**Selected Topics in Mathematical Physics**

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**Module - 2**
**Lecture - 3**

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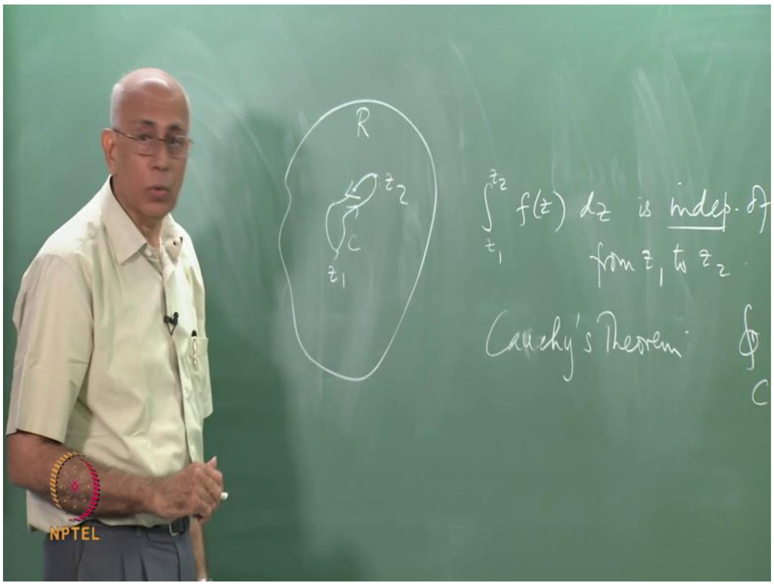


We have seen that the function is analytical some region, then the ((Refer Time: 00:15)) condition are satisfied the derivative is uniquely defined at every point the derivative itself is an analytical function, and the function has infinite all possible derivatives, and they are all analytical functions. Now what about integrating this function? We turns out that if your function is analytic in some region R, say some function f of z analytic here, then the integral of this function from a point z 1 to a point z 2 along some given line along some given contour c integral from z 1 z 2 f of z d z turns out to be independent of the actual part you choose between z 1, and z 2 as long as it is a connected path from z 1 to z 2 that never ((Refer Time: 01:10)) the region of analyticity. So, this thing here is independent of the actual path from z 1 to z 2, its independent of that. This should remind you immediately of a similar thing that happens in physical application when you are looking at the conservative vector field for instance, which can be written as a gradient of a scalar field.

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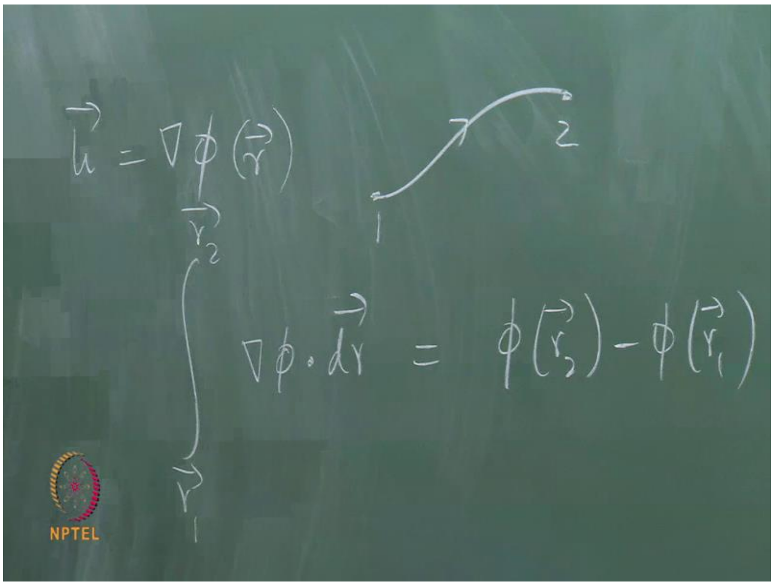


Then if you have a vector field of u which is a gradient of some phi of r say, and integrate this field from some point r 1 to r 2 like this, from point one to point two along some path, then grad five dot d r, this thing here is equal to phi at r 2 minus phi at a r 1. That is the whole point about using potential when you want to describe a conservative force for instance, then you can deal with this potential, because it is scalar as a portion of vector, and the field itself is founding taking the gradient of the scalar, but the integration along any path is independent of the actual path, and it depends only on the end points ((Refer Time: 02:39)). And immediate consequence of this is the an integral, then phi dot d r over a close path of some kind is equal to 0, and the same thing is true here, because if this thing here is a same as that integral. Then this integral plus that integral is equal to 0, and it immediately implies that the integral of an analytic function over a close contour is identically 0.

