Selected Topics in Mathematical Physics
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Module - 9
Lecture - 23
Fundamental Green Function for Del Squared (Part I)

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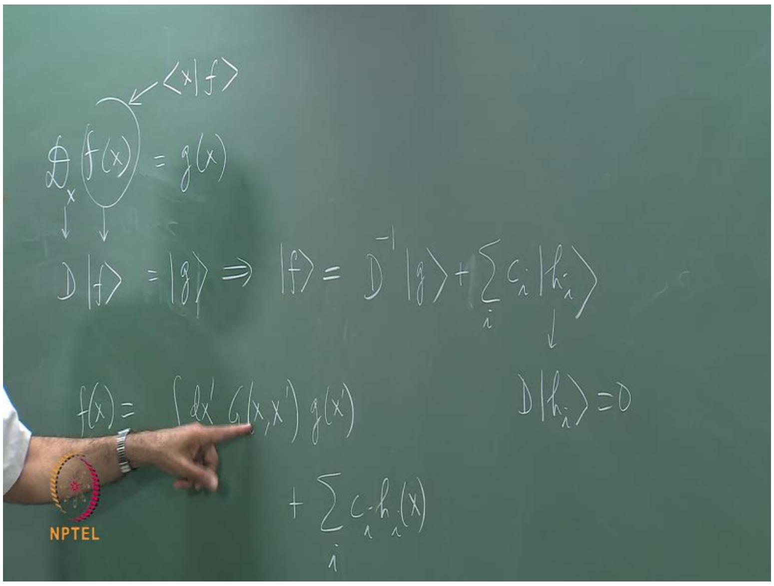

To will start today with just likely new topic which is the first of the partial differential equations of a physics the famous partial differential equations of traditional physics and this is Laplace's equations and of course if you have a term on the right side and Poisson's equation. So, the equation we have in mind is of a form del squared some function of r equal to sum other given function of a r just this Poisson's equation by the way the operator del squared itself appears in all these equations all the standard equations were on discuss namely pass on equation the way equation the defuse in equation the Helmholtz equation.

So, what and there is reason for it of course because del squared is a scalar operator it is in very in term the rotation, and these physical systems are situation that we account describe would generally is have rotational in variance of some kind. So, you have a second order differential operator del square which is the scalar operator has allot of fantastic property which called an partial differential operator in the classification of such operators and its absolutely fundamental when you generalize less del squared to high dimensional spaces curves spaces other manifold and so on, we terms out the if you know a certain I can value problem base on the del squared namely roughly speaking, if you know it is I can functions and its I can by the spectrum then you can say great deal about the underlines basic self.

So, very interesting fact that knowledge of the I can value spectrum of the explosion operator and that is generalizations gives you great deal of information about underline space itself I will come back to this point later on for the moment deal focus on to pass on equations the first of these equations and the question is how do we solved such on only such an linear partial differential equation when you given some function g of r will do this by sort of combination of physical arguments and mathematics in simple way all of your familiar with this equation from the most classic example of all namely electro statics were you have del squared acting on the electro static potential is equal to in charge density on the right hand side a part form a constant factor, and you know what the solution to what is essentially pluses proportion went up a lets that is the general solution.

So, be on to exploit that fact and I will give of a physical argument to see how you can extend this solutions of this basic equation to high a dimension, but the purpose here is restricted for all these partial differentially equations my fundamental my primary in is to fine the. So, call fundamental to solution to this equation namely the fundamental green function for such operators differential operators which will then help you to right down the solution for the general in to mechanics equation. So, it like to fine the green function for a del squared operator. Now what is a green function just to recall to use in you must already come cast is some courses on differential equation the idea behind green function is a fallen. So, let start with a simplest example namely single we a variable. So, you have a function f of x and a some differential operator d which involves x functions of x there it of is respect to x and so on. So, for acting on this f of x is equal to some other function g of x on the right on the side given function and the question asks it is what the solution to a such an equation and in homogeneous equation the answer is of course that will need to formally invert this operator and some of get it to act on this g here. So, let us do this in abstract notification see where it looks like in abstract notification if this d is the representative some operator d in some space some linear space and this f is represented by this kept vector on this space are used there are notification here this is equal to g on the right hand side this will employee formally that the solution f is equal to d inwards if it exists acting on g formally.

