Selected Topics in Mathematical Physics

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Module - 9
Lecture - 24
Fundamental Green function for 2 (part -II)

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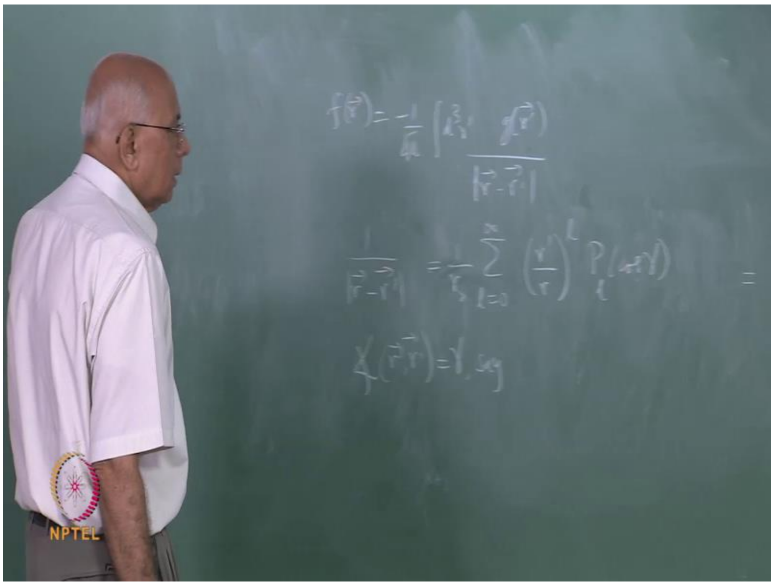

What I would like to now, is having got this 3 dimension, will ask now can be do this d dimension. Can we write down the solution to at least fundamental Green function in d dimensions? After all for this is square operator and it transform is minus 1 over k square. And that fact did not depend on the dimensionality, because it just said take del on operator twice on heat to the ... So, I could started by doing the following.

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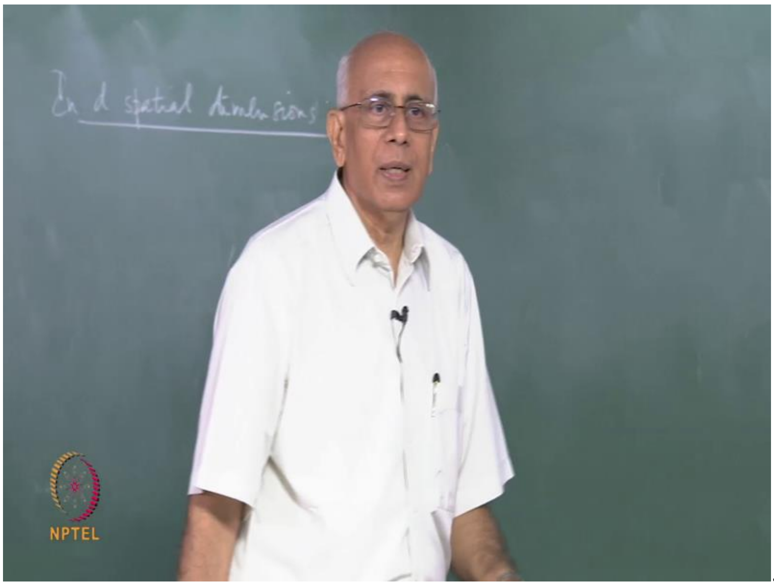

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So, in d dimension in d spatial dimension. And this is very important, because del square operator appears in all sorts equation and did like to solve it on all in all dimensionality. By the wave in 1 dimension, the del square operator; it just d to over d x to. And what does means to say that function harmonic in 1 dimension, what does it mean?

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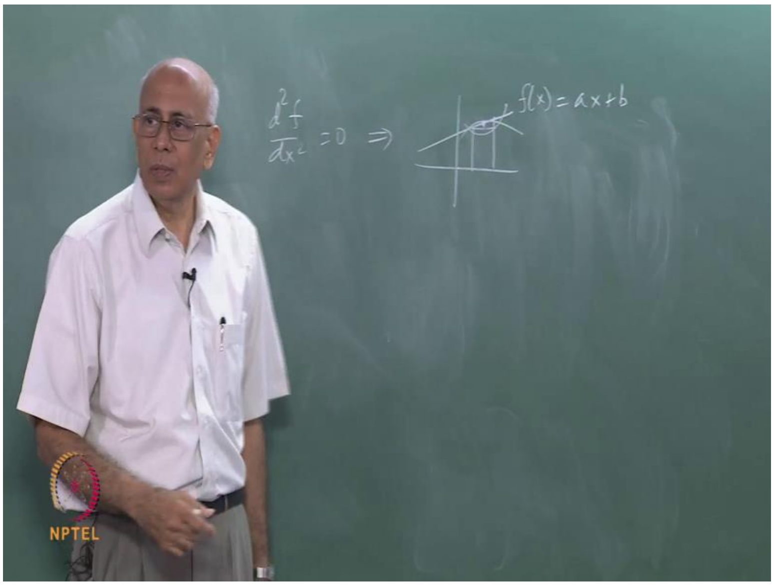

It says you have function of x and says d 2 f over d x 2 equal to 0, what does a same implies for this function f? Just linear function, it just linear function that is, no this function just linear. So, f of x is some a x plus b. What did actually means is that, the value at any point is arithmetic average of values situated symmetrically aborted. So, the value here is symmetric average of these to values as you can see from this triangular construction.

So, f of x is f of x minus a plus f of plus a divided by the difference between the 2 divided by 2. This follows, because is no ... had this function bin con x or con k, had it been by this or like this, that would no longer be true. The value in the middle is not the arithmetic means of these 2. Now this property generalized to other dimension, when you say that.

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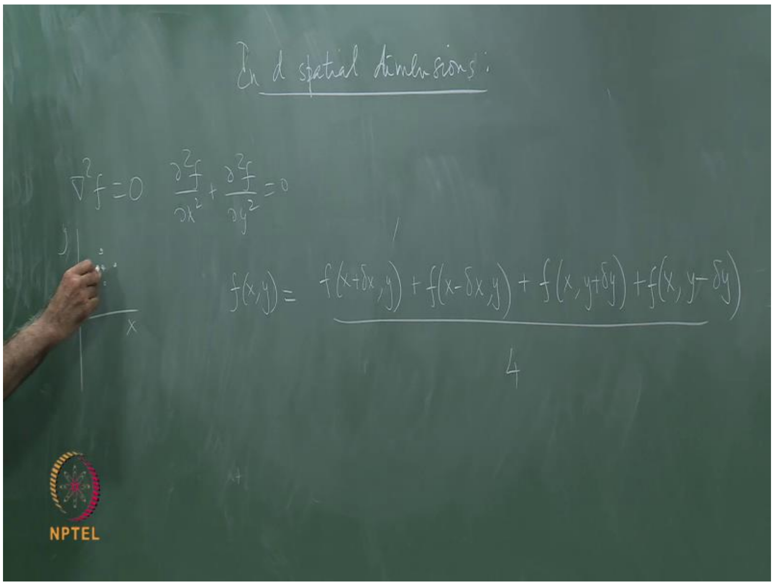

You have function del square of f equal to 0, what you really saying is that, the value at every point is the arithmetic average of values arithmetically situated about this point, in the region function satisfied this del square f is equal to 0, because for instance it will look at and 2 dimension. Here is x here is y and you are saying d to f over d x 2 plus d 2 f over d y 2 equal to 0. What you really saying is the following.

The value here; you could take a point here, a point here, a point here, point here in this fashion. And then, if I re write these in terms of differences, for in separation some kind, then, this is the difference of the difference. So, what I really saying is that, f of x, y equal to f of x plus delta x y plus f of x minus delta x y plus delta y divided by 4. If, I start with this and say this value is arithmetic average of these 4 points, now you can see I move the 4 to that side and then I get f of x plus delta x minus f of x and that is going go in the limit to the first derivate with respect to x, first derivate to respect to y. And then this is going to appear because plus here, so it is the difference of first derivate, which give to the second derivate after this equation. So, if you start with this, you can end up this limit, but, this thing is completely cervical symmetrical.

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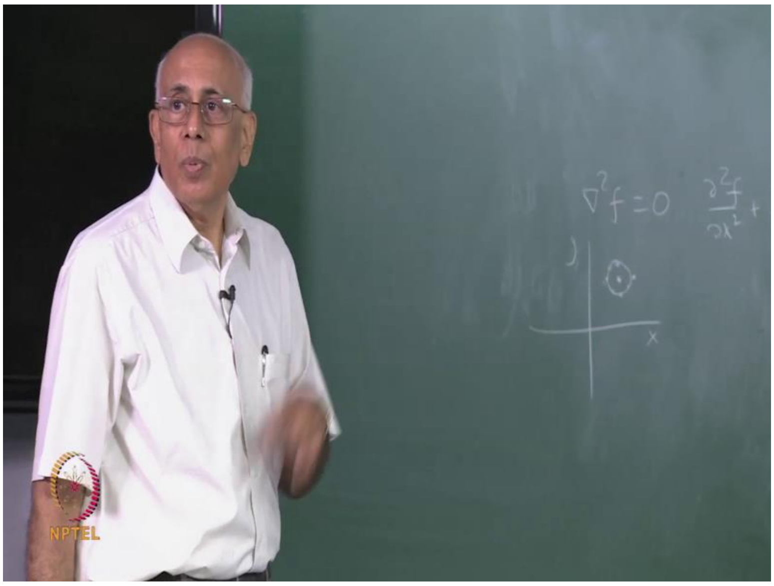

So, circularly symmetrical, because you could choose in this point, this point this and this. In fact, you could choose all this point. In the arithmetic average of the integral of this function over that circle, is the value in the middle. This is the property of the harmonic function in the any number of dimensions, this mean by value property. This does not mean in that the solution this linear function of x and y, that is the richness of the whole thing. In 1 dimension, the only function with satisfied that is the linear function.

