Selected Topics in Mathematical Physics
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Module - 11
Lecture - 30
Green Function for (Del Squared plus K Squared); Nonrelativistic Scattering (Part- II)

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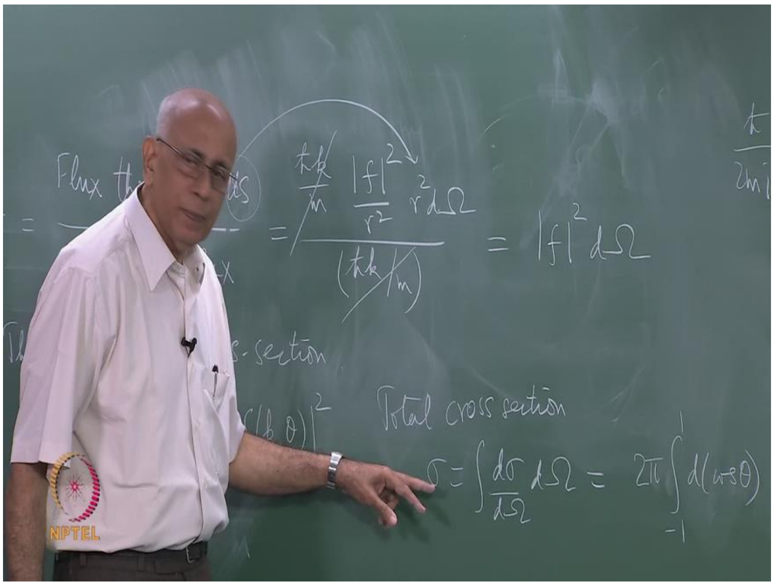

What is the total cross section equal to you integrate over all angles, everywhere then the total cross section is simply going to be total cross section sigma equal to integral d sigma over d omega d omega which is equal to, and this is over the polar angle as well as the angle there is no dependents on the polar angle here that gives you a two pi. So, this is equal to two pi integral minus one two one d cause delta more f square this is a sign theta d theta from zero to pi preferred to write in this form, what is the physical dimension of this quantity? What is the physical dimension of that quantity? Well go back here be wrote in a normalization be wrote psi incident as e two the k dot r, which is the dimensional quantity plus i scattered which has to be dimension less this is dimension less this guy here has dimensions of length. So, this has dimensions of length this scattering amplitude has dimensions of length.

And the cross section as modes square of that. So, this has dimensions of an area, and that is the reason for calling it as cross section cross section, we can compute what this quantity is for a classical beam of articles for instance literally calculated, what happens?

If I have for instance hart sphere of some radius or, and I have a beam of particles coming in what is the geometrical cross section of this problem pardon me.

Yeah, all you have to do stake a cross section is a pi r squared right, but if you do this with waves classical waves, then what is the cross section go into be you got to include diffraction effects. In fact, under diffraction there is going to be maximum at this point here by constructive interference at this point.

Classically it is in the shadow, but off course you have shadow scattering once you have waves, and then the numbers greater than pi r square will see we will try to find example, and do this for a hard sphere could in a actual hart sphere, and scattered waves of it, and see what this is, then will come this problem of does this radius capital r gives you us length scale k gives you a line scale now we have the question of is k r, because n one smaller than one etcetera, then you have different answers.

We will see what it is we will see that it can become as large as four pi r squared. So, once you include diffraction effects, but any way this is now the formula for the total cross section what is the next job we need to find this scattering amplitude we need to find out what is it, and then how am I going to start to the Schrodinger equation, and find this scattering amplitude explicitly.

In other words, all this is true regarders of what the potential is, but now we got to really find out how does the scattering amplitude depend on the potential for that yeah, omega is substance the fraction that is captured in the particular.

That’s right yeah.

So, the integral of it will actually will total amount scattered.

Total amount scattered, yeah to total scattering cross section namely effectively, what is this potential acting as suppose you did it completely classically how big an object is this potential. So, this peak for way which have a certain characteristics wave number k that is a question we will ask. So, let us write.