Of course, but that is not all, because you could also have a piece plus a pieces which is some better h i were d acting on this h i is equal to zero an. In fact, you must some you over linear combination of all such i can function. So, i can states such that if you operator acts on it it ungulate it you get zero here. So, i you can see this is the solution of the homogeneous equations d of x acting on some h of x is zero and this is the part at comes from the homogeneous equation and in the language of differentially equations this portion many you right in back in function space this portion what you call the practical (( )), and the rest of it call the complementary function, now of course we can right this back in the opposition space see what happens here?

But remember that this. So, called f of x is nothing but the representative of this cattle actor in the x basis it will like its stands for act then what is this equation really look like it is says f of x equal to and the some operator g acting on d inwards acting on this g and now we can ask in the in the i terms, we can see what is it going to be this is a differential operator some kinds. So, the inwards of it in general would be integral operator in general. So, this in would be of the form and will take real over be explain sum g of x comma x fine little g of x by clause, of course summation c i h i of x were these are the representatives in this bases x bases or of this abstracts specter information base, so this g, here represents this d inwards.

In fact, make it one know size what is this g of x comma x one the... So, call green function for this differential operator this think here is nothing, but x d inwards x one its not be a from this notification that the beside what it is I leave here to your figure this out a minutes very do one minute to this see outs turnover put on next on either side you get x f that is equal to f of x that is equal to x, and then there is d inwards g plus summation over i c i x h over right this is occupied called h i of x. So, that is what this thing is and here now insert a Complete set of state and a complete set of state is integral d x prime x prime x prime that is they identity operator i in set it in between then of course I get x the inwards x prime that is a function of x in x prime between called x the green function integrated over d x prime, and then x prime acting on g that is g of x prime.

So, this is all it is this is all writing it on this abstracted notation its stabile writing down here is this one more step take the metric Salomon between x and explain an that is it this yes yeah we have to settle very about been in exists. So, if it is exists then this is through. So, will have to ask what are the condition and the which these been the were exists etc and what I am going to little later when we talk were integrally equations it is a show you under what conditions you would have this remember exists, but this operator for the moment as you moment exists. So, this thing here is called a green function of the differential operator d x its nothing, but they inwards of the operators expressed in this basis and we need now to find out what this series, but that is very simple we need a in equations for this g that is out had to find.

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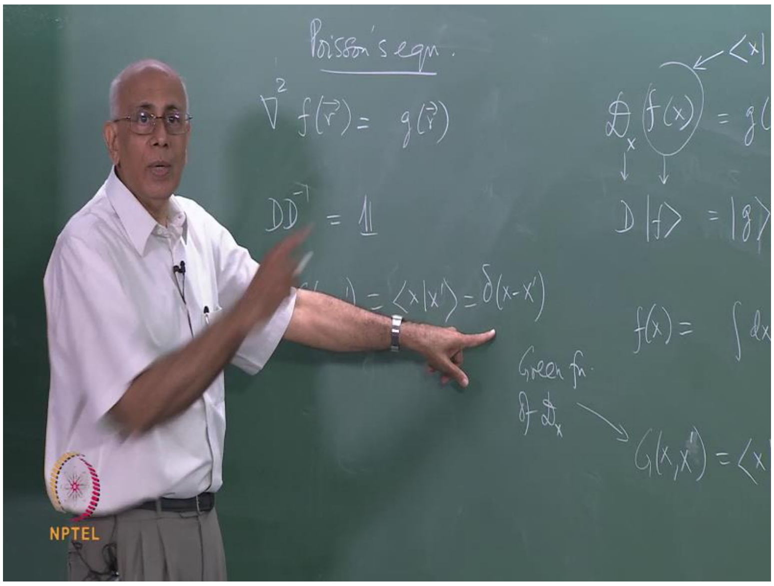

Because notice the d d inwards must be equal to the identity operator, and if I express this in the position basis what could happen this thing here is represented by d x and this i here g of x comma x prime.