Now something which is linear x and y would satisfied this equation of up course, but, you have much greater possibility that, this may not be 0, that may not be 0, but, they could cancel each other. So, that is why all this huge spectrum of harmonic function which are highly non truer, has the dimensionality increases, but, they all satisfied the mean value properties still. Now, the consequence of this property is something like Gauss's law in electrostatics. It says you take the flux round swell and whatever inside there the charge or whatever. It tells something about the charge inside. Now let see what this says for the coulomb potential in d dimension, which we define to be the inbred of the Green function which is minus 1 over k square.

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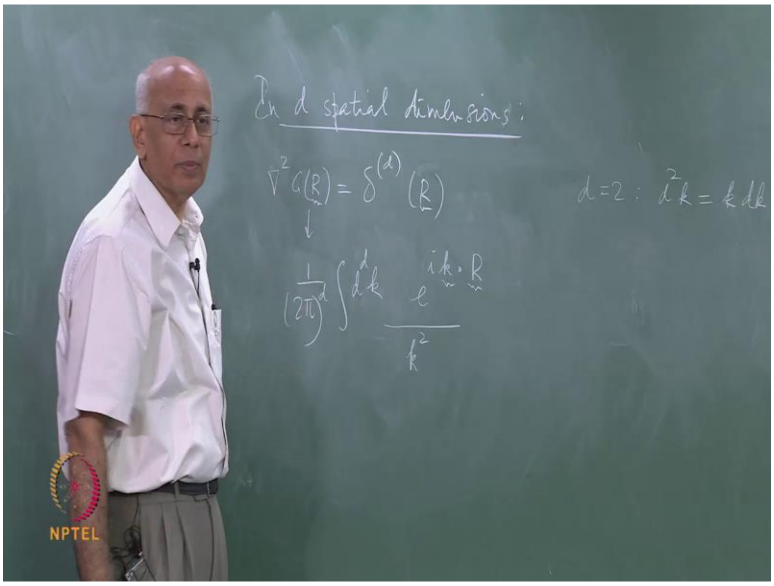

Let see what is says. So, I have del square g equal to delta function d dimension of R. So, let us call this vector R and this g is the function this R. It is in any number of dimensions. So, I put underneath to denote of vector. And the integral that I have find it do is 1 over 2 phi to the d integral d d k e to the i k dot R divided by k square. This is the integral I have to do. As usual, it sensible to use coordinance, but, let us look at what happened in 2 dimensions first. In 2 dimension what is this d d k? It is k d k d phi, phi is a as angle 0 to phi follow angle and that is it. What happens to this? This thing is bounded and you get k d k and you get k square and the integral transfer from k equal to 0 1 onwards, what is the happen? Does the integral exist? No, not as I stand it.

So, it says in 2 dimensions, we are in trouble. This is formerly infinite, something wrong, this is formerly infinite, diverges in 2 dimensions in 3 there was no problems, because this had a case quite d k, in that this. In 4 this would even better k q d k. So, want that we good no problem, but, in 2 dimension we are in trouble. And this trouble is occurring for this function first small values of k, k is like wave of vectors, how if you look thing in terms of frequencies, because these 2 are related to each other.

So, this divergence in feel theory is called an ... It is happening first k or small frequencies, all long wave lines if you like. On the other hand, had the divergence occurred, because k goes to infinity it would be called an ultra divergence. This terminology as come from obvious connection with fortunes, but, that is stander terminology. So, what we discovered is that, the Green function for the del squared operator, has a formal divergence in d equal to, but, for I had dimension it is 3 4 5, that is etcetera. So, let us tried to do that in higher dimension first and then come back in this 2 dimension problem. We like to choose farcical co ordinance in d dimension in what ...

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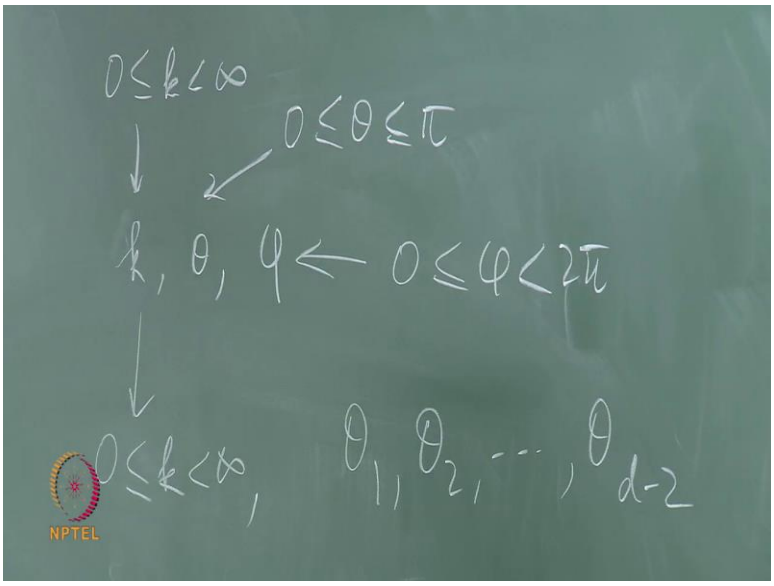

In 3 dimensions, off course you had k theta and phi. This stand form 0 less then equal to k less than infinity, this line 0 less than equal to less than equal to phi and this 0 less than phi less than 2 phi, those of usual farcical folo coordinate. What happens in d dimensions higher than 3 4 5 etcetera? How do you define these folo coordinate? In exactly the same as be this thing, return out it is exactly same as this. What you need is a k, which would run 0 less than equal to k less than infinity that is the magnitude. And then you had theta a 1 theta 2 dot dot up to theta d minus 2. So, in 3 dimension is just 1, these are the angles.

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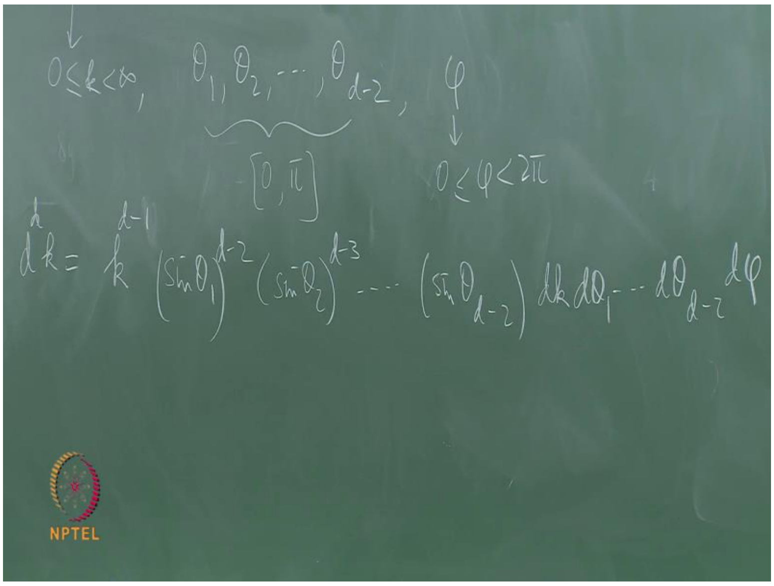

And these guides all run 0 to phi and then there is a phi, which runs as before 0 less than equal to phi less than 2 phi. These are called high per co ordinance. So, make name just say mention generalized whatever happen in 3 dimension. And what is the volume element? This quantity d k give you first k to the power d minus 2 d minus 1 and then there as sing 1 to the d minus 2, because remember 3 dimension 3 to just d 5, but, in d dimension d minus 2 sin theta to d minus 3 dot dot dot, all wave d minus 2 to the power 1. And then you have; d k d theta 1 d theta d minus 2 d phi, that is the volume element in hyper spherical harmonics.

So, what I like to do is to put that in play the same before, namely choose this r, choose the polar access along r and then k dot r becomes; k magnitude k magnitude r times cost theta 1 and all the other angle scan done. All other thing is can be done.