A consequence of the Cauchy's conditions, this is called Cauchy's theorem integral, and I will denote a closed oriental contour by this symbol, I have to tell you the directions in which I go always whenever I ((Refer Time: 03:32)) contour could be the direction, it does not matter f of z d z equal to 0. For any close contours equivalently from any point to on it any other point on it, the integral is independent of the path itself. So, you can actually deform this path like a rubber band, you can shift this all over the places as long as you do not cross the region of analyticity, you do not cross any singularities, and so on; this thing remains 0. And the so power full a statement as a very powerful statement, and we are going to use that exploit this all the time, this idea that you can deform or dis taught the contour of integration without changing the value of integrate, that is going to be a sort of central ((Refer Time: 04:19)) in very large number of things that I am going to say that this definition of the contour does not change the value of the integral, that is going to be exploited to that most. Now before we go on with this I want to show you how this can be exploited, we need to finely come to terms with the idea of a singularity of analytic functions. And let us do that I just reminds you, since many of you already come across simple examples of singularities. So, let us start with that them, the first and simplest class of singularity which I am not going to pay attention to all the one the really should in a regular scores is so called removable singularity.

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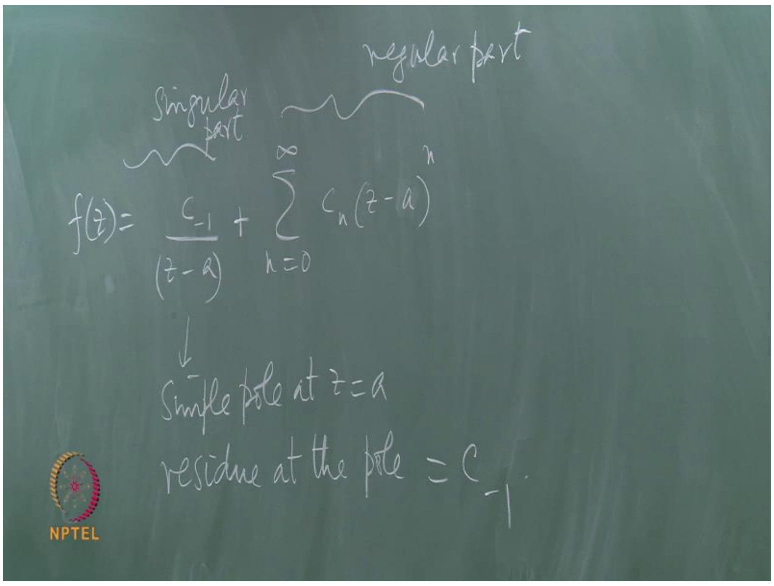


But will get over the simply strategy, and as well see these are the removable singularities for instance would be those singularities, which look like the function as singular, but if you define the function appropriately is not singular at all. For instance suppose you had sin z over z this, as z tends to 0 this thing becomes in determinate this quantity becomes in determinate. But it has a finite limit, it has a limiting value, and the limiting value as z tends to 0 is equal to 1.

So, I could actually define this function as f of z equal to sin z over z for z not equal to 0, and equal to 1 was z equal to 0, then I ensure continuity by doing that, and then this singularity that z equal to 0 does not exist at all its remove its gone. So, by this trivial example make this little more ((Refer Time: 06:09)) I am going to get rid of removable singularities of this kind, assuming that the function is defined appropriately by limiting value, so that continuity is preserved and so on so forth. So, we want to any more attention to this kind of thing, look at really actual singularities. Now analytic functions themselves, we call that non singulars once, we wrote in the form n equal to 0 infinity some coefficient c n z minus z not to the power n by the way.

Let us we can leave this z not as it is for the movement, but since I am going to use z not for the other purpose, let me use the symbol a here. In other words, I am describing the function near the points z equal to a, and the claim means if it is analytical z equal to a, then I have a power series, Taylor series convergent in sites some circle of convergents about the point a. To this nonsingular or regular part I could add portions which have specific singularities, and simplest of these is when the function looks like f of z equal to some coefficient c divided by z minus a. So, an addition to non negative powers of z minus a, there is a negative power to power one into remind myself that this is z minus a inverse minus one. Let us call this co efficient c minus one plus ((Refer Time: 07:46)).