And that is equal to the of x with the get of x prime which is nothing, but delta of x minus x prime. So, solve this equation with the delta function of the right hand side, and then you can solve this generally equations by plugging that n here, and then doing this integrate it gives the particular integrate. So, this in a sense is the green function method all your doing this inwards some operator righting it some particular presentation and its satisfies this differential equations and engineer I used call in this unit impulse function awards, I will it is alright then your super power all such response to the unit impulse function multiply the whatever you have there the given function and you have your solution.

Now how does this practice well, if you look at the very simple example let us look at just for to refresh remembly let us look at extremely a simple example oh first one question it should ask our how much they should you put piece put in what decides that yeah all the boundary condition you will decide what these co efficiency are and what that mixer of the complementary function as to the particular integrate as will seen of i am a simple example. So, let us to this example first and come back to persons equations for example, is a standard one I start with a self a joined differential operators.

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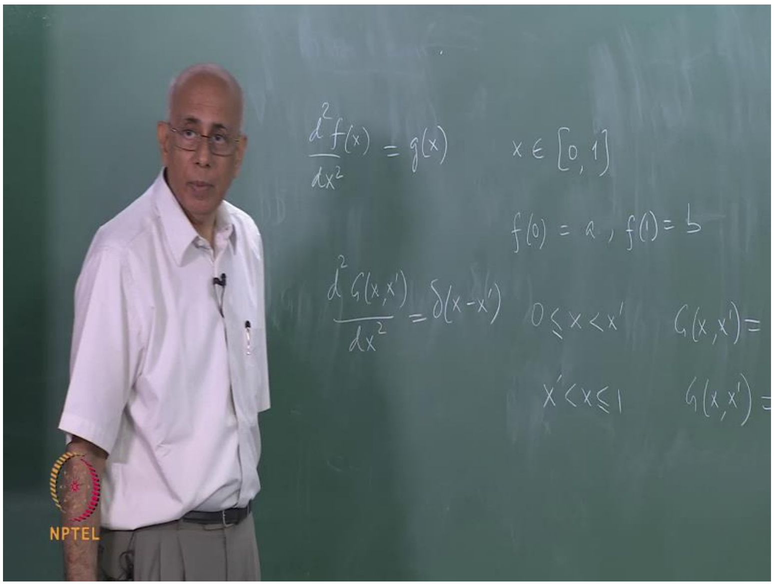

So, it is of the form d two f over d x two, if is a f of x equal to sum given function g of x and like to solve this equation is the simplest of example when we some boundary conditions in in a to say what the range of x etc. So, let us like unit interval a lets a x is a elementary zero comma one. And a need to put some boundary conditions on this f. So, typically would say f of zero equal to something on the other f of one equal to something on the other some constant some other constant call this a call this b. So, subject to these boundary conditions the question is what is the solution to this differentially equation what is particular in here. So, let us right to find out what the green function of this operators and the green function satisfies d two g over d x g d x two equal to delta of x minus x prime in this form. So, what will says is that the second derivate to of this with respect to x is essentially zero expect one x equal to x prime. So, if the big jump of a delta function jump at x equal to x prime. So, let us first solve it in the reason x less than x prime.

Then we solve it in the region x greater than x prime. So, zero plus than equal to x less than x prime what is a solution this equations well this is zero here. So, it says the second derivate of this g as a function of x with respect to x is zero which means that g is just a linear function right. So, this is equal to this implies that zero x comma x prime equal to sum a one x plus a two a one and a two could depend on x prime. In fact, they vote in general depend on x prime, because we just integrated with respect to x alone and in the other region x prime less than x less than equal to one g of x comma x prime equal to some other constant a three x plus a four now we have four constants to a determine.

