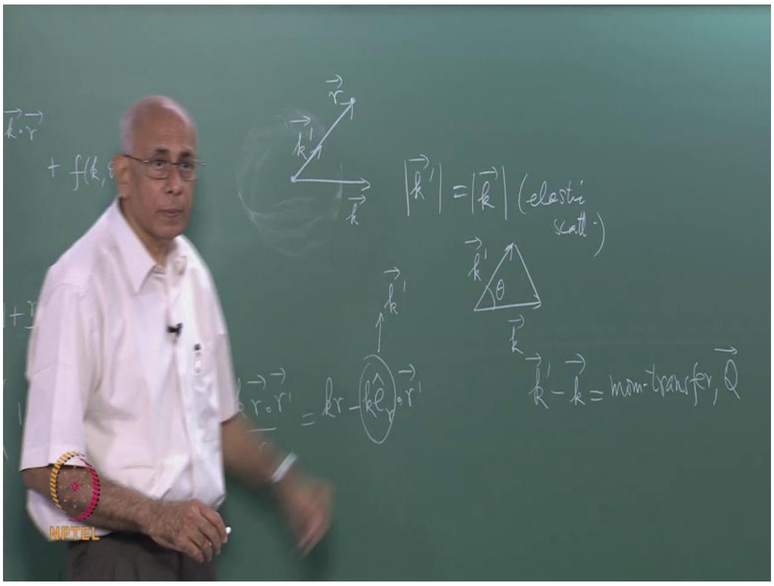

Let us look at the geometrical of this once again, we have the incident direction k, and you have some directions with polar angle theta you looking at k prime, the magnitude of this 2 vector are the same, energy is h dot square and k square over 2 m. Incident momentum, scattered momentum is the same in magnitude multiplied by h cross gives you the momentum. And there is a momentum transfer, so k prime minus k equal to momentum transfer Q, let us call that Q and that is the scattering angle citric. So, what is this of function of this vector k prime is now designated is now characteristics by it is magnitude, which is k itself or the square root of the energy in some units. And the directions in the directions is precisely the scattering angle theory.

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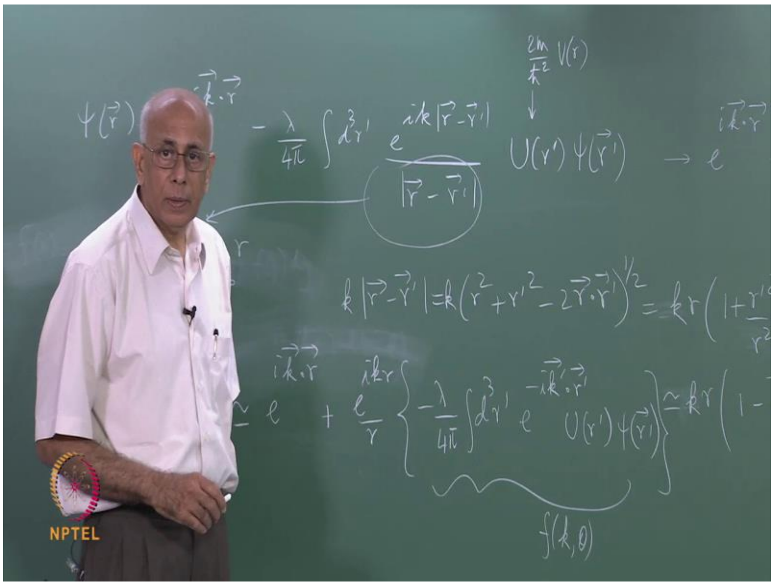

So, this quantity here is a function of k and theta, and that precisely the scattering amplitude, because that the coefficient of it is the i k over r. So, we have in exact formula will come back to this equation, but we have now exact formula for the scattering amplitude.

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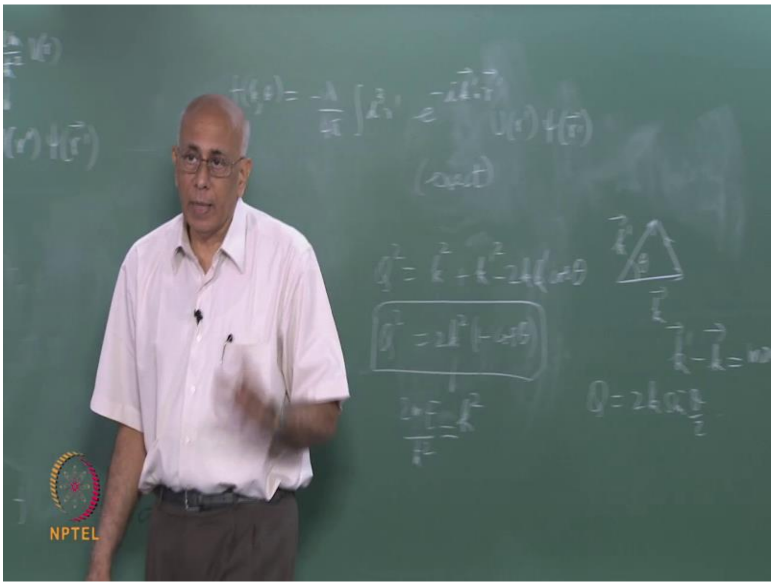

And that is f of k comma theta equal to minus lambda over 4 pi integral d 3 r prime e to the minus i k prime dot r prime U of r prime psi of r prime, this is exact no approximation at all. The catch is to find this quantity, you need to know the solution to the Schrodinger equation everywhere. So, it is back to the square 1 some sense, you make to know the solution in order to wave it to find it asymptotic form, but of course once you know the solution we read of the asymptotic form, but you can do a little better than that.

They thing about emphases is that this is an exact expression for the scattering amplitude. So, now going to word approximation I make put that in here, I get in a approximate equation for this scattering amplitude. And notice also as you resemble expect this depends on the potential everywhere in space not far away not asymptotic, because this integral is over entire space. In particular it will be strongly dominated by those region where U of r is large clearly. So, in nearer the scattering center is more contribution in some sense. And we need to be able to manipulate this, but that is an exact answer here. Now, what are we going to do about an approximation?

Well, we need to use the fact that this lambda can be true when I said; we will keep tracks of powers of this lambda. So, in some sense you could ask question, what happens if I have a very large kinetic energy it will start with. And then I have scattering which is weak compare to the kinetic energy, in other words the potential and some sense is weak compare to the kinetic energy.