So, if in the neighbour hood of z equal to a some point a of function has this kind of behaviour namely most of it is Taylor series, which is analytic at z equal to a, but there is a portion which diverges becomes infinite, and z equal to a, and it has a behaviour which goes like one over z minus a. Then this function is set to have a simple pole at z equal to a, and the residue at this pole equal to the co efficient c minus 1, and we will see little later why this residue so crucial. In fact, the whole of the complex integration is sometimes called the calculation of the residues, because we are going to be evaluating the residue the various functions. And this part is a singular part, and this part is a regular part. Now this singularity may not be overt may not be always obvious, may have to work little bit in order to extract this residue.

For instance, suppose we have the this function sin z over z square, what sort of singularity does it have near z equal to 0, it is clear this function is blowing up z equal to

0, because it is in extra power of z in the denominator, but what kind of singularity does it have. What we should do? We should take the sin z which we know as entire function, and expanded in a power series in z, and then divide by this one over the z square, and see what happens? This of course, is equal to z minus z cube over three factorial plus dot dot dot divided by z square. So, gives your one over z minus z square over three factorial plus regular dot, this function. And it precisely of this form, these are non negative powers in the form the regular part happens to converge for all finite z in this case. But this portion is singular at z equal to 0, and what is a residue? It is the co efficient of one over z minus a, that is the residue at the point, and what is the residue on this skills? Is just one uniquely in this case.

So, in more general circumstances, if we had f of z equal to say g of z over h of z ratio of two smooth functions say, and a little function say, let say at g of a is not equal to 0, and h of z has a simple 0 at that point. So, this means that h of z is a form h of a which 0 plus z minus a h prime of a plus higher orders, and g of z is the analytical source at g of a plus other terms of conditions g of a 0, and that say. Then what can we say about f of z at z equal to a, what sort of singularity ((Refer Time: 11:40)). Well in the neighbour hood of z equal to a, this goes like g of a plus z minus a g prime of a plus dot dot dot. And this kind of here in the denominator goes like z minus a h prime of a plus higher order terms. So, it is clear that limit us he go to a, this term dominate this comes out, and it has a co efficient, which is g of a over h prime of a.

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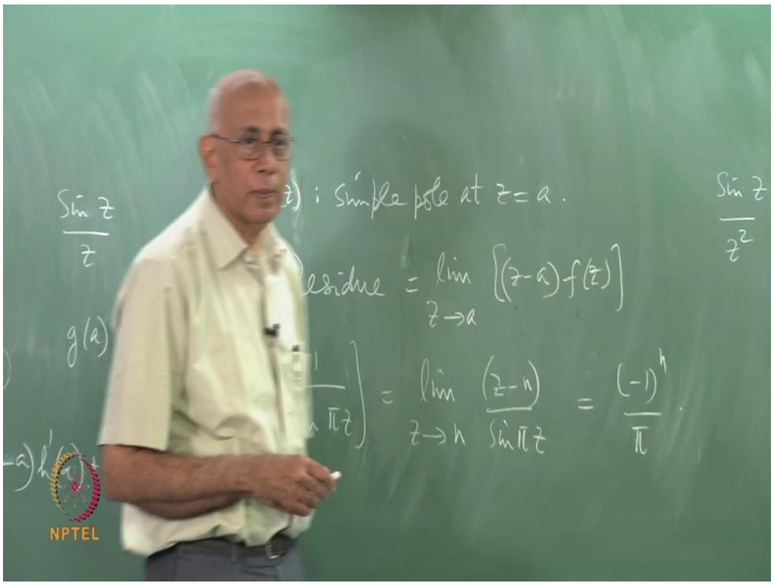


So, what sort of singularity does it have simple pole, but what is the residue g of a over ((Refer Time: 12:26)), that is the very useful think to reminder. So, one could ask what is a general formula for this function, suppose I say f of z has a pole, simple pole at z equal to a, residue what is the residue at this point? How do do this? Well remember it is of the form c minus one over z minus say plus regular part. Now extract that c minus 1, the obvious think to do multiplied by c minus a, and then let z go to a. So, it is immediate obvious the residue is equal to their limit as z tends to a z minus a times efforts.

Because this fellow will have c minus 1 over the z minus a which will cancel this, and the next term will be a regular part, but if put z equal to a after multiplying by z minus a vanishes. So, that is the quick way extracting the residue. All you have to do these to do this, where does this function one over sin by z where it is does it has singularities sin pi z itself is an entire function we know this. And it vanishes sin pi z vanishes whenever z equal to any integer 0 plus minus 1 plus minus 2 and so on. So, where this function has singularities, this is cosine by z, where does it have singularity at all integers at every integer its singular blouse up, what sort of singularity is it? And only one ((Refer Time: 14:27)) simple pole. So, the singularity is a simple pole, but everyone of these point. Because sin pi z as a simple 0 at that point, simple zero and z equal to a means, it is half the form z minus a, and not z minus a squared or cubed or anything which are higher orders zero.