To give us the full g and what away pieces of information we have tell the first pieces this. So, we put in the boundary conditions I need to put in boundary condition in here to, and then there are two other Pisces of information needed to get on the what the constraints are and what are those Pisces of information what would p well first one is the continuity of g the reason as a such a g is continuous is because it second there a better as a delta function jump. So, what it is mean for the first derivates this is got in infinite jump out here. So, what is it means for the first derivater it means a first derivater was a finite jump of some kind right, because if you integrate this equation if you integrate this equation with respect to x from some for a fixed x prime.

Say x prime minus absolon to x prime plus absolon to integrate d x and integrate this kind here then we says d two g over d x two this is the equal to d g over d x fact x prime plus absolon minus d g over d x an x prime minus absolon immediately if are left on x prime on the right of x prime my integrate about small range you may get is and what these equal to i integrate this across the point x prime in x what is the answer what happens if you integrate delta function you get what this is equal to one. So, in a sense it says that the discontinued d of the first derivative of g at the point x prime is given inventive.

So will need to use that information here, I differentiated both side with respective x and put x prime x equal to x prime minus absolon the one hand process absolon the other and take the difference between in a two for the minus I take this in for a plus a take that and i equate the two at explain. So, that is give you one more condition and what is the last condition of all this says that that derivate about g has a finite jump. So, what is g itself have to have they has to a cast, because it has to be something like this, but as the solves a different and as a finite difference between them we to solves. So, the function g itself look like this, g prime look like this.

And g double prime will have a delta function jump and that on. So, whenever you have a delta function here in the second derivate we know immediately the function itself would be continues, but the first derivate has a known jump. So, that fixes all the conditions the first one is fixed by f at zero then f at the other hand at one then the continuity of g at the point x prime will give you one more condition between the four functions for quantity is a one a two a three a four and the finite jump here gives a one more condition. So, you can put thousand and work out what a one a two a three and a four are and that is green function you can do this for more general boundary condition is the most general condition would be f of zero plus some multiple of f prime of a zero is something else and similarly for one these would be mix boundary conditions both sides and you can right the green function in the general case.

So, this gives you some idea how you handle the green function, what we like to do is sidely simpl problem in some sense with like to do it high dimensions settle in partial differential operators, but it like to get the fundamental solutions. So, in general the boundary condition an gone to take is point to be the green function simply vanishing at infinitely your something like that. So, will take all of space in what happens the green function vanishes at infinitely. So, now, let us go back and look at possess equation we right the solution down by guess derived it more sense to be consideration.

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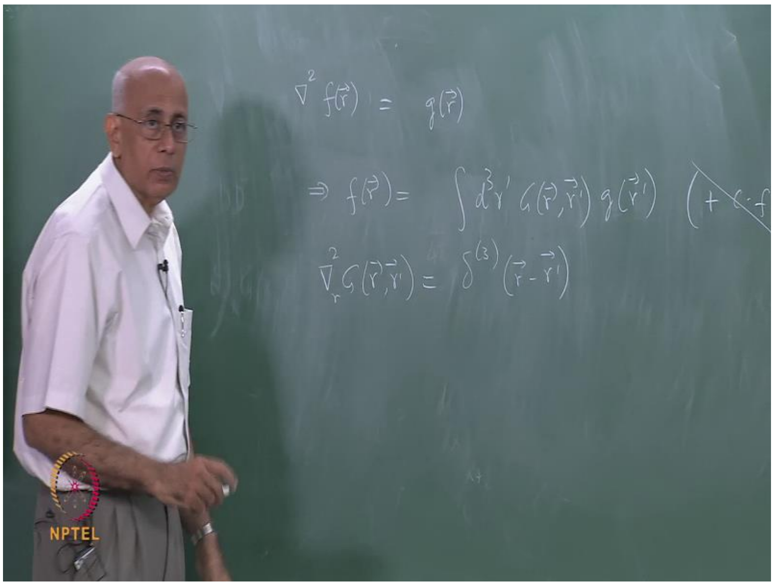

