Selected Topics in Mathematical Physics
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Module - 11
Lecture - 29
Green Function for (Del Squared plus K Squared): Nonrelativistic Scattering (Part I)

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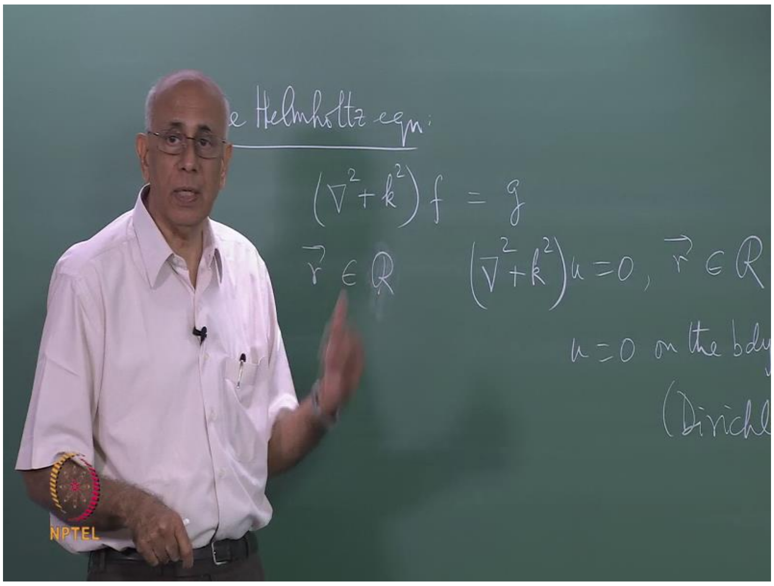

Today let us begin the study of next in this sequence of partial differential equations of physics, this is the Helmholtz equation, and the equation itself. It is really simple which is at the form del square plus k square the in homogeneous Helmholtz equation is something like del squared plus k squared on a equal to some given function of g the t you will recognize the homogeneous version of this as describing the normal no modes of a regions. So, if for instance your position rector r is an element of some region r when this equation del square plus k square on you equal to zero, in this region for r in this region is the homogeneous Helmholtz equation, and it describes the normal modes of vibration. If you like of this region are provided you solve satisfy certain boundary conditions suitable boundary conditions have to be specified the most common one would be this is for element of this region, and u equal to zero on the boundaries of r this accord directly boundary conditions for instance. If you have a two dimensional membrane like a drum head and then you clamp the drum head at the edges of the drum that cornpone to saying the displacement you is zero.

At the ends of this boundary at the boundary of this region, and then you ask what are the normal modes of the vibration memory what are the possible values of the quantity case squared. So, it is like an eigen value equation as you can see all your doing is solving del square due equal to minus k square u. So, this is an eigen value equation, and they allowed values of k square would correspond to the normal modes of this system whatever be there dimensionality of this region. Now of course, your familiar with this problem in the problem of the vibrating string is standing base on a string clamped at both ends is precisely of this kind or the normal modes of a drum head, which is clamped at the edges is of this kind and so on, that is one class of problem my immediate purpose here not to look at that is a well studied as a problem, but rather to try to find the particular integral for this equation here. The inhomogeneous equation more specifically I would like to find the green function for dell square plus k square the fundamental green function del square plus k square satisfying the boundary condition that the green function vanishes at special infinity.

So, this is in the same spirit as the green function be already found for Laplace operator the diffusion equation and so on. So, that is our problem, and to sort of give it a physical setting let us do this in the context of in application which is the quantum mechanical theory of scattering the simplest scattering theory in quantum mechanics non direct eristic particle scattering of a statistic potential. So, let us change get get into this little bit describe this problem of bit, and I will explain with were the green function comes in...