So, this function has simple pole that is all the integer values, and what is the residue? When the certainly has a singularity at z equal to 0 right that immediately obvious. What is the sin phi z do as z tends to 0 tends to zero, but always you have to tell me, not to limiting value, but what is the leading behavior? What is the leading behaviour of sin pi z as z goes to 0 pi z, pi z itself.

Now I have one over sin pi z, so what is the leading behavior, one over pi z right, what is the residue? 1 over pi. What is the residue of the other, other points what is the residue of this at z equal to n, you have to do this, you have to do this. So you have to find the limit z tends to n.

Student: ((Refer Time: 16:05))

((Refer Time: 16:07)) So, z minus n over sign by z. Find the limit of this, reason I am doing this, because these studies this ((Refer Time: 16:29)) rules for finding the limits of ratio which became indeterminate in high school, and after the got through entrance exam they ((Refer Time: 16:37)). What this equal to differentiate this, differentiate this, and said z equal to n in what happens? If differentiate this, we get one, you differentiate this pi cos pi z, and what cos pi n minus 1 to the power n. So, this thing is equal to minus 1 to the power n.

So, we are guarantee the cosec pi z in the neighbour hood or any z equal to n any integer n goes like minus 1 to the power pi plus a part plus regular, and z equal to n. The part that regular and z equal to two for instance will lead to a singularity at z equal to 3 at a contribute to the singularity elsewhere. So, you have to be careful at each pi the residue ((Refer Time: 17:48)) quite different as it is in this case. So, we will use the idea of residues finding residues of functions co extensively, but it should be very quick at finding this limits these points, but this is a very simple ((Refer Time: 18:06)) residue at the point.

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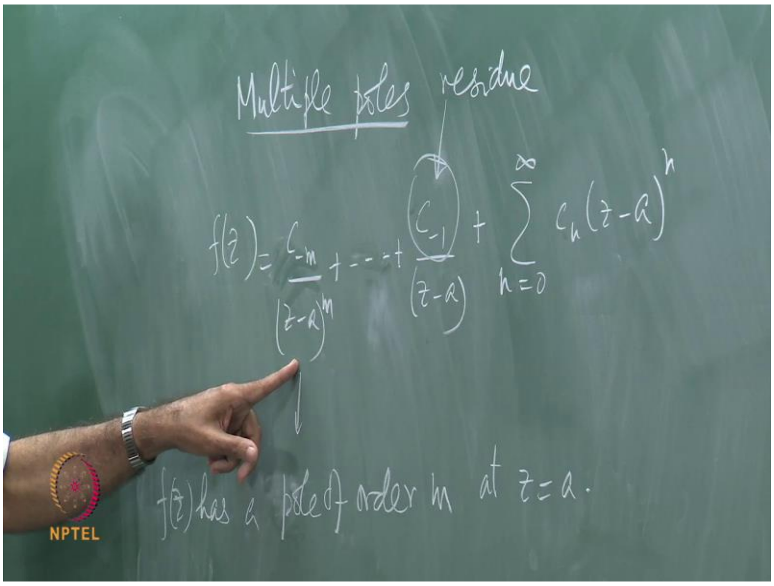


But now could ask we talked about simple poles. What happens if this function f of z has what a call multiple poles multiple of higher order poles, multiple poles. So, f of z is a form thus a regular part, n equal to 0 to infinite c n z minus a to the power of n which may or may not terminate at some point. We do not care, but its convergent certainly in some region some circle about z equal to a plus a portion, which is c minus one over z minus a plus, let say there are however, higher order terms here more negative parts, and its starts at some c minus m over z minus a to the power m plus dot dot dot, where n is a integer bigger than one h m, then this function is set to have the pole of order n, and z equal to a.

So, if it turns out that in the neighborhood such a point, the point a this function has a lot of negative powers of z minus a terminating at highest power some nth power z minus a to the power minus n, and then less singular terms and then a regular part here, then this function has an nth order pole at z equal to a. And for a region ((Refer Time: 20:00)) to become clear, here interested always in the residue, but remember in the residue was defined as the coefficient z minus a to the power minus 1, this term is important part this thing here, and be need to find this term. Now the question is how we gone to extract this term. Earlier we extracted the quite simply, because this thing had c minus 1 over the z minus a plus a regular part all I did goes to multiplied by z minus a, and then said keep limits z goes to a got c minus 1.