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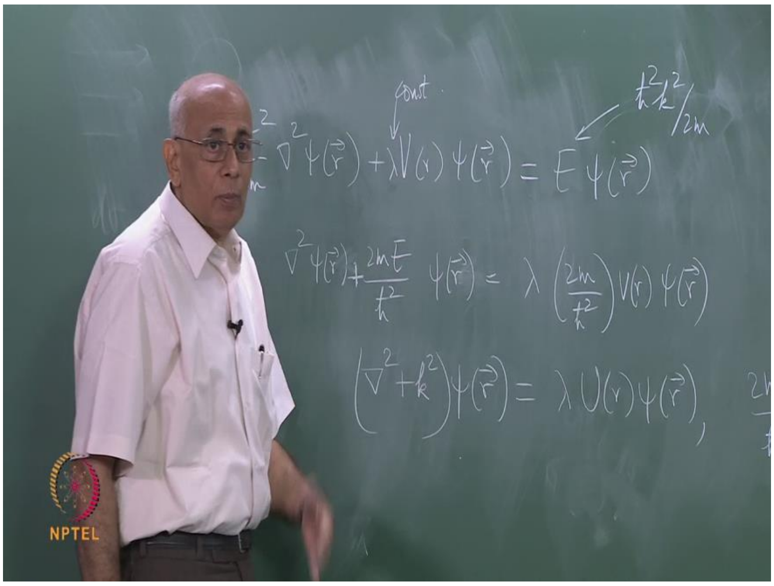

Now, got to go back, now we have no choice, but we have to go the shod nary equation, and write the equation down, and try to solve it understood time independent Schrodinger equation is of course minus h cross square over two m del square psi of r plus v of r psi of r equal to since we looking at stationery states, and there is an incident energy which is conserved that is the i n value of the energy over. So, e this guy here is given to you, and we are looking at those cases vary is positive that is the continuous spectrum of scattering states just go back remember what happens in the hydrogen atom problem if you plotted r verses the potential v of r this is a minus one over r potential an attractive potential, and it look like this, and this is an energy scale here if you plotted what are the values of the energy again values, then you have one value here the ground state which is minus one over one square in Rydberg units, and then the next state that n is once.

So, you have minus minus one, then next one is minus one fourth, and then minus one over nine there is one over n square spectrum with a minus sign for all the bounce states this is a ground state the first exited state etcetera, and then as you tan toward zero, then number of state there is an accumulation of these bounce states, and you can also have any pause positive value, this whole thing is a continues spectrum here those that correspond to this catering of an electron of a given energy of the proton without getting capture. So, that is the spectrum we are looking at. Now we are looking at the continues spectrum any given positive value of e, I want to know what this catering is what this wave function looks like. So, this is the problem we want solve, and the wave a you going to do there is a is to use something called perturbation theory. So, the assumption is that I would like to keep track of the powers of the potential the strength of the potential relative to the incident kinetic energy to facilitate that let me put a cupping constant lambda here constant here that measures if you like the strength of the potential, because I am very often going to be loose, and writing b f r i am going to write it as one over r which is not the right physical dimensions. So, this lambda constant will take care of all those dimensional Constance Besides if lambda is zero, then it is a free particle, and i as i turn strength of lambda up it becomes a stronger, and stronger potentials. So, I want to keep track of this if for that purpose limit put this constant here yes for book keeping.

Now, this is the problem you want to solve, and the way to do this is to first write this as del square psi let us take the minus sign there, and bring it back here plus two m e over psi equal to lambda two m by h cross square v lambda v lets write the lambda outside two m h cross square v of r psi this is a function of r plus i move the potential to a right hand side for the movement, but this you recognize is precisely the Helmholtz operator equal to i need a name for this follows let me call this lambda u of r psi of r there h s observes this quantity two m by h cross square v of r u of r this is redefined potential observing this constant.