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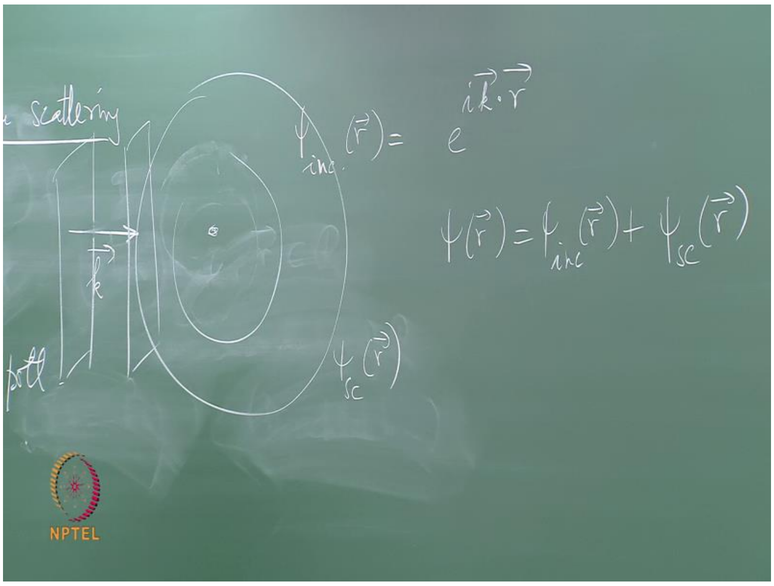

So, we going to look at non relativistic scattering it is a scattering of a quantum mechanical particle from some fix static potential, and non relativistic, because we are going to avoid the complications that arrive when the particle moves relativistic speed speeds comparable to this speed of light. So, we will use non relativistic approximation it is static potential does not the change the time, and the problem we are looking at is time independent in the sense that we have a you would like to know what happens when i show it a study stream of particles at such a potential, and ask what happens to the scattering in various angles. So, we just to put this in context if you have a scattering center here, and we will assume that this a particle has a mass m, and it sees or moves under the influence of a central potential about some origin which I denote by v f r.

So, there is a potential v f r, and this is a central potential in other words it is not a function of the angle or coordinate, but only of the regal distance from this center of attraction repulsion or a case may be, and then the question is when I shoot a particle of given energy at it what happens to it now classically of course, a familiar with this kind of problem for in sense if you shoot a beam of particles call emoted bam of particles with given initial energy kinetic energy, then these particle would in general come in scatter of this potential, and you can compute from this potential by using classical mechanic you can compute how many of particles from the incident been would go in a particular direction etc you can find everything you know about it, and your familiar already with this famous problem the Kepler problem, where is this potential is a one over r potential than the scattered particle moves in an elliptic orbit these are the open orbits for kepler problem the one over r potential, and the formula that gives you the scattering cross section in different direction is called the rather fourth formula formula this is the famous problem of a scattering of a alpha particles in the nuclear in the classical approximation quantum mechanically this is a slightly different problem, because as you know quantum mechanics particles do not have definite rejectories, because of the uncertain principles.

So, what you have to do is to specify some initial wave function or initial state of the particle, and then ask what happens to this after the scattering occurs. So, we going to look at the stationary states of the problem another words energy Eigen values in which I prescribe the value of the energy. So, we will look at in contrast to the problem of bounce. State for incense if you have positive charge positively charge nuclear, and then you have the hydrogen atom problem this gives you a set of discrete energy levels corresponding to the bounce states of the hydrogen atom, but you could also ask what happens? If I scatter in electron of a proton using this coulomb potential this thing is again as you know describable by a wave function, but in this case wave function it is not normalizable, it is a plane wave state to start with, and it goes off as in as another plane wave state. So, we have a picture were you have instead of this we have a scattering center spherically symmetric potential, and the incidence state has to be specified for me that would correspond classically to a set of a beam of particles coming in with the fixed momentum linier momentum constant linier momentum now what would that correspond to in the quantum mechanical language they would be plane wave states.