You cannot do that now because these things will block. So, how do I 0 in on this coefficient c minus 1, this is the residue. It is still the residue at the pole. How do a find that? ((Refer Time: 20:54)) what limiting procedure would a suggest do this? It is clear, I am multiplied z minus a to some power, and then I am said z equal to a, but any power less then n is going to ((Refer Time: 21:09)) trouble.

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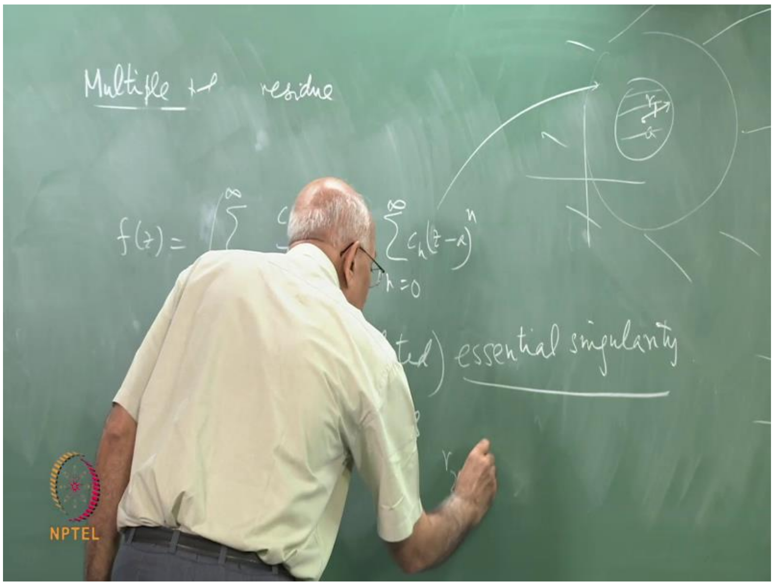


So, I multiplied by z minus a to the power of m f of z into a, and this series this regular part converges inside here, let call this radius r 1, the regular part converges inside, this is circle of convergences of radius r 1. Since this is an infinite series with negative powers, you should also ask what is the radius of convergences of this thing, otherwise it does not make sense it should converge right. Now this radius of convergences is specified region outside which the series convergences, because if put one over z minus a equal to w for instances, some other complex variable. Then it's a power series in w, and this see in positive non negative power of w, and that converge for mod w less than some number r 2 or if you like mod z minus a greater than these one over then one over r 2.

So, it is clear that this series actually converges outside some neighborhood or z is equal to a. So, perhaps it converges ((Refer Time: 22:28)) outside. So, we are said now you cannot write representation like this, because this portion requires you to be inside here, but this portion to make sense requires you to be outside there. The only way that we make sense out of such a representation is if this regular series converges inside some r 1, and other series converges outside some r 2, and everywhere else it converges outside. When in this overlap region between the two which is an angular region both series converge, and this series representation for this function make sense.

So, typically series of this kind they call by the way Laurent series, and they would typically converge in some angular region all these. So, you require that r 2 less than mod z minus a less than r 1, typically in region of this kind, it could be that r 2 ((Refer Time: 23:54)) to a point 0 alone, and could be that r 1 extend all the way to infinity, that is possible. Here is an example consider this function.

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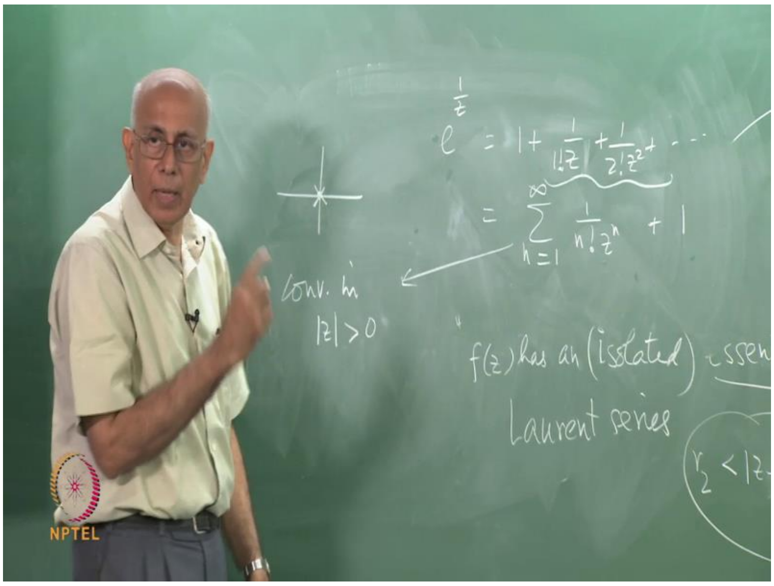