This looks like the Helmholtz equation they in homogenous Helmholtz equation with one difference this is not a given function this involves they unknown function itself right, because this is after all an iguana value equation just rewritten that iguana value equation, but suppose for a minute you pretend that you know the right hand side what happens, then the solution of this equation is exactly that for an n homogenous equation which some particular integral, and complimentary function.

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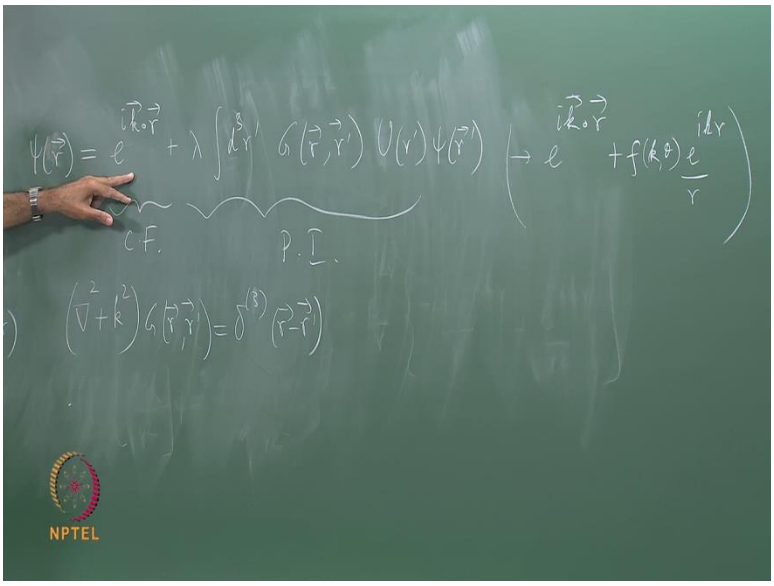

So, i would expect this solution of this equation to look like this psi of r is equal to, and integral d three r prime the green function for this operator g of r r prime multiplied by lambda sorry there is a lambda that is going to sit here, and then u of r prime psi of r prime psi of r equal to this that is the particular integral, but this g of r r prime is precisely the green function for this operator plus a complimentary function. So, this is the particular integral, and then there is a complimentary function, and what should that p well, I already know what the boundary condition is I want it look like psi incident plus this cater portion this cater portion comes entirely, because there is a potential that is a second term.

So, I want this guy here to be e to the i k dot r as i explain earlier the purpose of adding the complimentary function certain amount of the complimentary function to the particular integral this to ensure that the function satisfy this specify boundary condition my boundary condition is that as emphatically these guy had better go like e to the i k dot r plus f of k comma theta e to the i k r it should go in this function. So, the trick is to start with this find the green function put it in there, and then that it go to this limit when little k r is much, much bigger than one, and extract by comparing with that form what is the f of k that is going to be the strategy now this is not a solution for psi of r. We have to appreciate that very clearly, because the same psi of r appears on the right hand side what it is; however, is an integrally equation for psi of r a started with a differential equation, and once I find that green function, then I have an integral equation for psi of r which is as hard to solve as a differential equation , but I am not going really going to solve it immediately I am really going to try to find the form, but I would expect that whatever I do, I cannot get a way for finding this f of error exactly I cannot get a way without finding this solution to the Schrodinger equation.

I need to find this psi some, but what equation does this green function is had is find clearly its satisfied del square plus k square g of r comma r prime equal to three dimension delta function that is easy to see, because I have to do. So, apply this del square plus k square on e the psi apply that here, and the goes to zero this the homogenous equation, and that acting on this if it is a delta function will give me precisely this. So, its immediately clear that I need to solve this green function equation with some specified boundary conditions.

The boundary condition I want is the natural one namely g tans to zero as r tans to as is all those to zero now this del square operator is translation in variant, I can shift from r to r minus r prime in the del square to respect r does not change that delta function is translation variant it is a function of r minus r alright the boundary condition is translation variant it is a function of r mi r goes to infinity same as r minus infinite r prime going to infinity.