Then I can systematically say that there is an effect due to scattering to first order, and the second order, third order etcetera in this coupling constant lambda here. And how that work? Well, let us go back, because in notation here is the little mazy. Let us go back and look at what we had for the fredholm equation. Simply because a notation is easier just once small thing before that, let us try to express this vector k prime minus k, this guy here the vector Q in terms of k prime and k.

So, the momentum transfer I would like to express it in terms of this scattering angle, now it is immediately clear from here that Q squared equal to k square plus k prime square minus 2 k k prime cos theta, but k prime square is the same as k square. So, this equal to 2 k square times 1 minus cos theta or if you like so that is important, this square of the momentum transfer the magnitude is related to the energy here, because remember that the energy E was equal to 2 m E by h cross square equal to k square.

So, we have one relation between the momentum transfer the magnitude of momentum transfer, the energy and the scattering angle, so we give me 2 of them the third is returned. So, this also implies that Q equal to 2. So, if I write this as twice a sin square theta over 2 this becomes 2 k sin theta, and it is a positive number because theta runs from 0 to pi, that is what remembering that the scattering geometry here is worth remembering, that the momentum transfer magnitude is related to the wave number incident wave number, and the scattering angle by this very simple expression.

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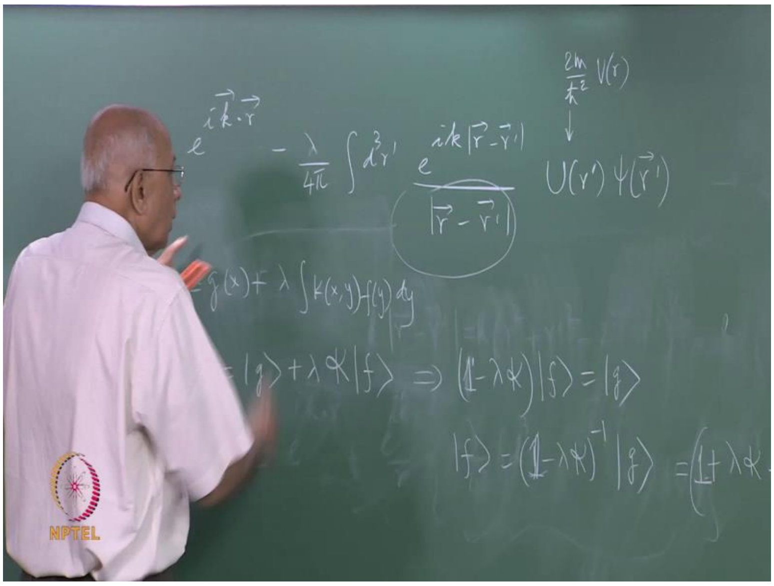

Now, let us go back and look at what happens to the solution of an equation like this or if you write in abstract form. Let us write this out as f equal to g plus lambda some abstract operator acting on if, an integral operator given by the kernel k of x comma y, how do solve such an equation here. Well one by to do that is to say this implies that 1 minus lambda k acting on f equal to the given quantity g, which would mean that formally f is equal to 1 minus lambda k inverse acting on g. And now, I do a binomial expansion of this inverse operator provided a certain quantity is much less than 1, it is less than 1.

So, this is equal to 1 plus lambda k plus lambda k square by the way this 1 is the identity operators. So, let me write it as 1 in this fashion the identity operator plus lambda square lambda cube k cube plus etcetera, this whole thing acting on f on g, when would such an expansion be convergent. Well, it would clearly be convergent I do not want to erase this, I need to retain this. It would clearly be convergent if in some sense this number was less than 1, but it is an operator. So, what should be less than 1?

Student: Norm.

The norms of this operator should be less than.

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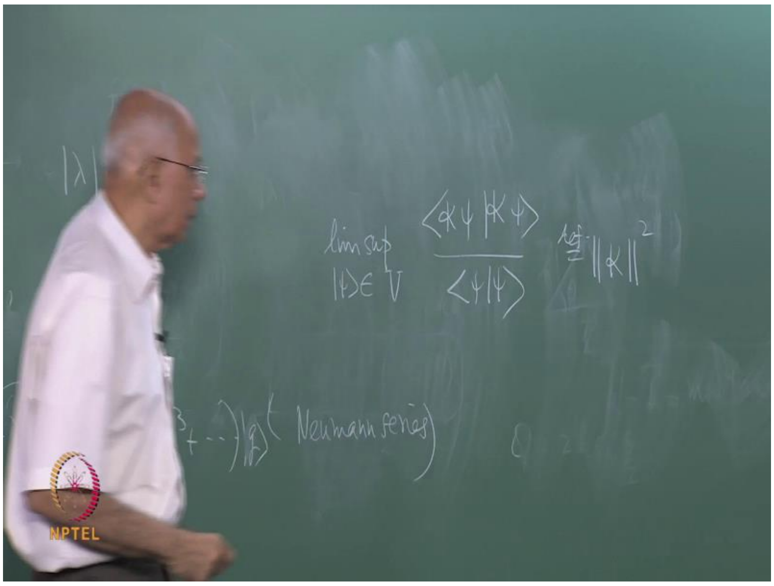

So, you would certainly expect this to be convergent, if mod lambda times the norms of this operator k is less than 1, when it is equal to 1 then you have a problem. Then what it implies is that this inverse there is in exit, and when it is equal to 1 and what is it imply. It means that this equation 1 minus lambda k acting on some Eigen vector equal to 0. In other words you get in Eigen value of this operator, and then of course, this series does not converge any more. So, if there is nontrivial non null vector in this function space h, such that h is equal to lambda k on h itself.

So, that lambda is at a special value one of the Eigen values, then of course the resolvent operator does not exist, and this solution diverges does not exist, this is nothing but crammers rule in plane some in the context of simple algebraic equations simultaneous equations, either the homogeneous equation as non trivial solutions or the in homogeneous one has a unique solution, and the homogeneous equation does not have any thing expect trivial solutions.

So, that crammers rule and that is exactly what it is here, expect this is called the fredholm alternative. So, if you are the assumption is if, the you are in lambda such that this is this condition as satisfy, which also implies by the way that mod lambda is less than 1 over norm of this operator k. Therefore in a sufficiently small circle in the lambda plane about the origin determine by the magnitude of the norm, this series is expected to converge and provide a solution for you, for this equation in homogeneous equation. This thing here is called Neumann series.

Student: What I that is the norm of the operator.