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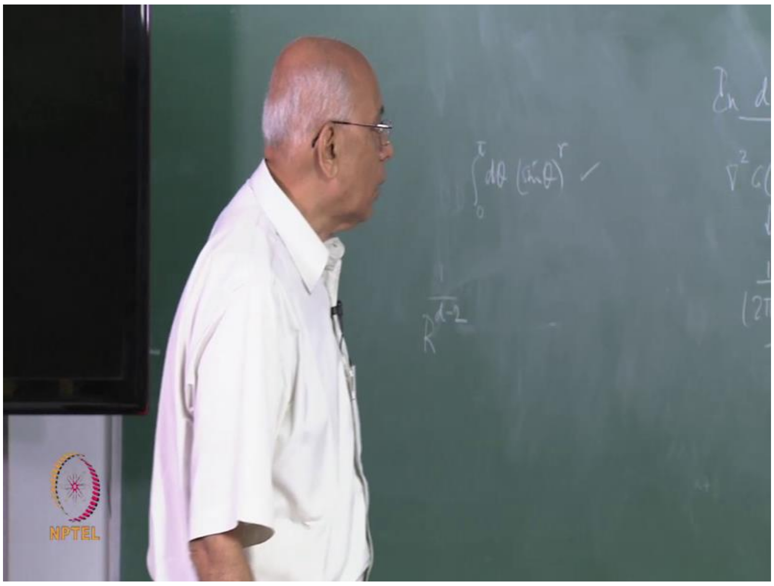

The integral can be done, because eventually you just have powers of you have things like 0 to phi d theta some sing theta to the power of r, this is converted 0 to phi over 2 phi over phi. And then up course this can be done, because easily this happens to is equal to root pi. You can write this special case function, its root pi gamma of r plus 1 over 2 over something all the other and this is in write down. So, we can do that for every one of these kind, accept these, accept this, because here you have an exponential factor also. And then, you have do the k integral itself over the magnitude over the k.

So, it is not a altogether all task; do it an all arbitrary number of dimension, you can do all this integral, accept the theta 1 and the k. And 1 should first do the theta 1 integral, the uncertain on base some function, which involves k and then you got put that in along this k to the d minus 1 factor and the integrate over k from the infinity. That too is the standard integral and you can read it of the table something like that. But, what I want to emphasis is there wave of doing this, if you knowing tables integrals, but, it like to little give physical argument and do this problem essayer way.

And above all, whatever we do, will be function d eventually and you should have to deal finally, with factor when you put d equal to 3, you should recover minus 1 over 4 phi r certainly and put t equal to 2 it should divert, should be a problem. And there should be way of getting, because we know then in to 2 dimensions, coulomb potential is actually well defined. What is the coulomb potential an 2 dimensions? How did define this? Imagine you have an infinite line charge and the parallel line charge, 2 parallel line charges, what is the potential between them? It is law right. If there infinite, then you can take any section and this independent were this section is taken. And the force between these 2; attractive or goes like loge of the distance between the 2 right.

So, in 2 dimensions, I accept the coulomb potential should log r, whereas had dimension is some powers inverse power of r and the answer will actually turn over to the 1 over r to the power d minus 2. So, in 3 dimensions, it turned 1 over r apart from the 4 5 factor. So, this should be the coulomb potential in all this d greater then equal to 3 and at d equal to 2, it switches to 2 is called marginal dimension in this case for this operator. We should able to extract to this information. In particular, we should derive this relation forwardly.

So, what I do next time is to give simple physical argument, which will help us did you this along with factors, that go with it and then see out handle the d equal to case. We will use method, which does analytical continuation, cheating the dimensionality d has complex number, is called dimensional regularization and is famous method in theory, which is the reason I thought it is worth then talking about it, even in this alimentary example of the del square of greater. So, it turns out that, you can make sense out of the formulate divergent quantity, by cheating the dimensionality of space as complex variable.

So, doing things as function analytic function of d and then extracting information from thus analytical function. We will see what happen when go to be equal to 2, see how dimensional regularization operate that other waves of extracting this information, but, will do this then check it out. So, we started this point try to evaluate this exactly in the simple organization and then checkout weather the right limit and point also.

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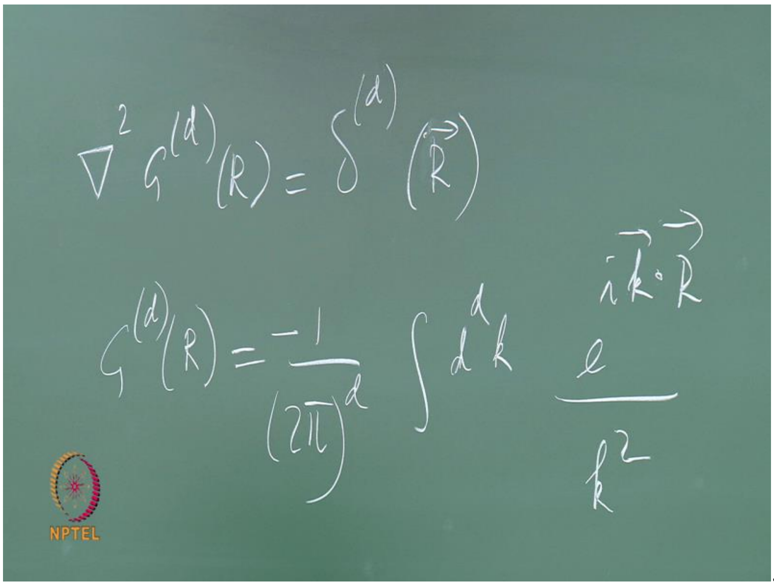

Try to do this in ultra-cervical coordinates, in which you have a magnitude k and then you have d minus 2 of this polar angles and 1 as middle angel. When you do this, what happens finally, is that.

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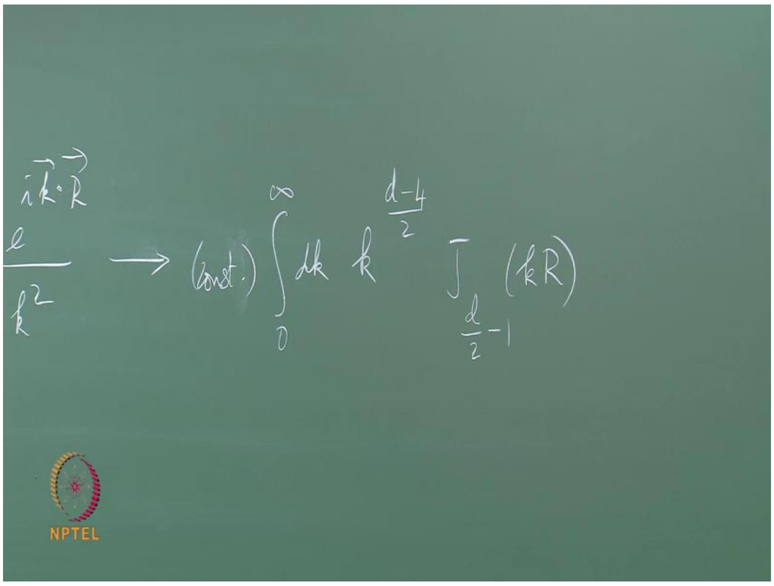

This whole thing reduces to a certain integral are due in all the angel which given in, but, after do all that, you can end of the just 1 is integral over the magnitude of this factor k and that integral take this. So, apart from some factor which I do not remember, there is an integral which is 0 infinity d k k to the power of d minus 4 over 2. And then blessing function appears here of first kind d over 2 minus 1 of k r. So, the whole thing, there are some constant. The whole thing reduces the things complicated, the integral over a Bessel function with an order which is some fractional the order d over 2 minus 1 and then a power of case setting here. And the question is; what the value of this integral and this also can be done this integral but, let see what we can say about the integral, without it.

The first point is this integral does not exist for all values of t. As you can see, you would like it to converge infinity and you does not like it to 2 barrel singularity of the origin, due to the integral. Now what are those criteria and depending on those criteria satisfy or not, you can make a session to where does it make sense as a function o g d, as a function of the dimensions d. Now this solve of argument, where we just down powers is called power counting and it is really powerful in most cases, but, in this case it not going work fully, this was partially and I explain why this is so. So, first point just look at the behavior at infinity.

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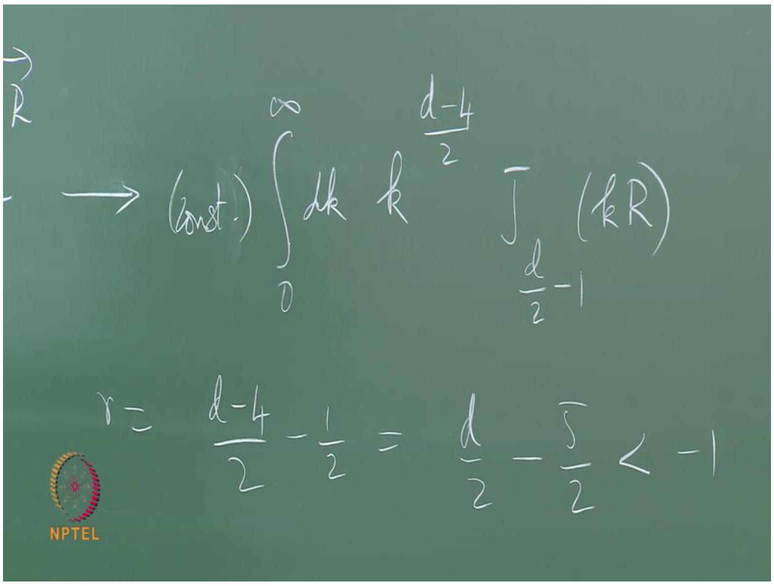

Now when does a integral like a up to infinity d k k to the power r, when does this integral converge? Clearly do not converge for positive power, converges only when r is negatives, we have it in the denominate. And in the denominate, tell the power should be at least bigger than 1, the 1 is make diversion right. So, it clear that, this integral is less than infinity if, r is less than minus 1, minus 2, minus 3, minus 1.5 is fine, but, minus 1 lives your long way divergent.