So, if you recall this equations f of r minus p of r over absolon not. If you recall what the solution to this equations this persons equation electro statistic what the solution to it is essentially coulombs law by the solution as f of r equal to minus an integral sorry integral d three r prime p of r prime over modulus r minus r prime one over four prime absolon right, that is the solution is coulombs law, and then the super position principle is used to add up the potential is to all the charges all point very charges the what as that implying for the general equation for the general equation was of the form f of r, this was equal to sum g of r the analogy with is which clear, but the solution we looking at this f of r is minus one over four pi G of r prime over this.

So, the green function in this problem for the delta square operator is nothing, but minus one over four pi mod r minus are point that is green function and we should get this, we should f to derived this regular see that this is in it these function for the three dimensional del squared operator satisfying the boundary conditions that this solution by the way possess equation is written down assuming that the potential vanish that infinitely. So, the boundary condition on the g is that as are goes to infinity which is g vanishes. So, this is the free green function with natural boundary conditions an alright see how to derive this. So, will start with these equation and right the particular integrate this implies that f of r is equal to an integrate d three r prime g of r comma r prime let a g of r prime plus, of course the complimentary function, but I am not interest in that the moment i just want to know what the green functions and what equation does g satisfy it satisfy the same equation as here, but I delta function all the right hand side.

So, you have del square g of r r prime is equal to an delta function a three dimensional delta function of r minus r prime in this case I need to solve this equations put the solution back in here, and then that is set that the particular now just remain myself that the derivative or del square is with respective r that just put an r here to show that it is with res respective the variable r the vector r rather than r r prime now i i said that this equation here impulse immediately that this g it is a function of r minus r prime and not r alone in the reason i do that is, because the right hand side is the function of r minus r prime del square del itself the derivative with respect to r is a same as a derivative the respect to r minus r prime for every fixed r prime is just a shift are these to statements enough to say that the solution are.

So, be a function of only r minus r prime are these enough del square is enough del square it is variant its quite. So, if I put if I put r equal to r minus r prime. So, let this we through then my statement is del r square is a same as del r square it is just a shift. So, this kind could be return del capital r this is delta three of capital r vector. So, can i now a certain this also is a function of capital r vector can I do that in the income the example we looked at the little earlier d to f over d x to we could right hand side was again delta of x minus x prime for the derivate second derivate operator, but that green function did not turn out to be within work it all fully, but it is not of a function of a x minus x prime why not the d two over d x two was also d two over d x minus x prime this again the derivate the delta of x minus x prime in function of a x minus x prime and react the green function the turn out function of x minus x prime what happened yeah, finite boundary is the boundary condition broke this translation in variation the boundary condition brocket.

So, you cannot shift are better, because your finite boundaries in this problem; however, we are asking for a solutions as this vector goes to infinity we are asking for something which goes zero right. So, r goes to infinity r minus r prime also goes infinitely for every fixed r prime. So, in this problem with these natural boundary conditions the boundary condition is also translation in variant the operators translation in variant they in homogeneous term in translation in variant and. So, is the boundary condition. So, we can a set there forth at the solution is also translation in variant a, this will employee now that the equation we have solve.