Now, we have an integral operator, so we need to know what the norm of this operator is. So, what is the norm of an operator in a linear vector space, what is the normal definition of so norm of operator?

Student: (Refer Time: 21:39)

If it is bonded if the operators bonded the norm is finite the operators bonded, if norm is infinite the operator is unbounded. So, what you do is to take I do not know what notation you are using, but let use direct notation. So, what you do to is to take this operator k act on every vector in the linear vector space. So, k on side let us is called k psi take this guy here, and take the inner product of this vector itself, but this quantity is 0 if and only if k psi is the null vector.

Otherwise it is some non 0 quantity it is in factor positive quantity, but it is value will depend on what the normalization of psi was you want to get rid of that. So, what you want to do is to divide by psi psi. And then you want to find out, what is the least up a bound of this quantity as psi romes over the enter linear vector space or at least the domain of this operator kappa of this k.

This lims so this least up a bound is called the norms in this case the norms squared actually they are two of them here is the definition. That is for in abstract linear vector space, what we have how over is an integral operator. So, the first we have to tell me what is mean by this square of this operator kappa square cube etcetera. So, what would you say is this is the operation of k on f this guy here, what is the operation of kappa squared on this guy.

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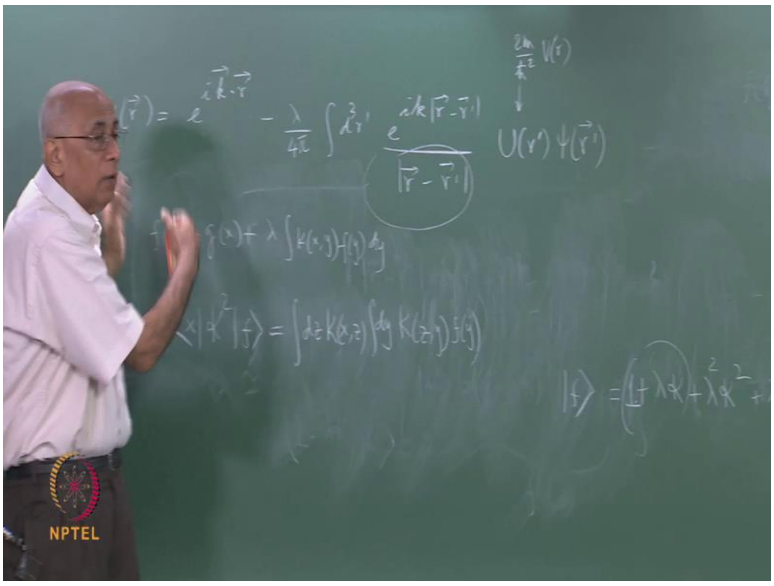

So, what would we the representative of k square of f, what would that look like it just says take this operator and act as second time on whatever is here. So, what would it look like, this thing here is represented by an integral, first you take k of x y f of y integrate over d y, let us call this z convenience, it is some function of z. Now, we take k of x comma z and integrate with respect d z, and the answer of function of x. So, this is precisely what is equal to in the position basis, this quantity here this abstract vector is represented by a function of x, and that function of x is given by this.

So, you see this form k of x comma z, z comma y and you integrate over z, and you integrate over y with the weight f of y. So, that is the definition of the square of the operator, what would be the definition of the norm of the operator? We have to translate this, you have to write this out in the position space in the function space explicitly, so that is going to involve a modules of k of x comma y modules squared the d x d y. So, that is what the norms is in these case, and then if it is bounded then this guy here it says here, that has along lambda is less than 1 over this norm, you are in good shape.

They are specific conditions on this kappa which will tell you under what conditions they are certain kinds of solution possible etcetera. We do not need those conditions right now, it is because we look at specific instances, but this is what the solution looks like this Neumann series looks like. Let us implemented here, let us put that in here, what would that mean? It would mean that the first term in the solution f is g itself; it is the identity operator on g that is already here.

And then this case, what you have to do is to take this solution psi, and replace it by this series term by term. This already a lambda sitting outside kappa constant, so the leading terms in this solution, which is of order lambda corresponds to replace in this psi with just the in homogeneous terms the g out here. So, let us rewrite this equation pretending that you are writing it for sign of r prime in psi.

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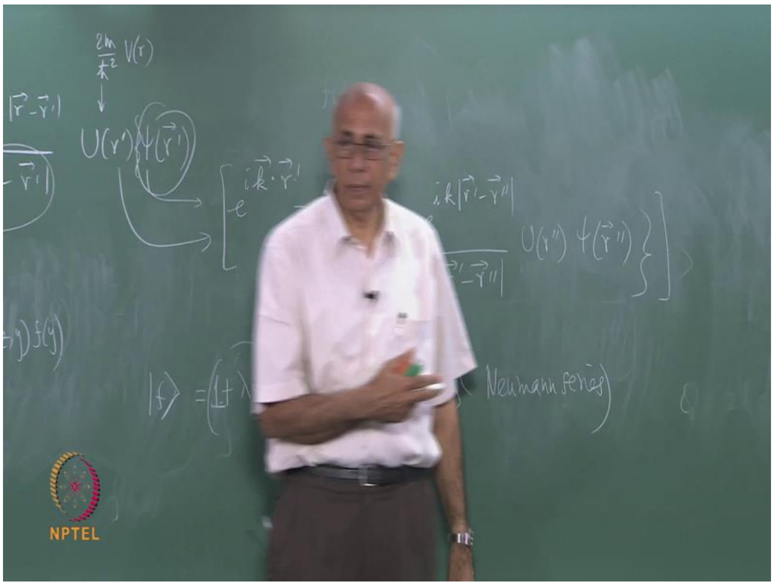

So, for this psi of r prime at this stage, we have to write e to the i k dot r prime minus lambda over 4 pi an integral over what, what should I use the as integration variable.

Student: d r double prime.

Double prime I should use another index, so it is d 3 r double prime e to the i k mod r prime minus r double prime over mod r prime minus r double prime U of r double prime psi of r double prime, that is exact no approximations that is exact. So, instead of this term I replace it with this whole thing. Now, I want a next term in lambda, so you can see now this solution is order lambda to the 0, order lambda to the 1, and then this lambda square times this guy here. And what will be the coefficient of the order lambda square term, clearly this thing replace by e to the i k dot r double prime, and then there is an order lambda four term etcetera.