Now what the behavior of this in for large values of k? You have a power of k sitting here and we also have some k dependence setting here, but, the Bessel function as a property that, whatever beats order, when its argument become extremely large, this function goes like simultaneously function sign go sign extra, divided by square route of k. So, the power k is minus half in such cases. And then this problem are equal to d minus 4 over 2 minus half, whatever b d, because it depends only on the argument here it goes like 1 over square route of k, that is it.

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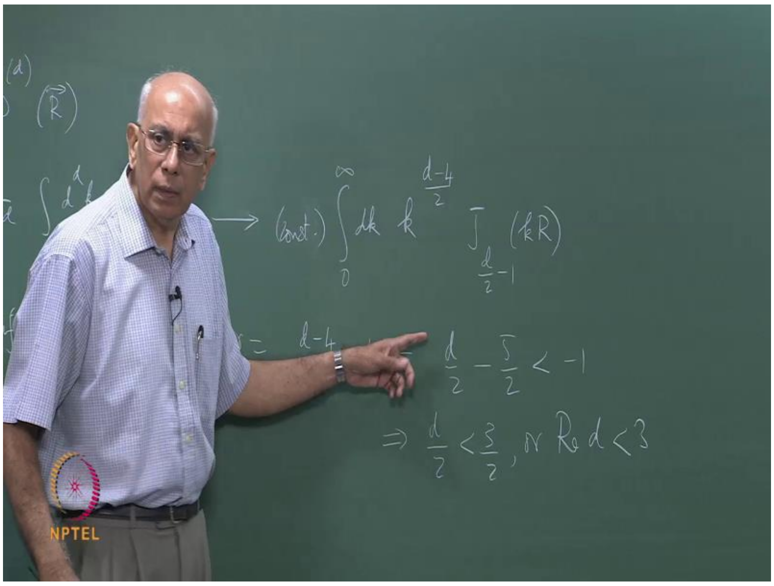

So, we would like to have this quantity. This is equal to d over 2 minus 2 minus half that is, minus 5 over 2, to be less than minus 1. And what is the say about d? ... over 2 less than 3 over 2 or I should now put in this real d less than 3, because if is d is complex has treated as a complex variable, the fact it has an imaginary power, it irrelevant for counting, because there are only. So, as you know in all such cases here, it is really the real part of index are matters, the real r will less than minus 1 or real d is less than minus 3.

So, even equal to 3 is not allowed, if we this route. On the other hand, he did integrate equal to 3, expressly we got this answer, but, I remind you doing that, we were face to that integral, did not converge absolutely. We ended up just remind you, this thing involved integrating 0 to infinity d k.

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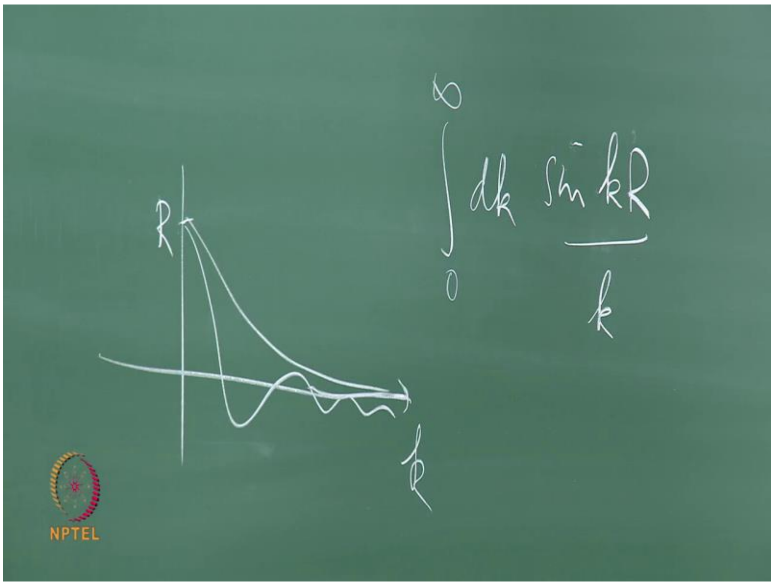

There was a sign k of over k. This integral does not exist in the absolute sense, if I take absolute values, then this is bounded by 1 and you have d k over k and it this point. So, this integral exists, only because the sign k r changes sign it takes plus and minus values. And you know the sink function convergence in integral, because at the organ of k, is equal to 0 and it does this as a function of k starts at r this thing here; at k equal to 0 and then horse late down dispersion.

So, the envelop of this function that dies on like 1 over k, this thing here. And that alone that in half that does not died on fast in half, actually would diverge diametrically that, because the function horse late in sign between plus and minus values and some cancellation between the prohibition from the plus and minus side and these areas cancel out against the which are some extent and then took convergence. And we know the answer here this equal to 5 over 2, this thing, for all positive r equal to 5 over 2.

So, already receive the symptom of the dieses. That is the one you have identified here, because it says that nearly less than 3 and this power counting does not take into account the fact into changes fact and 4. It is as if the whole thing and it is went down 1 over square route of y for than that. So, that was a reason. And we will work it out mind; we will see what happened in exact answer, now very short why. But, I wanted to bear in mind that, this integral formerly does not exist for real d greater than 3 or even equal to 3, so we evaluate d go made it barely made it.

So, there is a divergence and this divergence happens, formerly very large values of k. Now there is a terminology borrowed here from this theory, which is when such an integral divergence for a large k, it is large wave number or large frequently subsequently, it called an ultra verge divergences and ultra violet is a higher frequency side. So, this integral has an ultra formerly has an ultra violet divergence for the d greater than equal to d. What happens here is the small scale limit? Well something else happens

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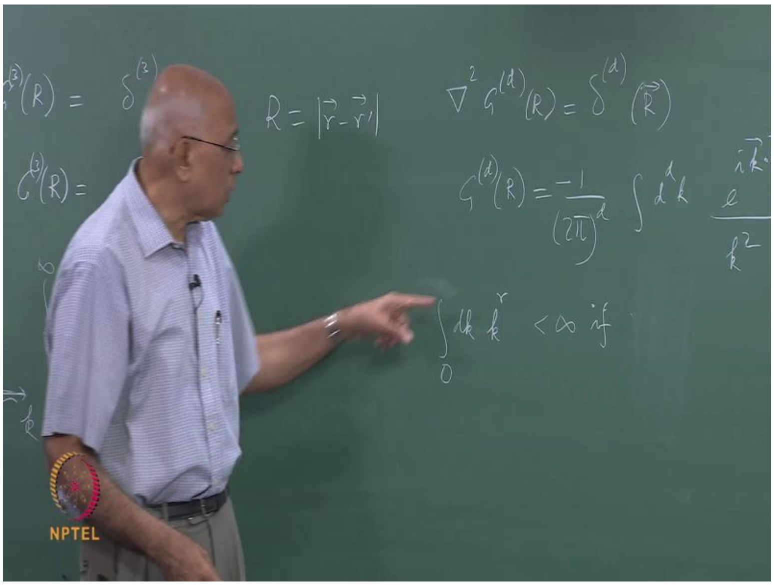

In the small scale limit, we should ask when does this integral converge and what would be the answer. Once again any positive value of a now is perfectly all right, this no problem. Even a negative values all right, as long as it is not more negative of 1 over k. The movement of a 1 over k, you have a log 0 and log 0 infinity and when you have 1 over k square, because very bad and so on. So, this power r should be greater than minus 1 this time for it converge right.

So, there should be real part of r greater than minus 1. The inequality reversed in this case. But, now you could ask how could this integral ... So, if I really formerly gave a nothing like this, does ever exist? Would this ever exit at all for any value of r? No, off course not, because of the lower rent is says that r must be bigger than minus 1 and upper than r must be less than minus 1 if, in both case. But, now say by the fact that, the behavior of this function is going to be different small value of k.

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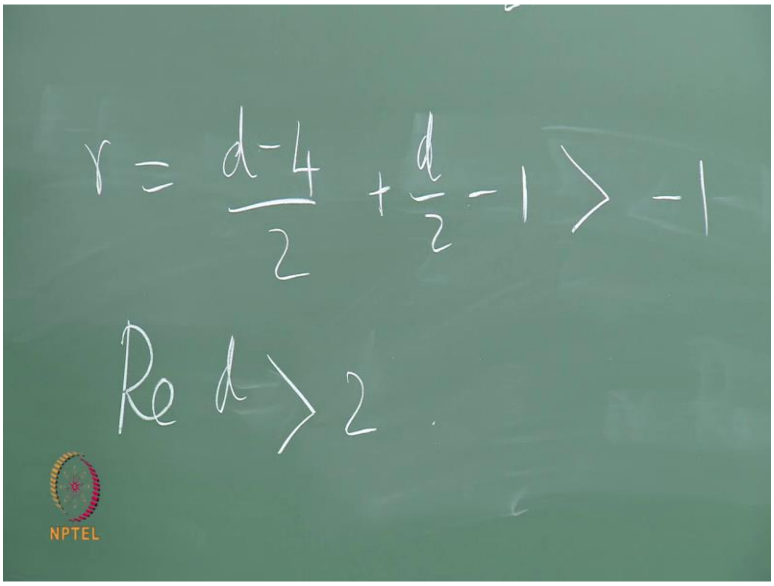

This thing still go like c minus 4. So, in this case r is d minus 4 over 2 no problem there, but, you see the Bessel function itself for small argument, depends on the order.