Now, just fix things let us call this the direction of the incident wave vector or momentum. Let us call this k this vector a later on use polo coordinates were we choose this as the polar access the direction of the coordinate, and then about this I will use polo coordinates. Now the incidence state consist of a set of plane waves these are planes constant face surfaces are plane waves, and therefore, let me write psi incident of r equal to a plane wave state with a fix value of the wave vector. So, it is equal to either the power i k dot r time some normalization constant here, but I cannot make this square integrable function, because this thing here diverges, but in some plane waves state multiply by some constant, and let me put that constant equal to unity, because as you'll seethe shortly this constant will get cancelled out now that plane wave comes in, and then it is scatters of this potential, and goes of in or direction, because diffraction occurs you like, and scattering occurs. So, what will go out are spherical waves these are fears the surfaces of constant waves are fears of from this scattering center, and that would be here size scattered of r, and by the super position principle the wave function at any particular point would be a super position of the incident wave, and the scattered state. So, this I total total wave functions the i of r equal to psi incident plus psi scatter we going to try to compute what this size scattered of r s ok.

Now, of course, you need to know what it forms going to be we have already assume that very far a wave from the scattering center in the classical case what I have to do is to put detectors everywhere I put a little detectors here in at all angles, and calculate what is the faction of the flex of the incident particles that falls into the detector at various solid angles that gives me information about how this scattering potential acts on these particles in the same way I would put detectors at various points, and ask for unit incident flux what fraction of it gets scattered into a particular direction here some given direction some solid angle in spherical polo coordinates, and ask what is the scattered flux in to a little cone of solid angle the omega this is the question, and ask, and the assumption is that you going to be able to put detectors wherever you like, and discover what is a scatter flux. So, this direction this is the problem yeah, scattered waves of spherical without really considering any of what the potential gets already.

I assumed, yes I assumed that this is just spherically symmetrical potential this is crucial when I look as in synthetically, and ask what happens to particle think of it classically if it is elastic scattering, then what is going happen is that the initial momentum is going to change that direction, but not magnitude.

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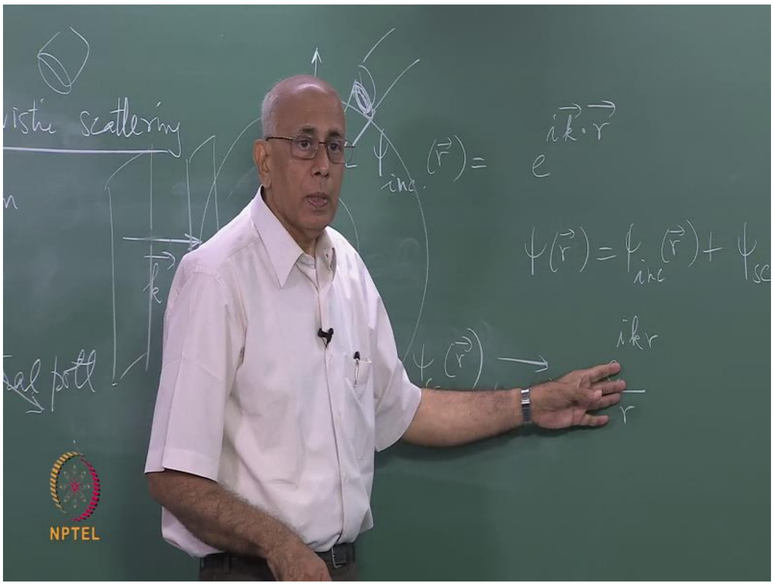

So, let us say that the initial or incident energy e is h cross square k square over two m h cross k is. In fact, the initial momentum, and h cross square k square over to m is just p square over to m that is the incident energy now it is elastic scattering, then all that happens is that an incident particle is with this momentum k, gets scattered in somehow bit a direction theta five into polo coordinates say, and this is k prime such that the energy in the final state is a same of the energy in the initial stage. So, this is also equal to h cross square k prime square over two m is elastic scattering, and this is all way going to consider just elastic scattering from a spherically symmetrical potential, then it is clear that infinitely far away there are going to be outgoing spherical waves we're not going to answer the question of what is happening very close to the scattering or region here that requires in a exacts solution of the problem.