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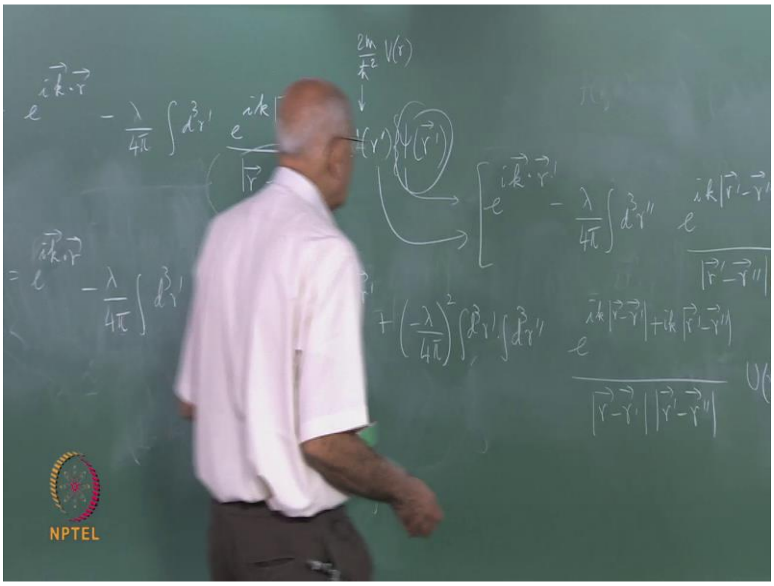

So, this means that your solution is an infinite series which converges for sufficiently small lambda and it look like this, even the i k dot r minus lambda over 4 pi integral d 3 r prime e to the i k mod r minus r prime over mod r prime U of r prime e to the i k dot r prime, k is a initial wave vector. And then the next term is lambda over 4 pi minus lambda over 4 pi the whole square right. So, this next term is the whole square integral d

3 r prime into d 3 r double prime e to the i k mod r minus r prime plus i k mod r prime minus r double prime in site divided by this two guys times U of r prime U of r double prime e to the i k dot r double prime plus dot, dot, dot etcetera.

So, that is the formal solution, and you can now pick of this solution to whatever order in lambda you like. I been a little cavalier about, when it converges expect to say that if this norm of k is found when you can sort that for sufficiently small values of lambda, this series well converge absolutely.

Student: (Refer Time: 30:39)

One form of it a another name is not so crucial yeah all you done is to taken in homogeneous equation, and found in iterative solution in the context of integral equation this is called Neumann series. In the contexts of scattering theory this is not quit disen series, this is called the born series. Retaining up to this point, the first order correction is called the first born approximation.

Now, let us come back and look at what it does to the scattering amplitude, by the way it is very rare that you can take of problem and actually find the all the term in the series and solved it getting clearly more and more integrate out here. They are problem which are solvable, but they are very few and far between, but it is a formal series of this kind. Now, let see what it does for the scattering amplitude for the scattering amplitude, we had a formula.

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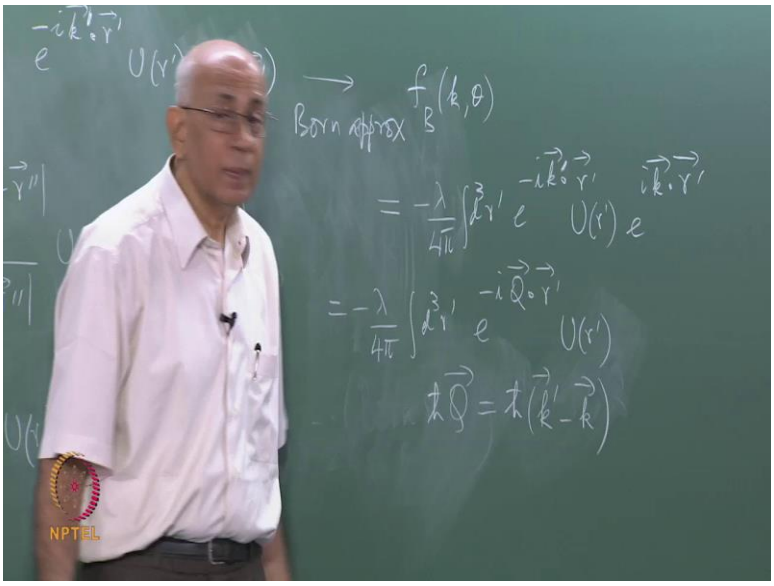

So, we have f of k comma theta equal to minus lambda over 4 pi and integral d 3 r prime e to the i k prime dot r prime with a minus sign, and then there was U of r prime, and then there was psi of r prime. Now, let us implement the born approximation the born series for the psi of r prime on the right hand side. The first term is e to the i k dot r prime, let guy here and that is already of order lambda, because there is a lambda sitting here. So, if I use this term in psi for psi of r prime, then I end up with the second order term order lambda square term etcetera.

So, this quantity in the born approximation goes to f let me denoted by b of k theta equal to the first born approximate to order lambda is lambda over 4 pi integral d 3 r prime e to the minus i k prime dot r prime U of r prime e to the i k dot r. So, in the born approximation, what you have to do is to take two plain wave solution 1 with k prime 1 with k, and sandwich the U in between, and integrate over all space, but you see you can rewrite this also as minus lambda over 4 pi integral d 3 r prime e to the minus i Q dot r prime U of r prime.

Well, Q is the momentum transfer h cross q is h cross k prime minus k, this is the final momentum and that is the initial momentum an different is the momentum transfer, what is that formula look like now? What is this look like? It looks like the Fourier transform of the potential, it look just like the Fourier transform of the potential with the conjugate variable being the momentum transfer, the wave number corresponding to the momentum transfer.