Consider the function e to the power of one over z, this function is of is of the form one plus one over z times one factorial plus one over two factorial z square plus etcetera. So, it can be written in the form summation n equal to one to infinity one over n factorial z to the power n, that is the singular part here plus one in this case, that is the regular part, and z is equal to zero. So, all negative parts negative integer parts of z appearance in series, and the series converges as long as well e to the power of z converges as long as mod z is finite, which means mod z is not 0 that is it. So, this series converges in mod z greater than 0, outside the origin, where does this series converge? It is a constant it converges everywhere including the point at infinity. So, this whole thing converges from mod z greater than 0. So, this representation of this function is valid in the region ((Refer Time: 25:49)) essential singularity at the origin, and isolated essential singularity as z is equal to 0, and this Laurent series converges everywhere outside the origin including the point at infinity. So, in this ((Refer Time: 26:09)), plane it converges everywhere. What is the residue of this function at z is equal to 0, it is the co efficient of one over z always what is residue? It just one.

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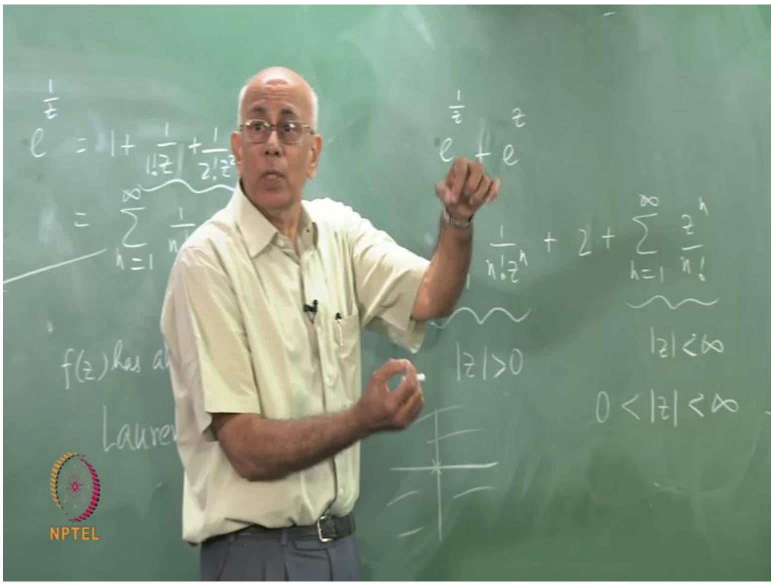


Suppose I added to this, suppose I added e to the power one over z plus e to the z. Now, what is the Laurent series look like. Well as before I would write this as summation n is equal one infinity one over n factorial z power to n plus n is equal 0 term here, n is equal to 0 term is one plus one plus summation n is equal to 0 to infinite, if you write it as one to infinity z to the power of n by n factorial, and then that is two. Now what does this converge for, where does this converge mod z is greater than zero. So, this requires the mod z is greater than zero. So, in complex plane it say everywhere except the origin at converges, and where does this converge or finite mod z. So, mod z is less than infinity mod z less than infinite. So, where does the whole thing converge? In an angular region of although you do not see the angular, angular region 0 less than mod z less than infinity.

So, on Riemann surface on the Riemann sphere, it converges everywhere except the south pole, and the north pole; except those two points south pole remember is the origin on the complex plane, and the north pole is point of projection is the point of infinitely. So, on that sphere becomes is easier to visualize, it converges except for the two pole, except those two point it converges infinitely this function. Now what about this function? What about e to 1 over z square, what sort of singularities does it have and where, at the origin yeah, it is in the origin what sort of singularities is it? It is a pole of finite order or it is an essential singularities, it is an essential singularity. What is the residue at origin?

Student: zero.

Zero, the residue of origin is zero in this case. There is no one over z 1 over z.