So, both the boundary condition as well as the right hand side r translation variant, and the differential operators translation variant. So, now I can assert that this green function is also function of r minus r.

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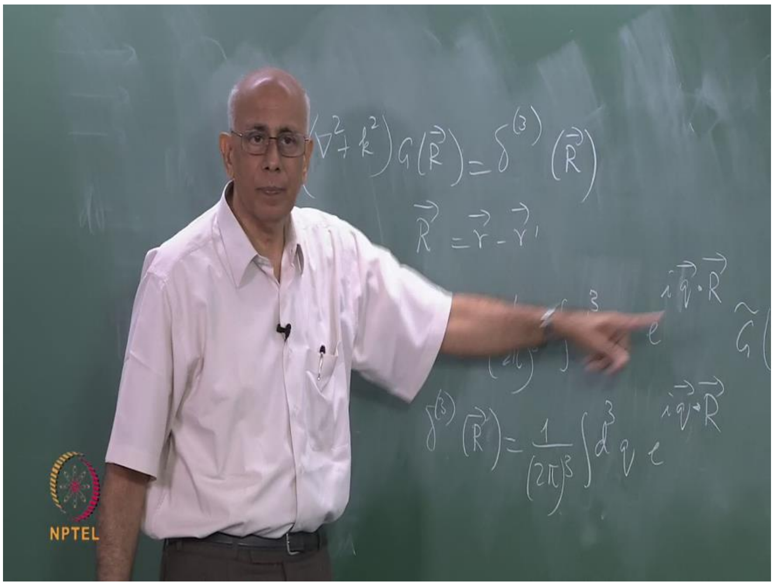

So, let us to that as we normally do write this del square plus k square g of r is equal to where r how are solve I think like this exactly as we did in the case of this del squared are know I do for a transform that converts the del square into a multiplication by this square of move wave number whatever it is right, I have already use k for symbol. So, let me write g of r equal to one over two pi whole cube integral d three term a use q instead of k for the a transform with respect capital r e to the i q dot r g tilde of q standard transform in three dimensions an off course we already know that delta three of r is one two pi cube integrate d three q e to the i q dot r, I put that n here what del squared acting on this guy this is del with respect to capital r minus q i q the whole square. So, its minus little q squared, and then i the usual game i put that in a argue to this is a complete set of function capital r. So, each for a coefficient must be zero. So, let us write down immediately.

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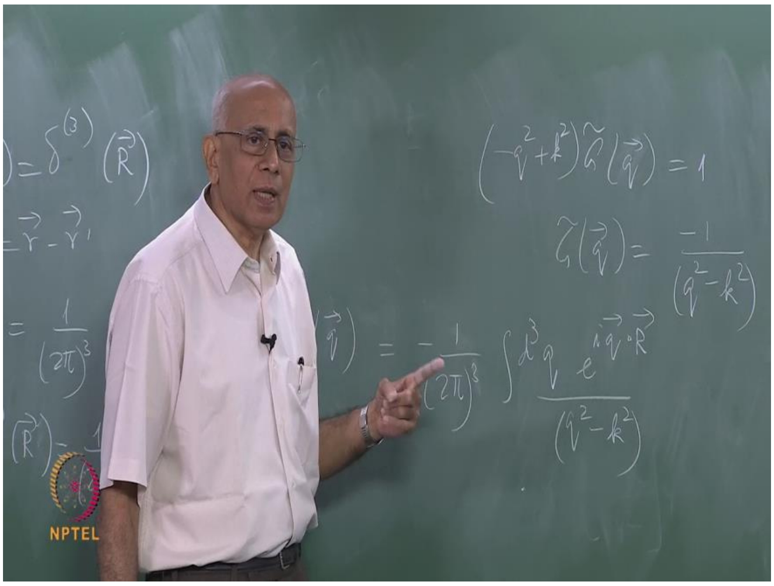

This will tell you that minus q square plus k square times g tilde q little q psi little q q equal to one, because that is the coefficient of the delta function. So, it is a its g delta q equal to one over q square minus k square with a minus sign, and therefore this quantity equal to minus one over two pi to the whole cube integral d three q e to the i q dot r over q square minus k square, and if I can invert it transform the job is done geo farms.