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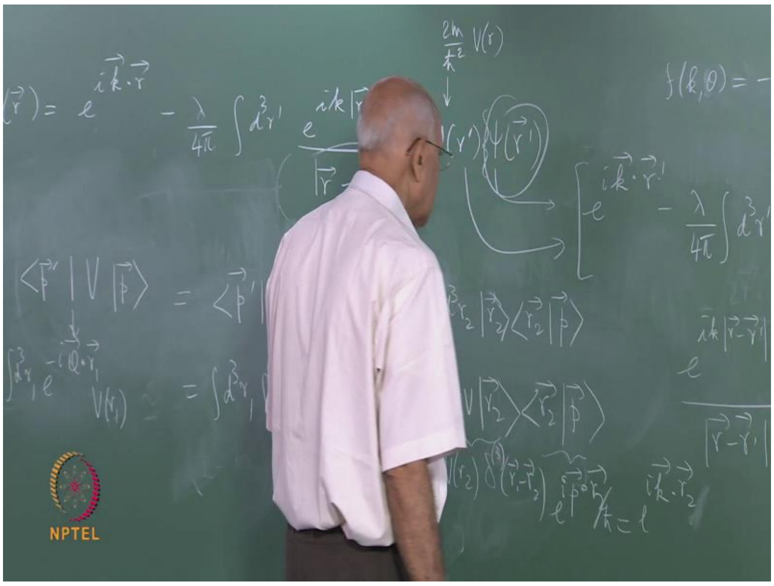

In fact, what it does in to better than that, look at what is this? What this quantity is, suppose the initial momentum is an Eigen state momentum vector p here. And you have the potential energy operator here, and you have the final momentum p prime, and you ask what is this matrix elements? V is a function of r, so clearly this is some complicated quantity this is not same trivial quantity had this means r this whole thing just trivial just multiplication on a position Eigen state, but we have a momentum Eigen state here.

Now, what is this quantity equal to let us try to write this out, this is equal to p prime and then I insert a complete set of states in between, so let us insert d 3 r 1 say let us put r 1 just because I need a large number of in this is r 1 r 1, and then there is a V. And let us insert another complete set of state, so d 3 r 2 r 2 r 2 and then p, I when are a 2 identity operators in between, what is this equal to? This is equal to integral d 3 r 1 integral d 3 r 2, and let us put all these guys together this is p prime r 1 and then r 1 V r 2 and then r 2, but what is this?

What is this scalar product for a plain base for a moment Eigen state is just a plain wave, let us just e to the i p dot r nothing more than that. So, this is equal to e to the i p dot r 2, which is equal to e to the i k dot r 2 divided by p is h over h cross for dimensional reason, that is e to the i k dot r, what is this equal to?

Student: ((Refer Time: 37:27))

e to the minus i k prime dot r 1, and what is this equal to you are asking for the matrix elements of the potential energy, the operator in the position bases. In the position bases the operator does nothing more than multiple by the corresponding function of r. So, this fellow is equal to V of r 2, it is a central potential and then a scalar product of r 1 with r 2, but that is a delta function.

So, this is delta 3 of r 1 minus r 2, I can therefore do the r 2 integral completely, and the answers of course is this quantity becomes equal to integral d 3 r 1 e to the minus i k prime minus k that just Q dot r 1 times V of r 1 and that is it. Of course whether I call it r 1 or r prime does not matter it is an integration variable. So, it is precisely this it is nothing but this quantity here, where was the formula which we had here. So, let us put in all this 2 or more average cross squares etcetera.

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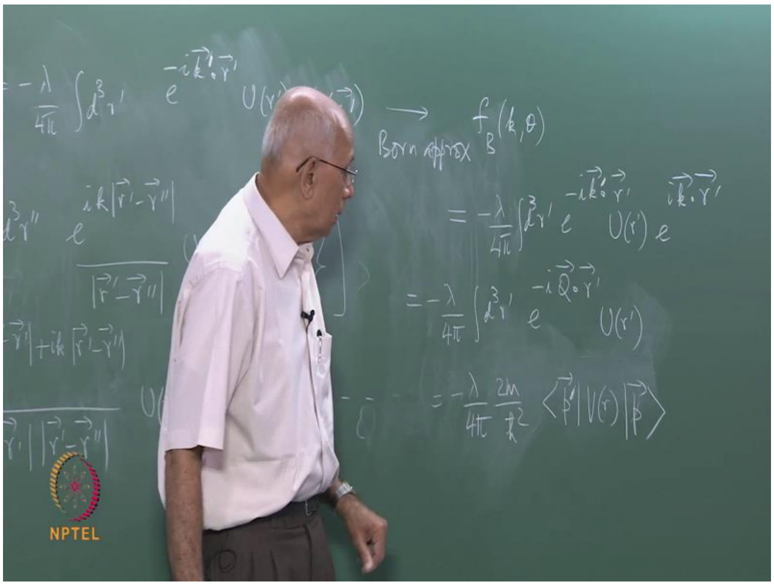

And we discover that this quantity is equal to minus lambda over 4 pi and then 2 m over h cross square, and that kind is precisely equal to the matrix element p prime V of r. So, in the first born approximation, apart from this constant here this scattering amplitude is nothing but the matrix element between the initial momentum and the final momentum Eigen state of the potential energy operator, that is in easy thing to remember. Of course we want the subsequent corrections then you have to put in nontrivial wave function here you have to put in this second order term etcetera, etcetera.

But let see where there gets us let us look at some simple example and see where this thing gets us, on the idea is that if some lambda is sufficiently small in some sense in the first born approximation is good enough. So, let see were that will get us, let us look at some typical cases. Now, we could do all kind of problems, we could look at harts wave potentials will could look at to Yukawa potentials potential etcetera. Let us look at very famous potential called Yukawa potential.

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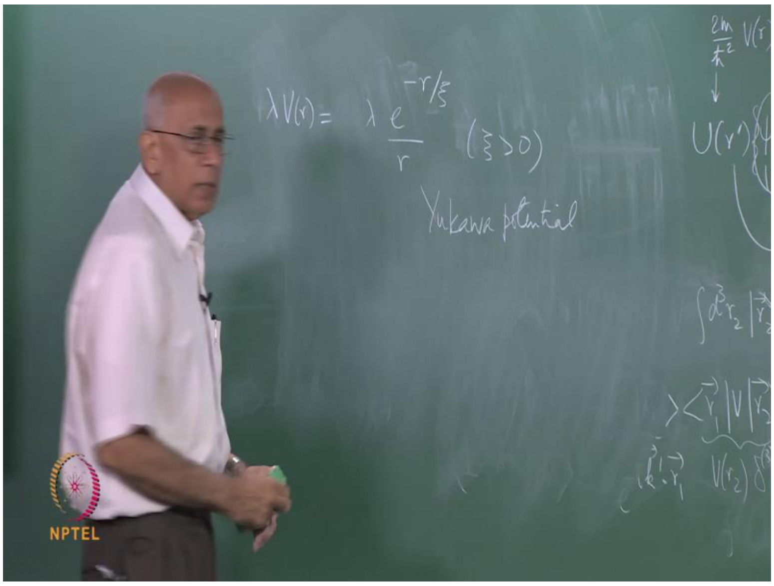

So, let us look at lambda V of r equal to lambda times e to the minus exponentially damped with some length scale psi over r, the psi is the positive constant this thing is called you Yukawa potential. It is a central potential for which the radial Schrodinger equalization is not expressible in terms of elementary function you cannot solve this very easily, it is called ((Refer Time: 41:24)) equation you can in principle write it in terms of some special functions then not interested in that here.