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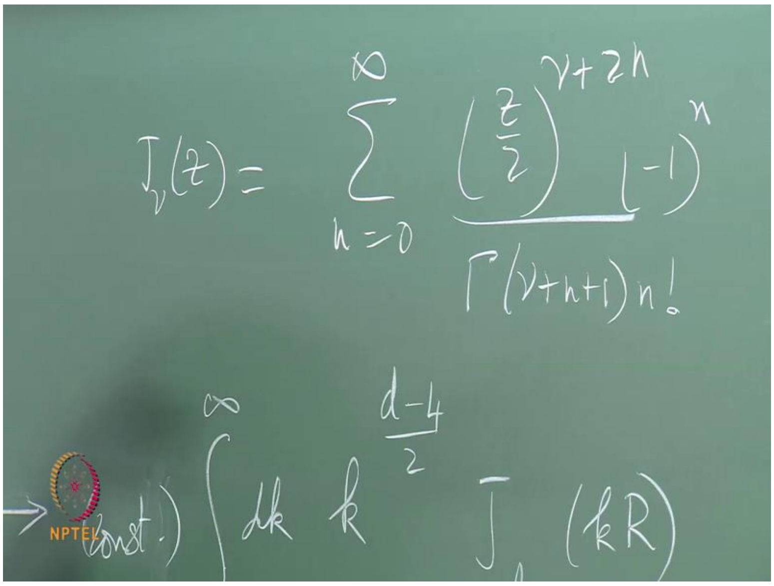

And if you recall what the definition of special function is; we assume of z of a summation equal to 0 infinity z over 2 the power new plus 2 n over gamma of new plus n plus 1 and factorial and a minus 1 2 the power 1, which disappear 1 you have the magnified thus the function. Whatever it is, it is clear that for small ends leading behavior is z to the power new and equal to 0 would term, that is the leading term and that goes like to z to the form new. Therefore, for small value of k near k equal to 0, this integral goes like k to the power d minus d over 2 over minus. So, you add that here d over 2 minus 1 and this amongst be greater than minus 1 and that says d this case real d must be greater than, but, you get d 5, 2 here again. So, this 2 plus 1 3 minus 1 is 2, really greater than 2. So, we see that this integral, this thing here formerly you right it down.

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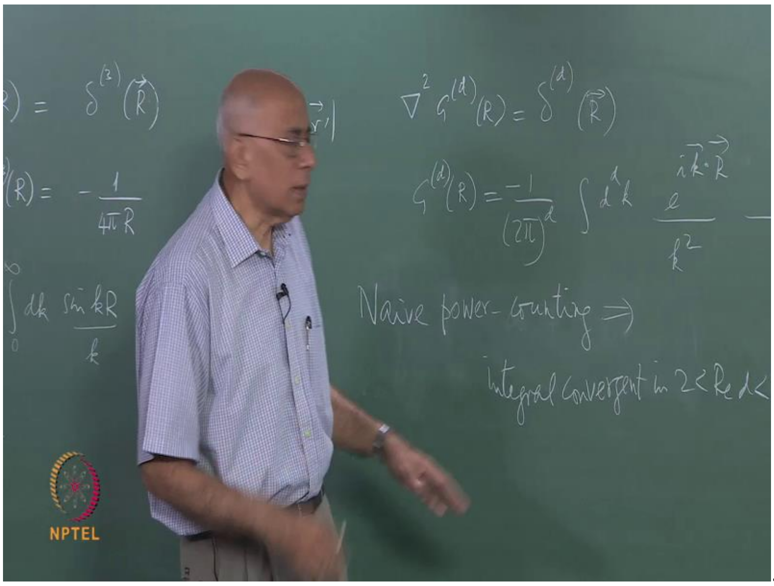

Power counting, I should not say power counting alone. We did a simple minded counting to the fact the integrant in assonant sign and so on. So, this is lot of knife power counting call, it knife power counting implies; that is integral convergent is convergent in 2 less than real the d. So, now I am regarding this integral as a function of d, a complex parameter d. Off course, my physical interest is an d equal to 2 3 4 extra, but, I am regarding formally mathematically, as a function of a complex variability functionality d. And then this integral is defined, well define a power counting in this region, in this strip between real d could 2 and 3.

The physical points of interest may be d equal to 2 3 4 and so on, but 3 we already valuated. Now I want receive the general d and it looks like the whole thing looks like only 3 and between 3 this case, so, here what is done, once it alright, very good. I evaluate this integral in the region between 2 and 3, get the answer function of d, its complex variable d and analytically continue to otherwise is of the if I can do so. And then, clime that is the correct answer for all the … This is what the beginning of a something called dimensional regularization is. And it also says something else. It says if you should faces singularity somewhere. So, where a single of part and keep the part which is nonsingular at any value of d at all. So, now what do you expect?

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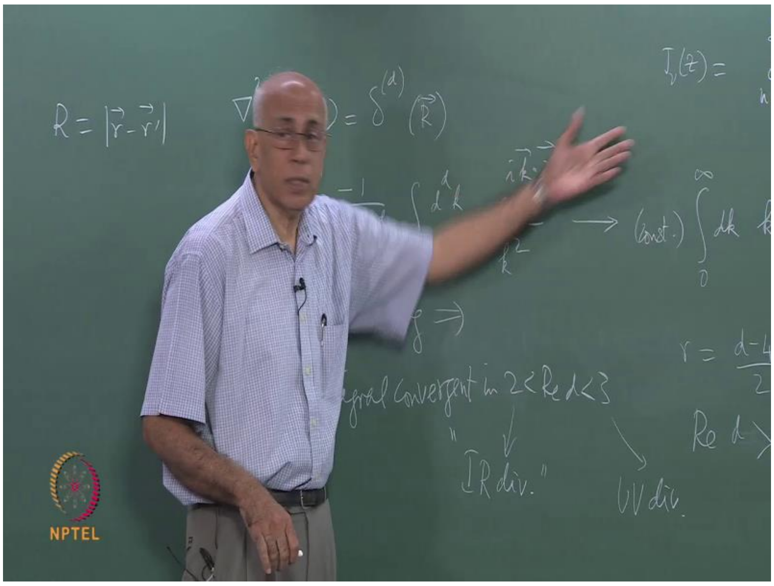

We expect that for d equal to 2, I expect a divergence of some kind. And d equal to 3, I expect another divergence and this is a ultra violet divergence. And this is happening, due to k going to 0, k small wave of number or large way of length or small frequency and the natural term t uses in and this point. This is called than divergence. So the technical term for such a singularity, is called and divergence lower end and then ultra violet divergence at the other end. And this guys since 2 have both, there is an as well as an ultra violet and divergence in the ...

So, we really have to exam in this further, where this come from we knows very clear, due the small and large k behavior respectively and we need to know how to deal with it. Now, what the prescription of a dimensional regularization does. This is a technique q theory of a making sense out of a infinite integrals, but, here it chosen that method of regularization, just because we know something about analytic functions and analytic continuation. And secondly it illustrates the general idea as you will see.

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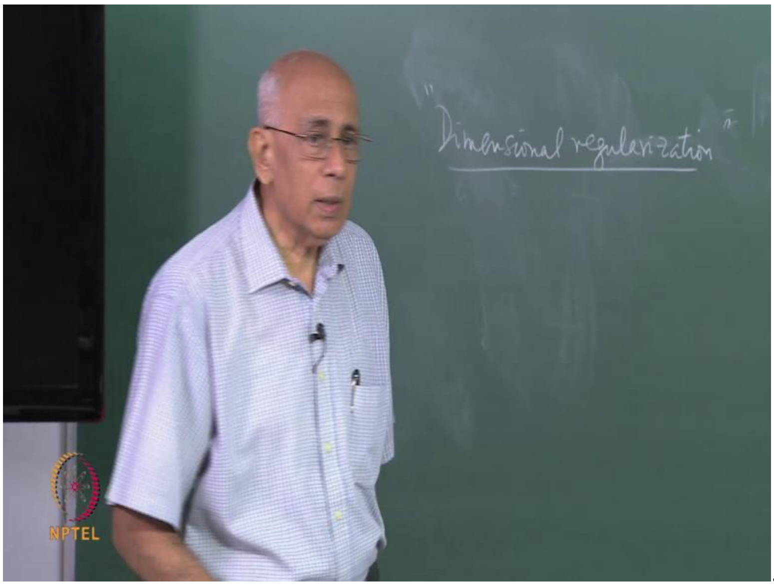

So, this thing here is called the technique is called dimensional regularization. And in a nutshell it is the following. We in evaluate is integrals; diversion to integrals, in the region in which their convergent, regard the answer as an analytic function of the dimensionality d and then analytically continue to your dimensional of interest, whatever is of interest to you. And should there be a singularity like a pool through, where the single apart and keep the regular part at that point and that gives you the value. That is the prescription.