Notice is almost identical to what we had for the Laplace in operator k was zero. So, we just for the del square operator we just had a q squared in the denominator that q squared canceled against the q squared from co ordinates here, and then we got a very simple directly integral to do, but now we have a case nominated ok.

We can still say alright that is to part of the integration, let us choose spherical polo coordinates ion q s space, what direction should I choose the polar axis alone were there is a vector sticking out the whole thing is the function of this vector right. So, I can choose without lose a generality this is a scalar rotationally in variant, I can choose the polar axis in little q along the direction of r what advantage to again by that I can do the five integration immediately just this made to find immediately.

So, there is thing here is equal to minus one over two pi whole cube integral q square d q zero to infinity divided by q square minus k square, and there is a two pi factor which comes in from the five integrals of this becomes minus four pi square, and then in integral of e to the i q r for theta from cross theta minus one to plus one right that is equal to two i sign q r over i q r as always. So, gives me minus one over two pi square r. So, that is have first hint that you have a getting somewhere, because remember for the (( )) in operator the green function was minus one over four by capital r. So, this is one over r coming in here game, but that is not end of the story, because you also have the i goes way, and then you have a q gets cancelled here, and you have zero to infinity q that is this is d q q sign q r divided by q square minus k square you still have to do that, but this is not very good news, because as its stands is integral does not exists right wise that yeah there are poles there are poles on the way axis there are poles on the q axis at plus k, and minus k, and you cannot get. So, this is formally infinite this is formally infinite, and we are the same sort of problem that we had in a other problems earlier right what you have to do is to shift the poles or indent to contour around the poles in such a way that the boundary conditions are satisfied. So, you really knack to find the suitable green function by using a prescription, and i epsilon prescription for shifting the poles away from the real axis in such a way that you get the right boundary conditions satisfied ok.

So, this is for a general lesson for all these partial there you have differential operators the green functions are themselves not analytic functions there boundary values of analytic functions obvious the transforms a boundary values of analytic functions you will need to specify some limiting procedure always. So, let see how that works, then will see there we can find the green function of…

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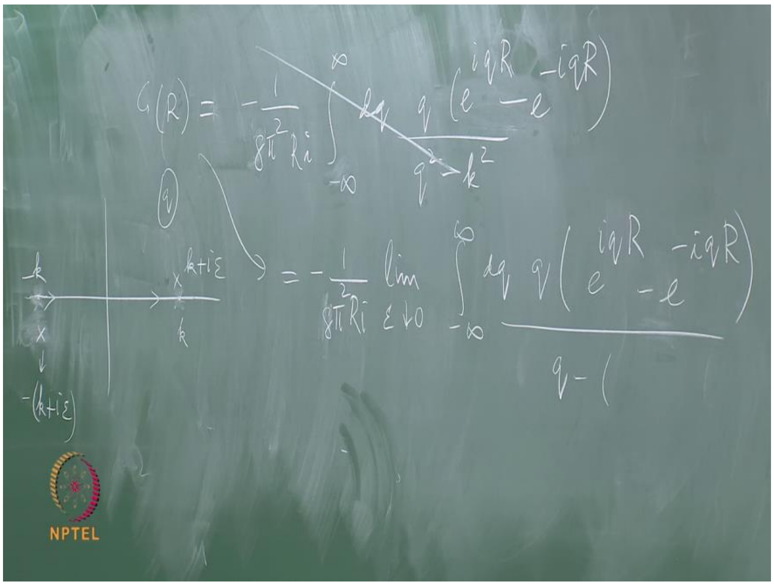