We going to have tried the equation down, and solve it there is no getting wave from it that will tell you the full wave function at all points, but the assumption is very far away from the scattering center, and we are going to look at only those potentials which die down sufficiently rapidly that this assumption is valid, then the wave front scattered wave has the form of spherical waves. Now what would it look like as in synthetically this will go for long distances a very large distances are telling to infinity if you like. So, let me put that in bracket we got to make it precise it should have the form of spherical waves which are out going waves. So, that you can gets into the detector write now what does it look like since we're working in three dimensions these outgoing waves must look like e to the i k r over r why is this.

So, because now when I take as fear, and ask at any particular point here what is the direction this the radius vector, and what is the direction of the normal to this surface it is like this we looking at the normal radial flux. So, you get in to the plus i k r over r, because k, and r point in same direction right that is going to be the one over r dependence comes, because the amplitude has to die down one like one over r in three dimensions, because a total amount of matter is conserve right, but this cannot be all it could be modulated by some fraction, because I certainly do not expect the amplitude in this direction to be the same as in this direction or this or this or this, this is now going to be direction dependant the r dependence is finish here as in, but it is got to have a modulating factor which is direction dependant. So, there would in general is some f is three parameters one parameter is it would depend on the energy itself after all even classically this is true that if I shoot a very energetic part it will it'll scatter less i expected to scatter less, then slow part it will right. So, therefore, this would be a function of energy which I can call k instead of the energy let me just call it k equivalent completely equivalent.

It would also be a function of theta, and five namely where are you with respect to this, but if this potential is spherical symmetrical I do not expect in a five dependence, because that dependence is this is completely spherical symmetrical this spherical symmetry is broken the incident direction, because you specified, and single doubt of particular direction namely the direction of the incident been, but i still expect axial symmetry about that direction. So, I would still expect this scattering here is a same as scattering in all these, and all these angles, but of course, it would depend on the polar angles remember this is the polar angles access. So, I do not expect any dependence here, but i for a central potential, but expect a dependence in particular if theta zero there is scattering in this direction completely if this for a herds fear, and you had classical particles that of course, there would be nothing here it'll been the shadow of this particle, but if these have waves you would expect a defection to occur, and therefore, you would expect in intensity some non zero intensity yeah, should not to be comparable to wave length of the...

We have taken point particles we have taken point particles, and this is a static potential it is termination everywhere in space the only week assumption being that sufficiently far large values of our this potential should die down.

My only concern is the happens only when the wave length is comparable to the

The fraction always happens.

Yeah.

It just that the pattern will change depending on what these happened fraction will always happened fraction is another name for scattering the precisely, what we are going to do? We are going to now ask what is going to happen due to the potential acting every were is acting every were in space.

So, even if this particle is if you like moving off like this, this still an effect of this potential right. So, it is not that this is a herd’s fear, and it has to make physical contact in order to it is going to scatter no matter what we will look at this I think what I am saying is that there will be some amplitude given in the forward direction just as there will be some amplitude in the backward direction also. So, theta equal to zero correspond to the forward direction, and theta equal to pi is the backward direction, and that is it it has the polar angle runs from to zero to pi. So, I hope you agree that is fare to assume that there is no pi dependence at all, because of fractional symmetry for as spherical symmetrical potential, but there is definitely going to be dependence, take a super position of actually to the proper.

Now depending upon how you prepare the system, if I prepare this with a velocity selected. And I say I prepare it call him about in such a way that there is essentially one in it is incident momentum that is fine, but if I prepare it with the spread in momentum initial, then I have to do this problem for each incident state, and then super the whole thing certain, I have to do that there is one more complication which I met as well mentioned here which will see immediately what is happening the incident plane wave is a momentum eigen state, we have assumed it to be that, but momentum, and angular momentum do not commute with each other in quantum mechanics, because angular momentum is r cross p, and r, and p do not commute with each other the same component do not. So, you have a little mismatch here you have initially moment eigen state, but you have a potential which is spherical symmetric, and therefore, conserves angular momentum the trick, then is to say the way to do this to take this momentum eigen state and...