We want to know what is happening to the scattering amplitude in the born approximation, which will be guaranteed to be valid provided lambda sufficiently small. There is a lengths scale in this problem that is given by psi characteristics lengths scale. So, we need to play with that, but before we do that let us take that formula and run with it a little bit, and see whether we can simplify it a little bit, because this is a little massy as it is stands.

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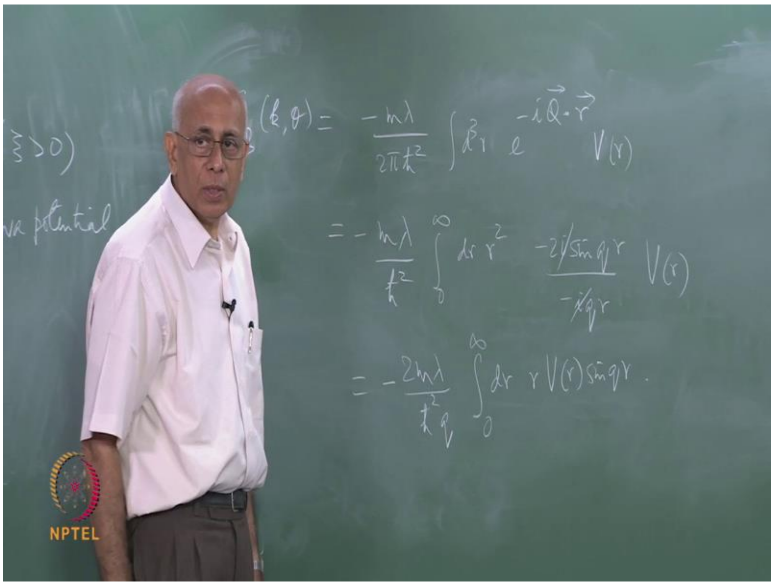

So, let us if we can do some other integration f born of k comma theta is equal to minus m lambda over 2 pi h cross squared. And then this integral d 3 r prime d 3 r might as will call it r e to the minus i Q dot r U of r of r, I need to do this integral for a given Q for a given momentum transfer here, what coordinate system should I choose?

Student: ((Refer Time: 42:58))

Spherical polar coordinates along which direction should I choose the polar axis.

Student: r Q

Along Q, whether get back sticking out, so I choose at long Q than the pi integral gives me a 2 pi is always, and this is equal to minus m lambda and this 2 pi cancels out in get h cross square the pi is gone. And now I have integral 0 to infinity d r r square that is it is here this case, and then e to the minus i q are cos theta and cos theta integrate minus 1 to 1. So, this is equal to minus 2 i sign q r over minus i q r, times V of r, and that is equal to so the i cancels out and then it is equal to 2 m lambda by h cross square q become minus sign.

And then the integral 0 to infinity d r r V of r sin q r, here it is a very simple formula. So, we give me a way of for in integrate and this do this integral, that is this scattering amplitude in the born approximation. The mod squared of it gives me the differentials cross section, and after that I have to integrate over theta the get total cross section in the born approximation. So, let us apply to this Yukawa potential this is your happens.

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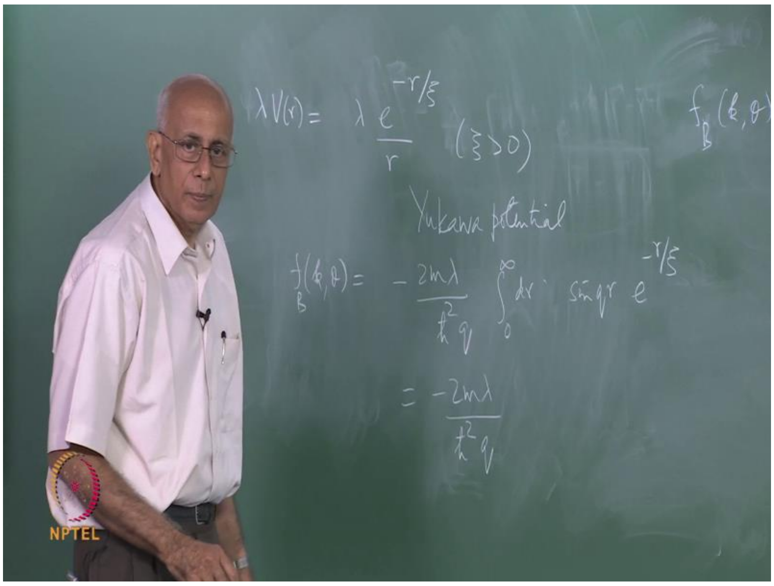

This thing here becomes f of k theta f born theta becomes minus 2 m lambda over h cross square q, then an integral from 0 to infinity d r there is an r sin q r, and then e to the minus r over psi over r. And obligingly this r cancels out that integral of course we know the value let us guy here. So, what is this equal to minus 2 m lambda over h cross square q, and what is the value of this integral e the minus a x sin b x, how you do this integral you learn this in first year calculus are something like that.

Student: 1 by 1 plus that.

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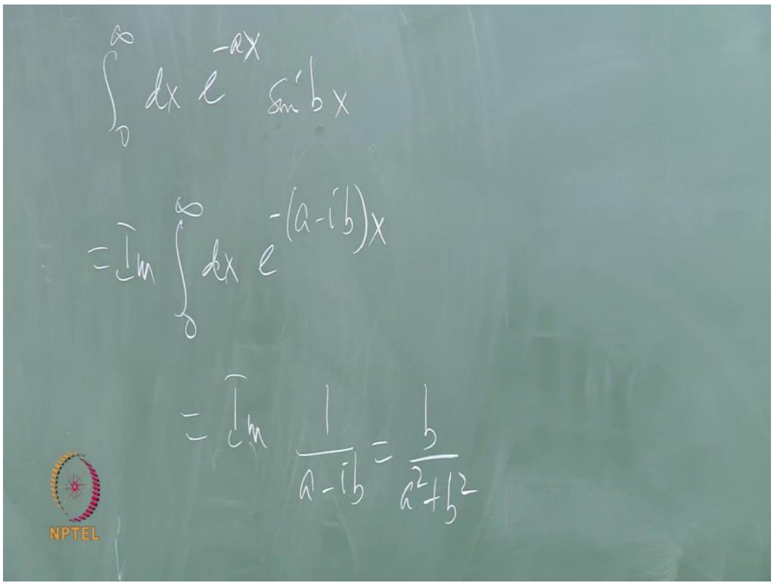

Integral 0 to infinity d x e to the minus a x Sin b x or cos b x, what other value of this integrals, you have to integrate by parts and then you bringing back and so on. So, what you get it the end of it you struck on ((Refer Time: 46:21)) and you do not have any thing no table of integrals nothing no mathematics nothing what you do? Well, this integral is b over a square plus b square, and this integral is a over a square plus b square no.