So, we have g of or equal to by the way its ample clear. Now, that I do not need to write a vector there, because it is a function of little a magnitude r alone. So, let me just call g f r its minus one over two pi square capital r integral zero to infinity d q sign q r over q square minus k square as its stands formally infinite. Now what is it I am looking for, and looking for a solution which corresponds to outgoings waves I am looking for something which goes like e to the i k r clearly there are four z q equal to plus or minus k. So, if I look at the contour in the q plain the integral runs from minus infinity two infinity, but there is a poles sitting here at k, and another pole sitting here at minus k you have to shift these poles appropriately, and then there is this term here sign q r, but let us rewrite this back again in terms of an exponential, because how to I do such integrals i do them by complete in the contour, and using contour integration, but I cannot complete the contour either in the lower or upper half plain as long as a have a sign q r, because this is got both e to the i q r, and e to the minus i q r, and one other tools going to blow up.

So, I rewrite this in terms of e to the i q r, and need not I have done this I could of retain this form minus i q r over two i. So, that is becomes a four point in this I hope a kept all the factors of two correctly otherwise we are in trouble. Now this term here, if I pick up the contribution from the comp from the polar q equal to plus k i am in good shape for this guy here for this, I am a speak up the contribution from the pallet minus k. So, that I get e to the i k r up, and top right, but for this since capital are is positive, I have to close the contour in the up a half play, and for this integral i have to close it in the low half play, because I want to add this well chose in zero in the limit this semicircle contribution which must goes to zero, and I need non zero contribution.

So, it is clear that this pole here should be shifted up there or if you like i indent the contour from below, and this pole at minus k should be shifted down here, and then I take the limit in other words is should write this not this, but rather as equal to minus one over four pi square r i limit as x epsilon goes to zero from positive sign from the positive sign integral minus infinity del infinity d q oh just one step, I should written this back again as minus infinity to infinity, and eight pi square, because I need to have a contour running all the wave from minus infinity to infinity close the contour. So, I use the fact that this sky here is an e one function of little q to write it as half the integral from minus infinity to infinity. So, this thing now becomes there is one eight pi square, and this become d q q times e to the i q r minus e to the minus i q r divided by q minus now this pole is is at this pole here is a k plus i epsilon, and this pole here is that minus of k plus i epsilon outside bracket.

In other words, as k plus i epsilon whole square, and take the limit as epsilon goes to zero. So, I do the integral, and then take the limit epsilon goes to zero. Now it should be immediately clear to you I have two poles on the real axis on the contour of a integration, and i can avoid them in four different ways I can put both in the upper half plain both in the lower half plain a this up this down for this up this down, then I am going to get four different green functions, but only one of them will satisfy the boundary conditions that I have want, and that is the one I have choose I leave it as an exercise to you to play with this, and do this for all possibilities put both up here, both down here.

I am see what happens to the other order, and see what happens you can immediately see what is going to happened intuitively, if I have put this up, and this down what would I have thought, and I got a green function which as immethodically goes like a e to the minus i k r over r that would correspond to ingoing spherical waves right, because instead of plus scare of chosen in the minus scare wrote here, and vice versa. So, this tells you how you have to have the right i epsilon prescription how this i epsilon prescription helps you to satisfy the boundary conditions, and identify the right green function now we are home, because this thing here is equal to minus one over eight pi square r i, and then lets right things down I write this as sorry q square. So, its q minus k plus i epsilon times q plus k plus i epsilon, and look at the first term this term here I am compelled to close the contour in the upper half plane that contribution goes to zero as a radius goes to infinity, and you got two pi i terms residue got plus k.