Now, let us see where there it makes sense here and then are try to give a physical argument, which will actually say why this prescription works in this particular case. Now first step is 2 actually evaluate this integral, but, now that requires a table of integrals to do this thing here; I am not going to do that, will short circuited I am not going to do that will short circuited, because I can get the same answer by direct argument. And will do that without even using for a transforms. I just go back to this equation and observe that, I want to solution which is farcically symmetric that why important. And the reason is I want to fundamental Green function. Now this equation here itself, the equation itself it is translation invariant.

So, if the origin the coordinate, it does not matter at all, it is a function of r minus r prime here. The delta function here is a function of r minus r prime and my boundary condition is also function of r minus r prime, because it is says a goes to 0 is r goes to infinite r minus r prime goes to infinite. Because of that, I know the solution is symmetric 1.

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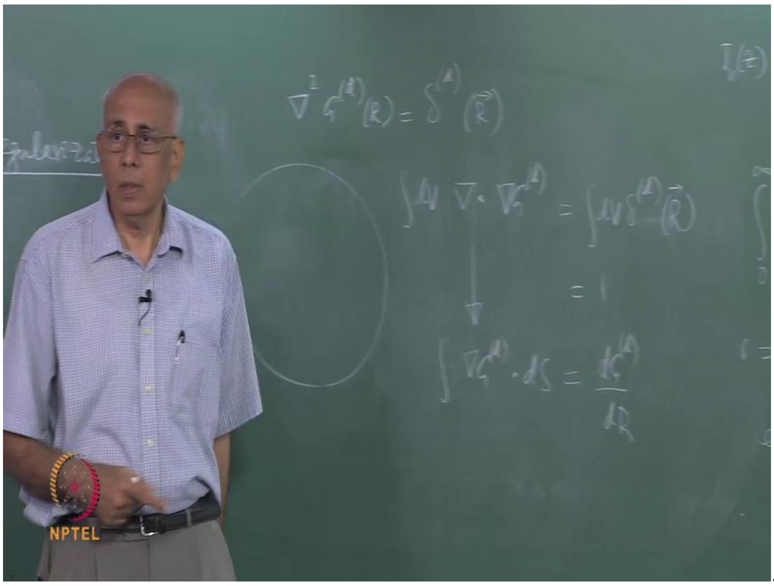

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But, let us right out by integrating over both side over hyperspace of radius R. And I am going to regard this equation as pose's equation, for a static charge. A unit charge if you like at the origin in some suitable units and the use Gausi's theory. So, let us integrate over hyper share of radius r, both sides and what is this says d v and that delta dot dell g that is, what else squad is dell or dell here, is equal to an integral dv delta d of r and that is equal to 1, because you are integrating unit impulse function and delta function over sphere of various r and the answer is 1, there is a unit charge inside distributed. But, I use gauss not here, use gauss for this guy. And what is that tell you; it says volume integral of a divergence is a surface integral is a flux of this vector appeal.

So, this is equal to an integral d s, rather dell g d dot t s. But, I want this spherically symmetrical solution. So, this dell g is like a force now, dielectric feel and that is radially outwards right. And therefore, dell g dot ds, this scale of product there is no cosign here, because d s the unit normal points outwards, this force also points outward in the same radial direction. So, I could directly write this as equal to whatever that force be, so it is d g d over d R. This is the normal component by the way, the radial component of the gradient is just a derivative with respect to the coordinate; the magnitude does d g d over d R times.

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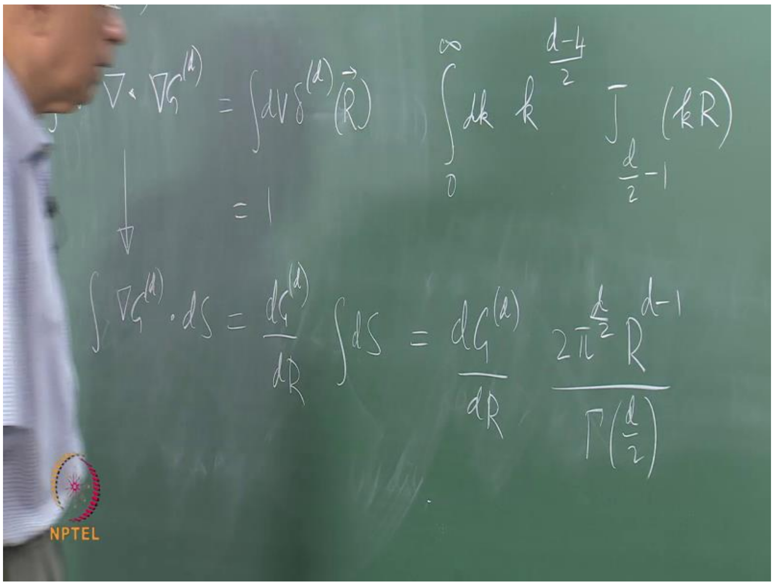

And integral d s over hyper share of radius r. This is exactly like doing gauss loop, using gauss long electrostatics to find, what the electric field is spherically symmetric distribution. You take a Gaussian sphere with center of symmetry and then you say or give that, the field is normal, outward normal, parallel to the unit normal to the surface area. And therefore, the field comes out; it depends only on the radial coordinate multiplied by the surface area of the Gaussian. That is all am doing here, except that this fellow here. So, this is d g d over d r times, this is now the surface area of hyper share of radius R in d dimensions. In 3 dimensions, it would be 4 pi square, in 2 dimensions it would be 2 pi r and so on.

So, clearly in d dimensions the answer is proportional to r to the power d minus 1. So, this multiply pi and the answer is 2 pi to the d over 2 r to the power d minus 1 over gamma of d over u. So, that happens to be the surface integral, which gives you the surface area of a units these are. We just have to check that, this guy is going to give write answer for the cases, we know. If you take d could 2, then this is 2 pi and gamma of 1 is 1 and that is 2 pi or not 2 pi or this is circumstance of a radius r if d. If d is equal to 3, then it is 2 pi root pi, it is there and then there is an r square and this guy here is gamma of 3 halves, which is the half gamma is square root of pi. So, that goes away against the root pi here and then the half here multiplies this, gives you 4 pi r square. So, it is in the cases and that is the general answer. So, now whether, because we find now d.

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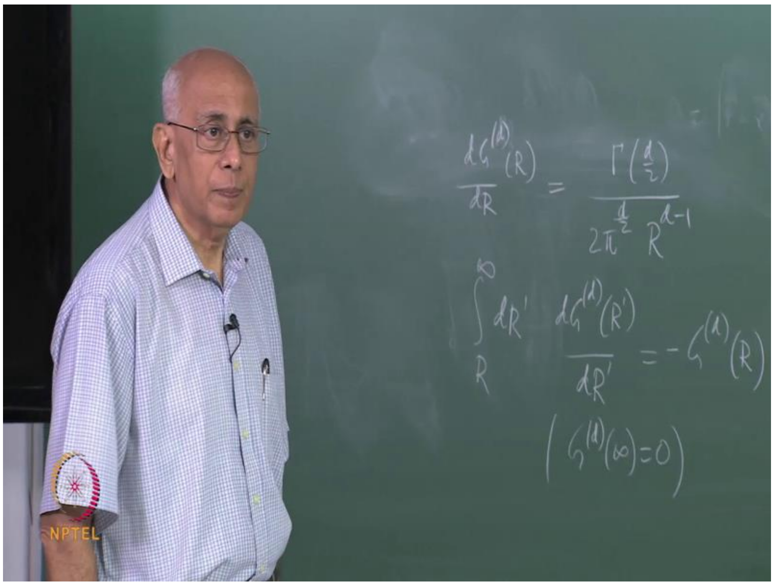

g d over d r as a function of r is equal to, because this integral is equal to 1. So, it is a d g d over, it just the gamma of over to divided by something on your 2 pi to the d over r 2 d minus 1. So, we have use ultra spherical coordinates, but, we use that, not to do key integral, but write this term, write the value of same. So, you cannot get away from doing those angular integrals, but, this much simple geometrical problem. By the by, what would be the volume of this, would you found the surface area. What would be the value of this fare? The question I am asking is s d of r bitness 2 pi to the d over 2 to the d over 2 r to the d minus 1 over gamma of d over 2. What is the volume v v of r equal to? It would be the integration up to whatever radius you want right. Similarly, you differentiate the volume with respect to the volume radius and you get the surface. That is the very definition alright. So, here we are. Now, the question is what this equal to? But, we need to impose a boundary condition and our boundary condition is that, g should be 0 what infinity.