All you have to do, it is the imaginary part of this guy here. So, write this as e to minus a minus i b, and that is equal to the imaginary part of 1 over a minus i b, which is equal to b over a square plus b square. And the course cos of course the co sign is a over a square, how do you remember which is which, if you had a cos here then the answer would be a over a square plus b squared the real part, and with the sign but an instead of doing this each time, how do you remember which is which, what happen to this when b is 0.

Student: (Refer Time: 47:50)

In vanishes, when a is 0 what happens?

Student: 1 0

Very tricky, but when b is 0 it vanishes the sin, but the co sin does not, so whatever vanishes that must be the sin, this is sin their other guys co sign.

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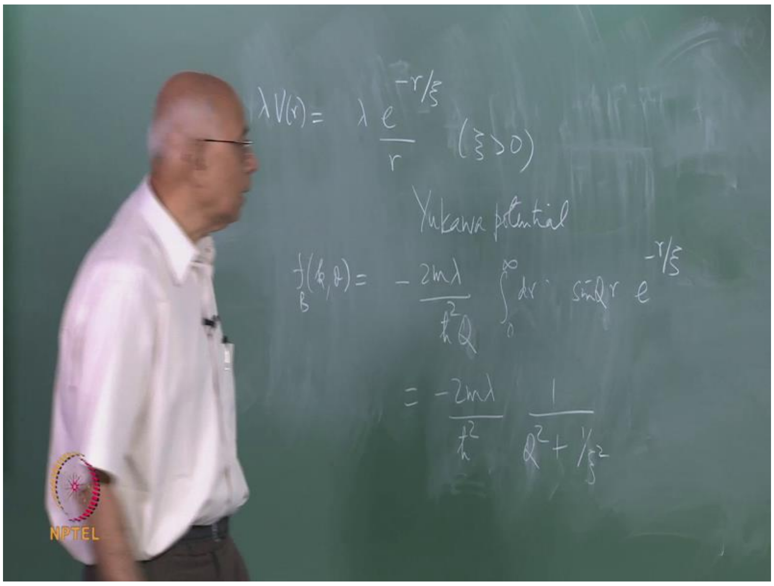

So, any way this thing here is equal to q over q square plus 1 over psi square, and the q cancels out that is it. So, that the explicit representation in the born approximation for the scattering amplitude from a Yukawa potential. Now, let us write this back in term this should be somewhere along the line a man mistake in notation, this was a capital Q everywhere. So, this is capital Q, this guy as a capital Q square of the momentum transfer.

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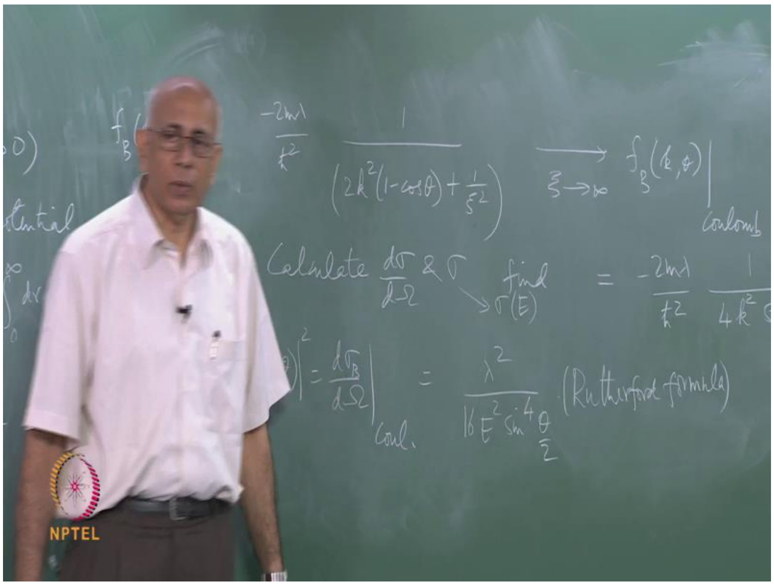

So, this implies that the born in the born approximation f of k theta equal to minus 2 m lambda over h cross square, and then 1 over and what is q square remember it was 2 k square in to 1 minus cos theta plus 1 over psi square. So, that gives you the exact dependence in the born approximation on the angle as well as the energy, as well as little k. So, calculate d sigma d omega and sigma total in this, the mod square of this is d sigma over d omega, and then you have to integrate it over all solid angles that is theta from minus 1 to 1, I mean 0 to pi and pi over 0 to 2 pi gives you just a factor of 2 pi. So, that gives you the total cross section, and expresses this sigma as a function of E.

Find the total of cross section as a function of the energy it is going to depend on k, so depends on the energy, and show that for very large values of the energy it actually drops like 1 over e very characteristics. Now, we can do a little more, we can actually do what the coulomb potential does. And that is a little tricky, because suppose I had the coulomb potential what was this integral actually, this was equal to d r there was r here, and there was V of r here in general. Now, for the coulomb potential you end up with just 1 over r that cancels this and gives you an integral over sign Q r which does not exist as it stands, because this function r oscillates back and forth.

But you can still make meaning out of it by saying over the coulomb potential is the psi going to infinity limit of this Yukawa potential, because psi tends to infinity this goes away and you end up just 1 over r and that we can extract the psi tending to infinity limit of this. So, this guy here goes psi goes to infinity to f born k theta for the coulomb potential the verse that equal to that is minus 2 m lambda over h cross square 1 over.

So, let us put all this fellows in, and then we have 2 k square, so that is gives you a 4 k square sin square theta over 2, and this is 0. Now what is k square? It is 2 m e over h cross square. So, let us put that in here also this is equal to minus 2 m lambda over h cross square 1 over 4 times 2 m e over h cross square, which is up there and then sin square theta 2 and h cross square cancels the 2 m cancels. And then you end up this statement that in the born approximation k theta modulus square equal to d sigma over d omega for the coulomb potential d sigma in the born approximation, this is equal to the mod square of that fellow, which is lambda square over 16 e square sin to the power 4 theta over 2, does this ring a bell.