So, you have this time at two pi i this q is replace by k k plus i epsilon, but I am going to let epsilon go to zero, and then e two the i k r on that side divided by well this is a q minus k, and a q plus k, and I have to multiply by this factor, and replace q by k. So, there is a two k, and then minus that is this term the contribution from this pole by closing the contour in this direction. So, is a minus two pi, because I am doing this negative sense. So, there is a minus two pi i very careful track of all the minus signs the contribution is a from the polar q equal to minus k. So, we got to put a minus k there e to the minus i q r, but q is minus k for this is e to the i k r divided by its this factor that you got to multiply by, and replace q by minus k that gives you a minus two k here. So, there is a minus k there lot a minus signs you have to be very careful now it is clear that four of them we can a make a all of them plus signs that goes away, and the contributions are identical. So, it just equal ent saying this factor goes away, and a get read of this all together the cancels all together that two pi square cancels you have a two cancels with this for the i cancels away nicely, and you end up with a results which is minus one over four pi e to be i k mode r minus r prime that is what r was divided by minus that is it ok?

As a check you put k equal to zero you should get the green functions which was minus four pi r if this rent exists the Helmholtz operators becomes operator, and you are end up with a right green function this boundary condition. So, all that is catering that this case squared in the Helmholtz operator does is to introduce the face factor. And now we are home, because we can write down what are a solution looks like at least what the integral equation looks like.

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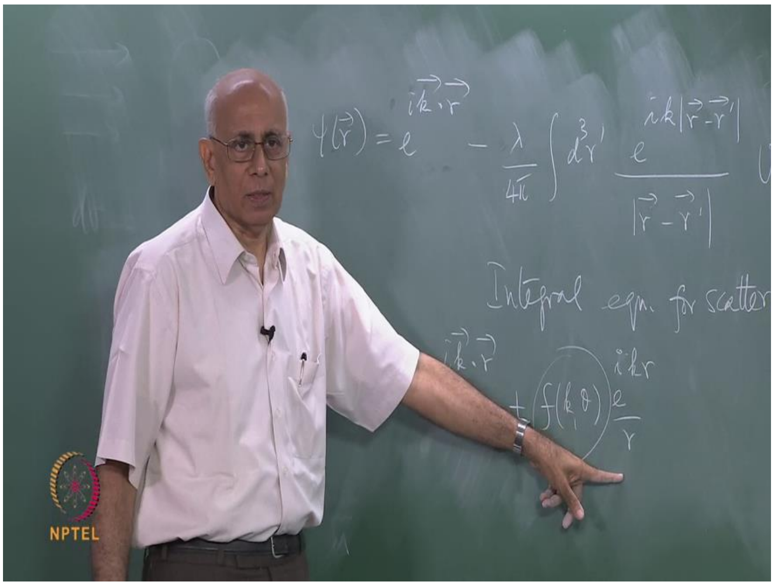

So, you have psi of r equal to e two the i k dot r minus you have to remind me of all those factors that was sitting around there is a lambda over four pi integral d three r prime ah green function. So, e to the i k mode r minus r prime divided by small r minus r prime u of r prime psi r prime that is the exact integral equation first catering it is not a solution for the wave function, because it sitting here, but it is the integral equation this portion is a indent waves, and it is this portion that is should give you the scattering amplitude we should able to identify that in the limit when little r k r becomes very, very large compared to unit will take that of tomorrow.

So, from this point on wards notice again that the entire effect of the potential residing here in this term with a lambda sitting outside is a cupping constant here this is an in homogenous integrally equation, because there is this term, and what we have to do is to deal with this equation here. So, will base of dealing method one of them would be to iterate will see what solution looks like, but the first thing we need to know is to extract behavior. So, I need to show that this in deep goes to e to the i k r k dot r plus f of k f of k comma theta e to the i k r i need to show there is, and extract this quantity this term well we are partly we can see immediately partly there is go in to happen, because if this potential dies down sufficiently rapidly, then when r is much bigger than all r primes i can take out this r outside do get this factor presumably this factor comes from here presumably, but this is a face factor. So, you cannot really neglect the little r prime compare to r you are to keep it very carefully.

We make an expansion, and extract this portion. So, will take it up from this point next time from the integral equation, and then I will say if you words about the integral equation itself do this. So, stop here today.