So, break it up in to a super position of angular momentum Eigen state, and then look at the scattering of each of those separately, and that is called analysis we want probably do that now right now, but we will see what happens we will see now what this f of theta going to do, and all the information is be buried about the scattering is going to be buried in this f. So, you'll permit me to say that this goes for large values of r, and now let us see when be precise about what pi large are, because you I just cannot say r ten in to infinity i have say r large compare to art, but it is the natural length scale in this problem we are far something which are not yet specified. So, they may the some length scalene that problem like it my die down exponentially characteristics length scale, but there is already a length scale in this problem, and what is that k the incident wave vector gives me number wave number gives me a length scale, so k or much much greater than one.

So, that is the your condition here r is much, much greater than k inverse for this the wave function will look like a to the i k d o r r that is the incident wave nothing happens to it plus f of k come at theta k to the i k r over r. So, that is my boundary condition am going to look for those solution of the Schrodinger equation which as in behave like that as the super position of the incident state, and the scattered state in the scattered state correspondence to outgoing spherical way's with an amplitude which look like which is function of the energy as well as the polar angle of scattering let us call the angle of scattering, and I want this as a function of theta. So, this scattering in all directions, and these quantities here not surprisingly is call the scattering amplitude.

And we will see immediately that it has a direct physical consequent interpretation, and is in factor measurable quantity this is not a emphasis again this is not the exact solution of the Schrodinger equation which we have in to written down yet, but I looking for that solution which as in behaves in this fashion that is my boundary condition in this scattering problem

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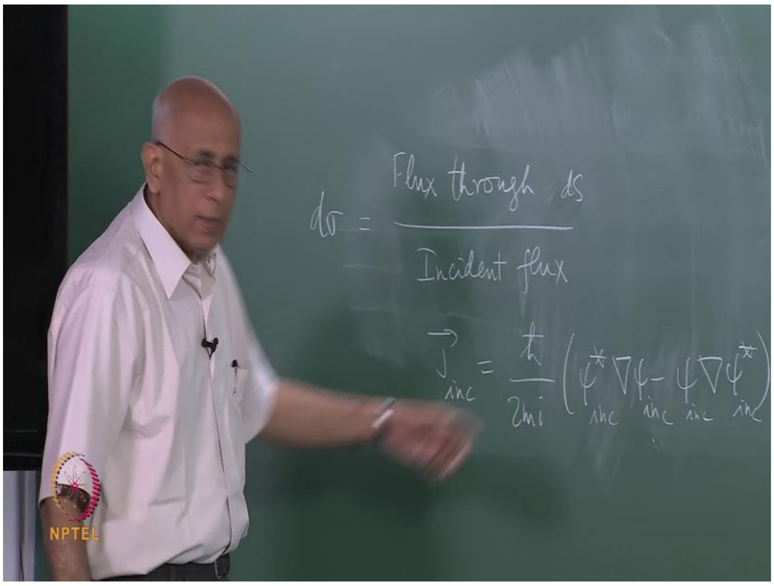

Now, let see how we are going to get at this a quantity first of all let us look at the significance of this f what? I had like to do is what is the flux; that means, number of particles crossing per unit area per unit time what is the flux scatter flux through through the surface through d s this surface element d s divided by incidence flux, this will tell us what fraction of a incidence flux is going through a at a particular angles solid angles some given small infinity solid cone of solid angle omega at some particular point theta come off five, this is equal to what is this equal to what going to define it as it is call the differential cross section how much of it is going through that particular cone. So, this is equal to d sigma by definition, and d sigma divided by not d s, but d mega is going to be call the differential cross section, but this is the quantity we want to compute. And now I want to show you that is directly related to the scattering amplitude f as follows first let us find the incident Flux.