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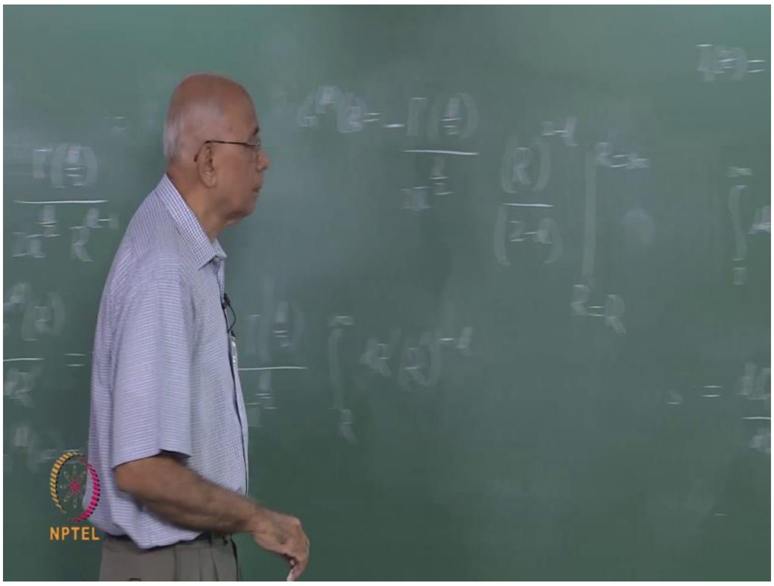

So, let us integrate both slides. If you integrate from r to infinity d r prime, just to avoid confusion, d g d r prime over d r prime, this is equal to g d of r with a minus sign, because it 0 at infinity. So, I show away that portion that is, my boundary condition and that is equal to gamma of b over 2 divided by 2 pi in the d over 2 and then an integral from r 2 infinity d r prime to the power r to the power 1 minus d or prime. It is the integral and that is an integral which is trivial.

So, it says g d of r is equal to gamma of d over 2 over 2 pi into the d over 2 minus sign, got be careful move this minus sign to the right hand side. So, this is a minus sign and then an integral of this guy here, at infinity it vanishes and why does it vanish at infinity? Well let us do the integral. What is the integral equal to its r prime to the power 2 minus d over 2 minus d. Let say r prime equal to r r prime equal to infinity. Why do I say it vanishes at infinity? d is greater than 2 i cannot get away from that the integral make sense only in the region d greater than 2 defiantly, I have to keep d greater than 2, otherwise is been does not make sense. So, no matter what trick I used to do is; I do not need to keep that; that is what we found earlier that you had a serious in front divergence.

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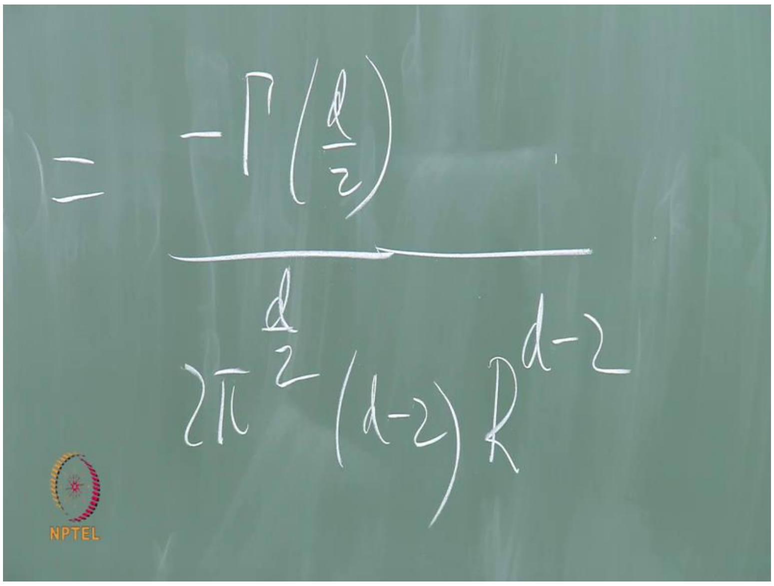

And therefore, this integral is equal to minus gamma of d over 2 divided by 2 pi to the d over 2. And then the only thing that contributes is the lower limit of integration, with a minus sign, kill that minus sign against this 2 minus d n right it has d minus 2 so d minus 2. But the other 1 remains, no question about it and then r to the power 2 minus d, which I will write as 1 over R to the power a minus 2 and that is the answer. That is the exact answer and it is an analytic function of d, as you can see and nowhere, it singularity is are and what is the most obvious singularity has a function of d. This is a simple pole at d equal to 2.

So, we said right away, there is an divergence in this problem a d equal to 2 and in deed you see that, a d equal to 2 there is a singularity. So, as it stands this expression does not make sense for d equal to 2. First of all if you put d equal to 2, you get an infinity here, you also get a constant here. So, it says there is no are dependent, but, that mean o is not true right, this got to be dependent in these functions.

So, it is clear that this formula is not right at a equal to 2, there is a singularity and we need to deal with that. Is there singularity somewhere else, what about this gamma function? What happens if d is 0? There is a pole, there is a pole of this gamma function. What happens if d is minus 2? There is again a pole. So, this thing as a Green function of in operator, appears to have singularity at d is equal to 0, minus 2, minus 4 etcetera, but, we are not living in minus 2 dimensions right. So, this not a physical interest, it is there formally it is there, but, we not worried about it. But, d equal to 2 we are worried about, because we have planner problems and we need to know what to do about d equal to 2, But, let us first settle the ultra violet divergence. Is there a singularity at d equal to 3? No.

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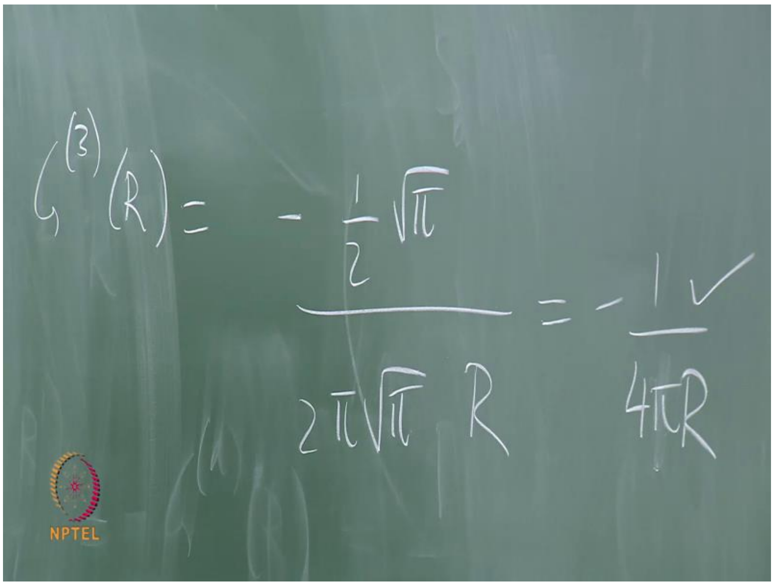

What happens if we put d equal to 3 here? It is minus and then there is a gamma of 3 over 2 that is equal to half square root of pi divided by 2 pi square root of pi and then there is an r minus 1 over 4 pi R which is right. So, you see you recover the d equal to 3 result brightly and there is no ultra violet divergence in the problem. For that matter, you could put d equal to 4 or 5 and you get the analogue of the in those dimensions and is clear from here, that analogue of the in d greater than equal to d dimensions is 1 over proportional to 1 over art of the d minus 2 r to the d minus 2.

So, in 3 dimension the column potential is 1 over R, but, in 4 dimension if you 4 special dimension for column, potential is 1 over R square and the 4 should be 1 over r cube. So, whatever the roll is played by the inverse square on 3 is special dimensions is played by the inverse q blog in 4 special dimensions and inverse 4 power in 5 dimension so on. Now you begin to see also why gooses law is works in 3 dimension in the inverse square of potential, because you see the surface area increases R d minus 1 and if you 4 also decrease like 1 over R to the d minus and you will cancel each other and give you something work, which is independent of the actual shape.

So, this is why the in verge square law is special in 3 dimensions, if no other law force law would have misbehavior. This form of Gooses laws, but, a calculate applicable directly tell you something about what end side, such very special. Now what have be done here, we defined or colo potential in a number of dimension to be the solution of equation or rather, they inverse full a transforms of 1 over k square or rather the potential with satisfy for the equation in glass equation, well a later. This is what we defined as a column potential in a name of ...And you see the special role laid in 3 dimension if 1 over R.

Now let us come to d would to know or ultra violet dimension in this problem and we know the physical code and code physically convey you know the knife power counting is misleading in this problem, because it works as long your integrant has a definite sign. And in all case we had a special function and it did not have a definite sign, kept oscillating. So, therefore, the convergence is better than what knife calls you. So, we do not have to very about ultra-violent, but, in for their valuable of this.

Now what the prescription again, go to the prescription of dimension of regularization and ask what the prescription. By the way, this result you can get in d equal to 2 equally easily by writing down, what the actual potential for a line charge in 3 dimensions. And using gooses law as you normally would for a static line charge that, get exactly the answer immediately. What would you say the answer here in... Let us do this argument did about integrating this directly in both sides.