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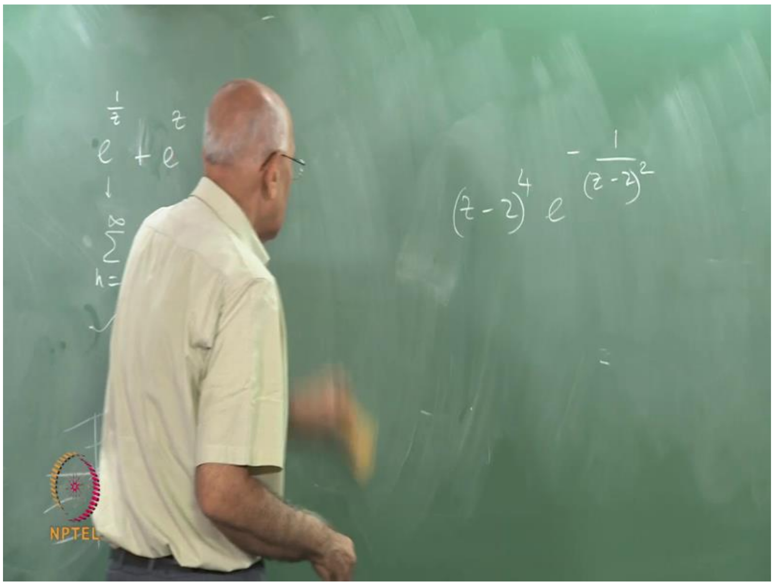


What about this? What sort of similarity does it have? Essential singularities exactly equal to 2, and the residue is 0, there is no co efficient one over z minus 2. So, the residue is 0, what about this?

It is got an essential singularity at z equal to 2, because those negative powers can be finished by this single power of z minus two, and what is the residue? Minus one minus one, because you got leading term here one minus one over z minus two whole square, and that gives you minus one over z minus two. What sort of singularities does that have essential, it is residue 0, there is no one over z minus two term possible in this case, and so on. So, you can compute residues in varies cases, let us come to why this residue so important? Why I am making such big deal about this residue, the reason is following the reason is ((Refer Time: 31:00)) in a property of integral which is very, very simple to state, but extremely powerful. Let me state this, I think here.

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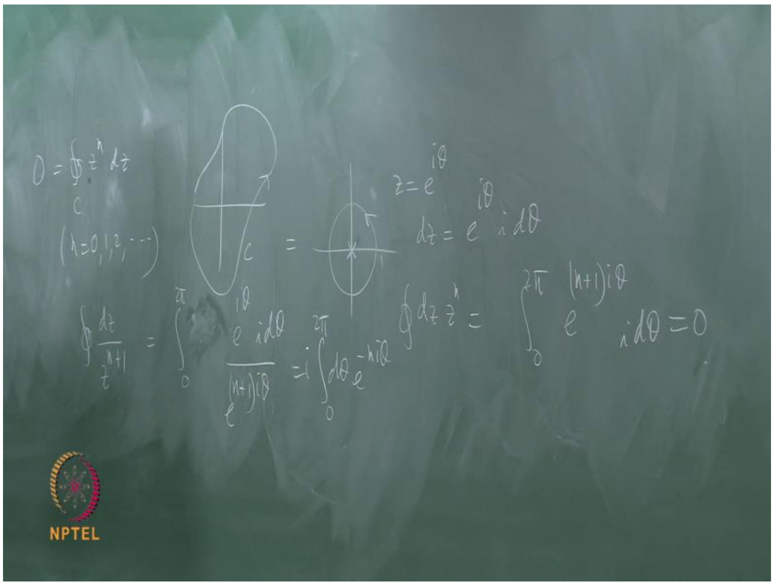

So, consider this integral, integral z to the power of n is d z over a circle which inside over a close contour which encircles the origin. By the way this function z to the power n is an entire function, it is for positive integer value of n, non negative integer value is n, it is an analytic function everywhere except at z is equal to infinity. So, it is clear that if you encircle the origin once in this passion in a close contour c, this is equal to 0 for n is equal to 0, 1, 2, etcetera, it is clearly equal to zero.

Now we can prove this in many, many ways; one of them is to say well as long as you are in region of analyticity you can distort this contour, so you can actually sink to a point, and then it disappears altogether or you can if you like say that it is a same as an integral over the unit circle, this thing here is equal to an integral over the unit circle, and on the unit circle on this circle z is equal to e to the power i theta, because r is one on this circle, and then d z is equal to e to the i theta i d , and this integral d z to the power of n is equal to integral 0 to 2 pi e to the n plus 1 i theta, and is equal to 0, because you cosine and sin from 0 to 0 to pi, and everything vanishes.