Now the flux is given by the current j incident, and as you know when you have the extraordinary equation the current itself as an extremely simple formula for the current which is h cross over two mi psi minus psi star, and in this case we looking at the incident. So, incident that is a standard formula mechanics is everyone familiar with this yes it is easy to derive, because all you have to do is to write the Schrodinger equation, and derive from a at a equation of continuity.

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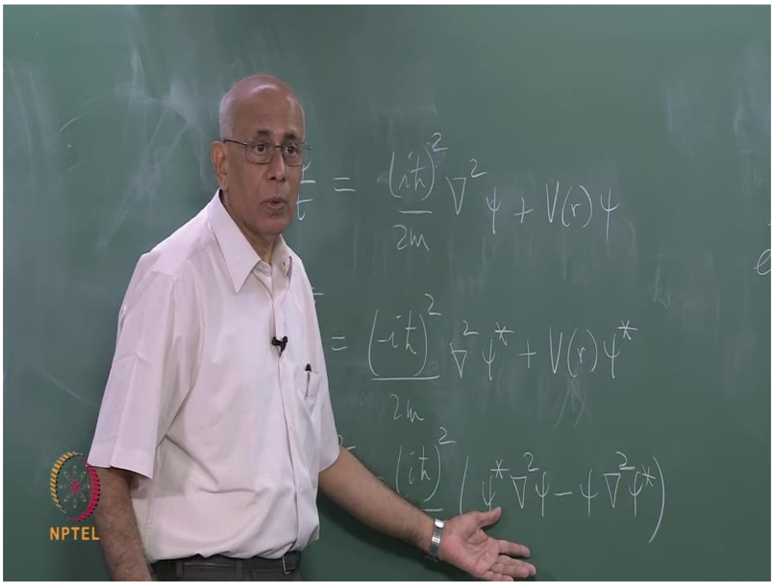

So, let us write Schrodinger equation down which is kinetic energy to start with i h cross theta psi over delta t equal to this is, now the full extraordinary equation, and time dependent Schrodinger equation time dependant Schrodinger equation i h cross theta psi over delta t equal to the Hamiltonian acting on the wave function, right the Hamiltonian in this problem consists of h cross square. Let us write this is i h cross for square over two m delta square that is the kinetic energy operator time psi plus v f r psi, this is the full Schrodinger equation for the time dependant wave function right that immediately implies that it complex cultivate satisfies delta size star over the delta t equal to ih cross whole square with the minus in del square size star plus v f r size star v f r is a real potential in a is a operator they could be cases were the potential becomes complex for instance I have absorption of something like that you can modulate with a complex potential, but here in this case probability is conserved completely there is no absorption of the incident flux at all, and from these two equations i multiplied by this size star multiplied by this psi an sub track one from the other, and this immediately gives me i h cross delta over delta t square this is the probability density if you like is equal to it cross whole square over two m, and then you have size star del square psi minus psi delta square, and this portion cancels out precisely, because we are far is real, and that can be start than off course it would not happen right. So, this cancels out, but you can write this as a gradient you can write del dot. So, you can write this equal to...

So, let us get it a one cross this is equal to this, this is equal to minus del dot j, and then matter to read off j to be just this, because I take the delta out of this that is going to be term which is precisely that, and then there is going to be del size star dot del psi, and that canceled out. So, from that it follows that the current probability current is just, but for the incident case what is this going to be equal to well psi incident is e to the i k dot r, and the gradient of e to the i k dot r is i k dot e to i k dot r you going to get a minus sine here, and there is an a extra minus sin here. So, this going to be two times i k the two i cancels, and this is gives you h cross k over n which is what you expect, because h cross k is the momentum for classical particle p over m this is what you expect be the flux. So, this incident flux on this side in magnitude is h cross k over m now we need the scatter flux up here, and what flux are we looking for we looking for the one that goes normally to the surfaces of constraint are. So, we looking really for h cross over two mi.