Student: ((Refer Time: 53:19))

This is the Rutherford formula for scattering from a coulomb potential, it is precisely Rutherford formula many remarkable feathers of this formula. The first one is we will evaluated it in born approximation in quantum mechanics, but it happens to be the exact classical answer, it also happens to with exact quantum mechanical answer in this case, because the higher terms vanish in this potential all the wave have and prove that here. And what is most remarkable about it is that it is a quantum mechanical scattering problem, but there is no plank's constant.

So, plank's constant does completely disappear which is wide make sense to say it is also the classical answer, because otherwise the more chance of coinciding. And it is exact this formula as exact, there is this famous cosec to the 4 theta over to sitting here, what is it as theta goes to 0, it blows up. This means the forward scattering amplitude is blowing up. And of course the total cross sectional blow, because you integrate this is not an integrals similarity that also blows up.

Now, the reason for that this is, the reason for that this is the fact that this scattering cross selection blows up the forward scattering amplitude blows up etcetera. Is that the entire formulism that we started with is not valid for a potential which goes to 0 as slow as 1 over r is cot go to 0 faster than that. In moment in does this when the assumption that the initial state was a plane wave state is no longer valid, because the potential really acts everywhere, and you have to be much more careful you have to start with a proper initial state, which tends out to be an exponentially modulated plane wave state.

So, there is a plane wave state which is modulated and behalf to use that there a logarithmic corrections and so on, phase factors. And will do that, you get an exact scattering answer you get an answer for the coulomb problem very regress, but the fact is that the Yukawa potentials actually in the limit psi goes to the 0, and sub giving you the correct Rutherford formula here. Now, by the way a word about this Yukawa potential in quantum field theory, the forces between particles and mediated by what are called gage fields.

And the mediated by the quanta of these fields, and in the case of electromagnetic interaction it is photons, in the photons have 0 rest mass and that what corresponds to psi going to 0 here. Because, this quantity here has the dimension in the static non relativistic approximation, this exchange of particles in quantum field theory will lead to a static potential of the Yukawa type.

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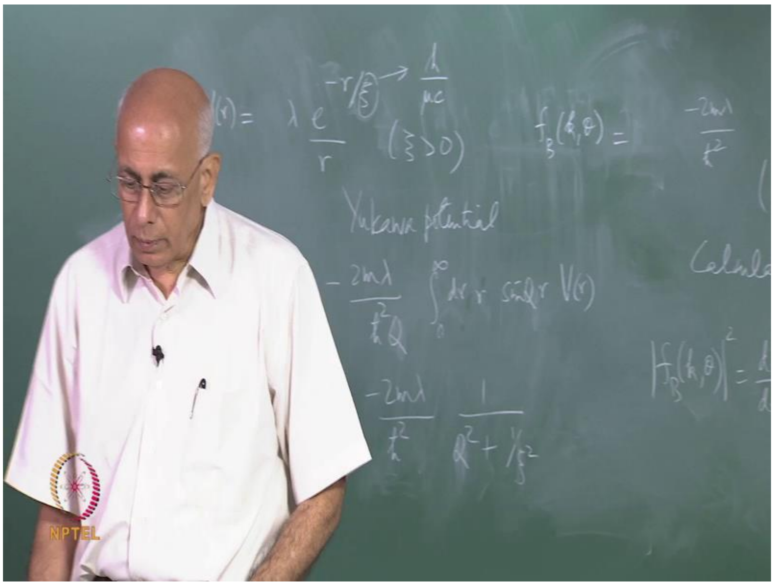

In which this quantity here has the dimension of the Compton wave length of the exchange quantum, and what is Compton wave length? It is h over the Compton momentum, and if there a rest mass of the particle is mu, the Compton momentum is mu c. So, this guy here is h over mu c and the mu goes up on top, and it is rest mass of the quantum, and in case of photon that 0. So, that disappears becomes unity and you end up with the coulomb potential.

So, there is a direct connection between the rest mass award of this exchange. Then this phenomena logical Yukawa potential the form of this potential, and the Rutherford formula as you know is valid in the case of electromagnetic interactions, because the exchange Compton is the photon and this rest mass is 0. So, that is why you get a coulomb potential from Yukawa potential.

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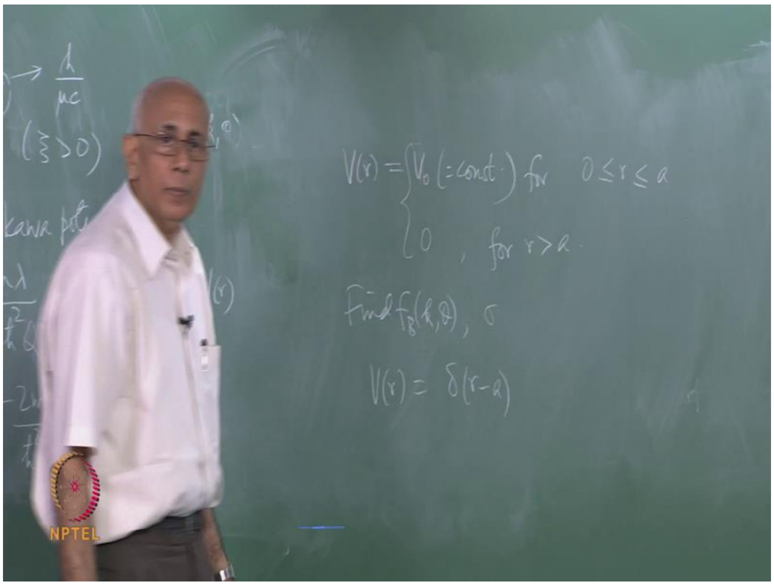

Now, we can do this in the case of other potential also in fact I urge you to do the following, do it further following potential. So, take V of r to be equal to V naught some constant for 0 less then equal to r less then equal to some a, some fix number a, and 0 for e greater then a. And find f B of k theta sigma the cross sectional in this case. So, this is like saying that I have potential barrier, and I scatter of this barrier. And also do this for V of r some v naught some constant do you do not need that constant here actually, because we can observe it in lambda delta of r minus a. The delta function at some value a, nowhere else just a spherical shell and you find out what is scattering cross section is…