So, for all non negative integer value is n, this integral is 0, What happens, if we have negative powers. So, consider d z divided by z to the n plus one, where n is 0, 1, 2, 3, etcetera. What happens to this integral? I play the same trick, but here now this is a pole at z is equal to 0, what is the order of pole? n plus one order the pole the order n plus one whatever with this n. And now write that integral down which is perfectly all right, because singularities are at this point, and the is not passing through this singularities. So, it perfectly well defined in that region. And then what happens? you get equal integral 0 to2 pi d theta, and then there is in e to the i theta i d theta the z divided by e to the n plus 1 i theta, and theta which is equal to integral 0 to 2 pi d theta e to the minus n i, and then i. What is the value of that integral, but the input n equal to 1, 2, 3, etcetera is certainly. What happens if you put any n equal to 0 then its 2 pi then its two pi clearly. So, this integral is also 0 except one n equal to 0, and then the answer is 2 pi i.

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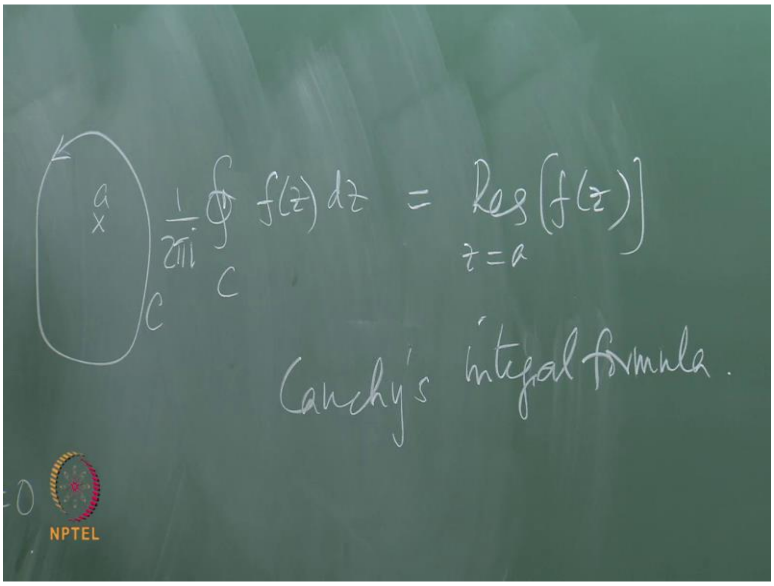


So, it follows that this integral 1 over two pi i integral over a closed contour encycling the origin once in the positive sense, that is why you run from 0 to two pi not the other way above d z over z to the power n plus 1, whenever n the 0 the answer is one, and when is n any other integer positive or negative the answer is 0, this is equal to ((Refer Time: 35:30)) delta of n is 0 is a curial result. So, vanishes except when it is one over z in this form. And of course, you can immediately generalize this, and say this contour we look at point a, and look at close contour which I am circles in the singularities at a alone, and you could replace this z minus a to the power n plus 1, and this contours c include the c. And then for except for terms which goes like except is in goes like one over z minus a, the answer is zero.

So, now it immediately follows that you have a function with poles, and z is equal to a some poles, and you integrate f of z d z around contour, simple contour which encloses the pole once in the counter clockwise sense, what will be equal to because it is called of poles or may be a essential singularities that point, you expand this in Laurent series of about the points are equal to a. And then as long as you remains in the this contours, remains in the region, the regular part this till regular, it is immediately clear that all the terms are going to be 0 in then singular part, and definitely in the positive regular part, there already 0, because of Cauchy's integral theorem. The only term that will contribute on the negative side is the one that is coefficient one over the z minus a. And what's the coefficient of one over z minus a, the residue. So, this is equal to one over two pi i equal to z the residue equal to a. So, this is ((Refer Time: 37:44)) integral formula, that is going to help us evaluate the large number of integral.

So, what it is does, it is reduce the evaluation of line integral, contour integral to algebraic operational, just finding the residue of the integrant at the point. We will see how powerful this statement is, you have do is do evaluate this, Suppose this contour ((Refer Time: 38:22)) around twice, what would happened the answer would twice. If it went around twice in the negative sense what would happen minus two times this. So, it would has the winding number of r around this point, there r is the positive or negative depending on the sense in the ((Refer Time: 38:42)) then the answer is minus two pi times the residue at the point.

What this equal to e to the power 1 over z integral around the origin d z, So, this is an essential singularity here, and this contour looks like this. What this equal to this is called back singularities at z equal to 0, the essential singularity, we do not care. All that your interested in is a residue at that point. What is the residue at that point? One. So, what this answer equal to 2 pi r, the guarantee it that. What this equal to 0, the guarantee it that ((Refer Time: 39:36)) is equal to 0. We look at the large number of many more complicate that example, but this is a very, very simple way of ((Refer Time: 39:49)), always do is find the residue at that point. And they going to the large number tricks for changing this contour to other contour and so on, all ((Refer Time: 39:57)) that it is deform of this function.