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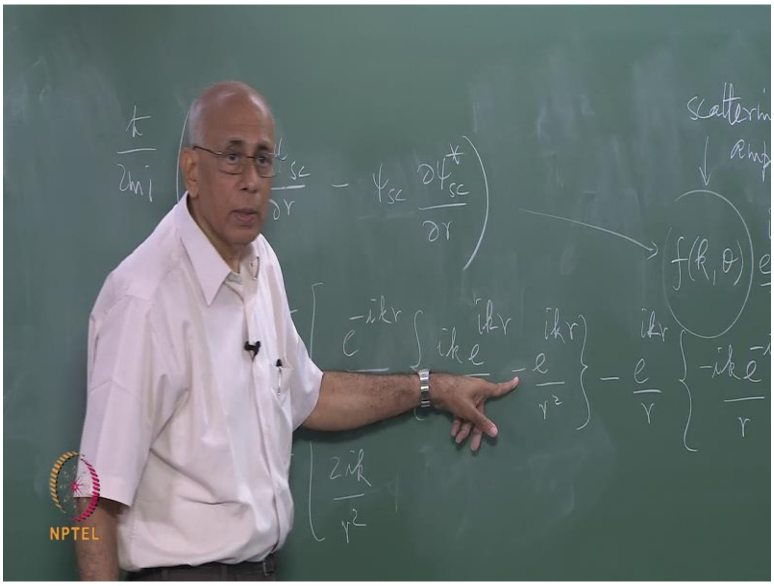

So, for the for the scattered flux this is equal to, and you looking for the radial component. So, this is equal to size scatter star delta size scatter over delta r radial component minus. Size scattered times delta psi star scattered star over delta that is the quantity we want to compute this gives you the current the flux radial flux, and you have to multiplied by the are d s to give you the flux through the d s, and this is going to give you on r square d omega right if you have a solid angle cone of solid angle d omega at a distance are, then off course the areas r square d s d s d s is r square d omega. So, what is this equal to we need to use this I need to use this information for size scatter. So, this gives me h cross over two mi times. Now everywhere there is going to be this factor wherever size star appears f is going to become f star, because there is no reason why should be real.

In general some complex number and... So, you have to put in f star there, and that is function of angle alone does not get differentiated. So, it is clear that from this, and that you going to get a modules of f of k theta whole square times this quantities now let us right that down this is equal to size scattered is e to the minus i k r over r this portion minus, because complex, and then delta size scattered over delta r is going to be i k into the i k r over or that is the first part plus or other minus e to the i k r over r square that is the derivatives of one over r with respect to r, then the other term is minus in to the i k r over r that this portion, and then I have to differentiate it is the minus i k r over r with respective r. So, there is a minus i k in to the minus i k r over r, and then minus into the minus i k r over r square that is the full derivative out here, and this is easy to simplify this gives me h cross over two mi mod f square, and then the first terms add up. So, there is minus here, and minus here these two guys add up, and therefore you left two i k this factor in that gives you unity divided by r square, and the second terms cancel this is minus one over r q, and there is a plus over r q.

So, that term cancels out, and that is it this now gives me h cross mod f square h cross k over m, because two cancel, and then mod f square over r square. So, this thing gives me again h cross k over m, and then mod f square over r square, and then this area element is r square d omega. So, the incident flux completely cancels out now you can see that instead of e to the i k are, if I taken amplitude a times to the i k there are, then of factor mod a square we should cancelled out on top, and below this equal to mod f square given

a. So, this immediately gives us crucial formula which says the differential cross section d sigma devised by d omega that is in definition of differential cross section this is equal to module s of f of k theta of square. So, the detector by measuring the flux actually directly measuring the module square of the scattering amplitude. So, the job is now to compute this quantity for a given potential, and then compare it what you see under observation.