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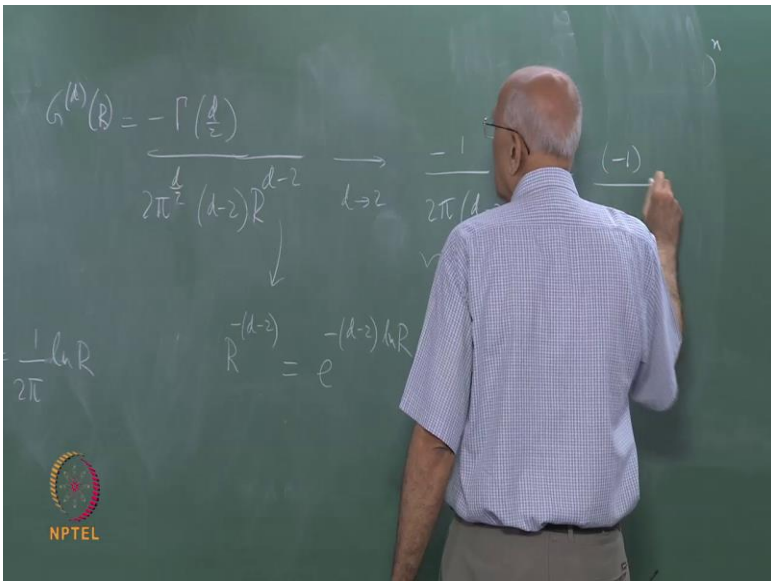

So, we had d g over d r in 2 dimensions over here r multiply this surface area of whatever is the unit of furious radius are and that in this case 2 pie r, where the circumstances that is equal to 1, that is it. So, what does telling about g 2 dg to over d r equal to 1 over 2 pie r. What is that implies g 2 apart from constant? You have to integrate this with expanding at law. So, this is equal to plus a constant depend on the boundary condition and so on and you come to that, but, this law. And as you know, if you tools uniformly charged, if you took a uniformly larger line charge and static line charge and what the potential actual distance r from this line charge, the answer is proportion to the law constant in 2 dimensions; what exactly this is.

So, what in special of about 2 dimensions? What is special is that, when you integrate a power here, remember we integrated a 1 over r to the power t minus 1, we got 1 over R t minus 2, that 2 is known as power here, which you integrate is not equal to 1 movement if 1 is denominative that of law right. So, this apparently simple fact that, the only the power 5s a lot when you integrate is 1 over r. This apparently simple fact, this found very deep and it has all kinds implication everywhere in the quantum, even quantum (( )). But, in just 2 dimension you have something very special happen and its traceable ultimately to the fact.

Now let see where we get formal answer here by analytical continuation. What is supposed to do is; to take this answer and write out here what happens d equal to 2.

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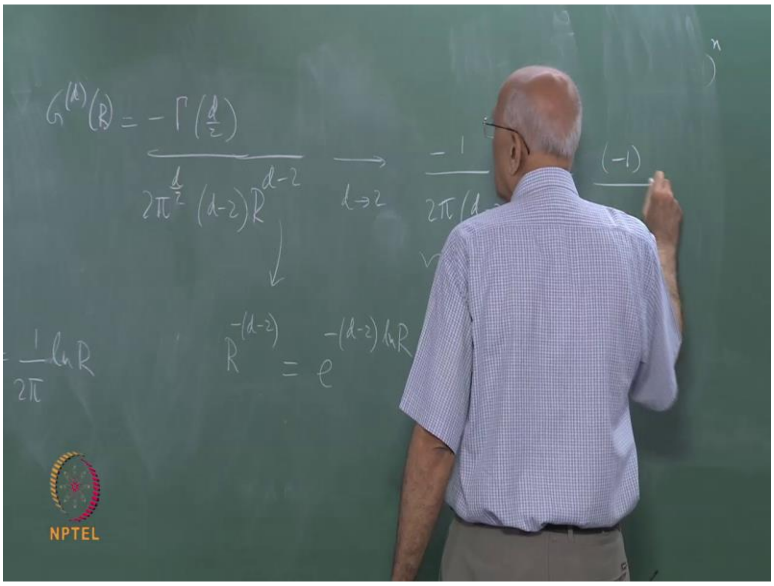

So, this is equal to minus 1 by d over 2 divided by 2 pie divide to 2 R d minus 2. And now I ask, what I going to do, what is the going pole is this go as a detail 2. The pole is already setting here to write the reset down this pole, you got multiply this d minus 2 and the limit as go through, which as the same as taking set d could 2 in all other fact. So, this is equal to gamma of 1 which is 1 and minus 1 over 2 pie comes here and then d minus 2 that is it, that the singular part. So, this is the singular part plus now we got look for the regular part.

The regular part will have a term proportional to constant, no dependence plus a proportion to d minus 2 to the power 1 and 2 3 4 extra; positive power. We are interested in what happen to the d goes to well. All you need to do is apart from other factor which you can expand, which you can expand every 1 of these nut tailors series about d equal to 2. Put everything out divide by this and ask what happens, but, we really interested in the portion that depends on R and only the part depend on R setting here.

So, write this sky as r to the power minus t minus 2 equal to d to the power minus d minus 2 log r identity and expand this tailor series d minus 2, which is the same as expand the power c gives already up there right.

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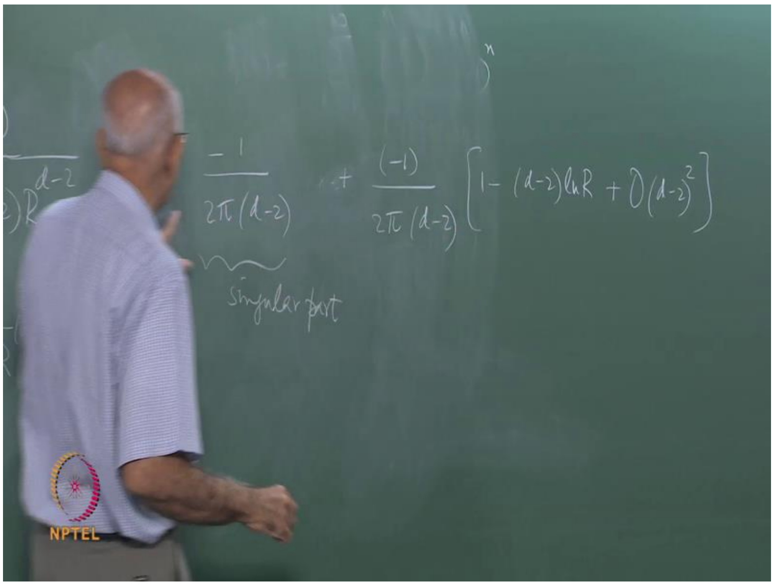

So, you have a portion that goes like that this factor, by the way every else we can said d equal to 2 and get plus or a minus 1 divided by 2 and then a pie and then d minus 2. And then, taking this and expanding a, you get 1 minus d minus 2 log r plus order d minus 2 whole square and this cancels out, this portion you already have here, I should not write it again.

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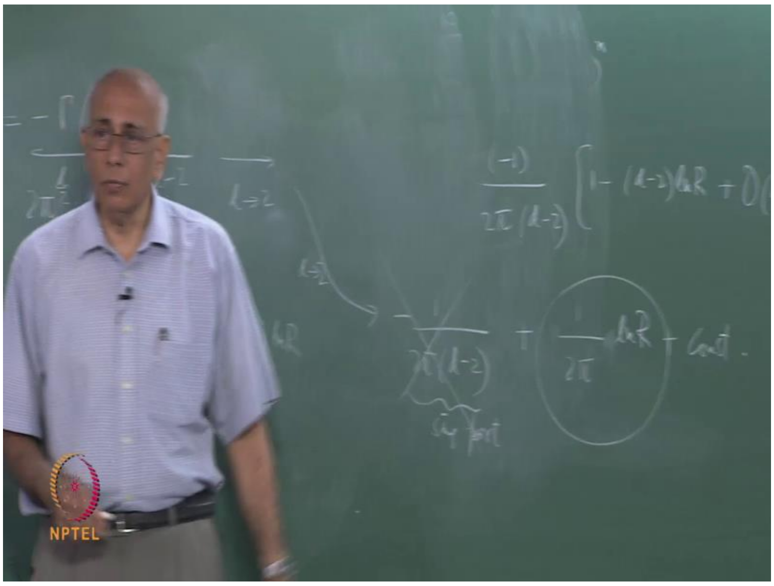

You already have here. So, this whole thing goes to minus 1 over 2 pie and d minus 2, this is the singular part plus 1 over 2 pie log r plus some constant as the d goes 2. The concept will not whole war plus higher order is give minus 2 and they all vanishes the ... Now the prescription of dimensionalization says through this away that the singular part, it does not involved R in any case and you left with 1 over 2, by which long as we knows that, apart from a constant.

Now the point about the constant is that, I as you know in a 2 dimensional problem, even in the problem of the line charge and electrostatic dimensions, you cannot choose the potential of 0 at infinity. This particular problem that boundary condition cannot be chosen, because it going charge distribution itself is going all the way infinity some direction. So, you cannot choose the potential to be 0 infinity. What we do in the case of a elementary electrostatic? You say well I know the potential at some particular point, some radial distance and I say that difference in potential between the potential there and where I am is proportional into log over a, that is, a is the point whose potential you know. So, row minus row of a is proportional to log row log of r and this exactly what happens here.

So, they operate a point is that, are the operational point is that in 2 dimension Green functional that propositional the log of a. And dimension given that neatly in 1 shot, in all dimension, including the d 3 answer and then it also example says what happens 4 5 this is; no ultra violet divergence in this problem, 1 can inflate as emergence with which does exist, there is a similarity. And that have happens potentially because; equal to 2 is a very special dimension with this integral happens to be log r. So, where a power of changes over to a log make log, you will always solve the problem exactly. This is a marginal dimensionality and they have lots and lots of interesting properties. In 2 dimensions this is the one of the interesting properties.