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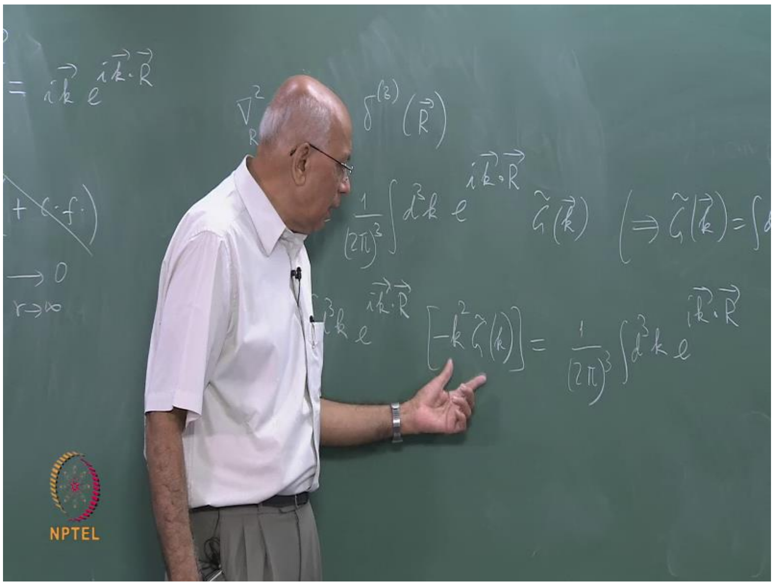

So, we have to the have to the careful therefore,. So, the equation you have to solve is dell are square g of are equal to delta three of the r that is the equation of we have to saw what should be do we should do something which converts the dell square into multiplication in; obviously, to do immediately. So, let us define of a trans form g f r equal to which three an mentions plus one over to pi q t d three k i q r g delta of k and of course, as a industrialization this will also imply that g delta k equal to d three r into the minus i k data r g of r, let us put that in here the substitute representation of g in here and the dell are does not act on that it acts only on this what is dell do in acts on this scalar is to the i k of r what is that you it just being i k vector.

All the does it bring down an i k to be is plane wave is the the represent plane wave the normal to the wave plane is the k direction and the dells always acts is always in the direction maximum rate of increase. So, this is i k. So, what is dells square dell dot this i. So, we got do the diversions of this, but this is not dependent on r and we get on the diversion on the grade this once again. So, it says dell square this minus are. So, that is put that right way it we says one over two pi for cube integral d three k with the i k dot r minus k square d to will dot k is equal to on the right hand side the delta functions, but we can right the delta function itself in terms of the explanation right, that the delta functions we are the transformation just one. So, we can right this as one over two pi whole cube integral d three k it to be i k dot r and this founds completes set of states for all k integrated over k it forms a complete set of states it is in the r space and your equating these to rights. So, component by component they must be equal then you equate to vectors there equal component right component. So, the coefficient of each i to the i k dot r on the two side must be equal which says minus k square g dell dot k must be equal to unit .

So, you called dell of k which minus one over k square that is it. So, ultimately says that this solution yes this quantity is equal to that yeah. So, dell square around this give will this minus k two is, and then I do not this this the delta function this is the free dimensional delta function. So, if I move this to the left hand side it says this minus one is equal to zero. So, it says a certain vector certain function is zero identically and when is the vector zero when every component is zero, I am what are unit specter in this problem it to the i k dot r in function space they are unit specters and for every k this quantity must be zero that is says g dell of k is minus one over k square. So, they argument is completely regulars, but it has to both to you have to both in the reason you can equate for a coefficient it is, because these things form a complete set of states therefore, you can then everything to the left hand side and say an certain vector is zero which means every component is zero. So, each for a components must be zero and therefore, once side put that in here this is minus one over two pi for cube integral d three k e to be i k dot r over k square.

That is g of r, if I can evaluate that integral then i were in expression for g of r i put back in there g of r comma r prime in that is vector which mean the fundamental green function. So, let see how do do this integral what we do to tackle such integrate if you took an coordinate system you are in trouble, because its means k vector point some were in to integrate all direction in k the r vector point some were else we got integrate over on the angle between the two, because in to the i k dot r it would be i model a k model a less r the course sign of the angle between them and that is big is not real do this integral that all if you go do in to the hard way, because if you take two r vector.

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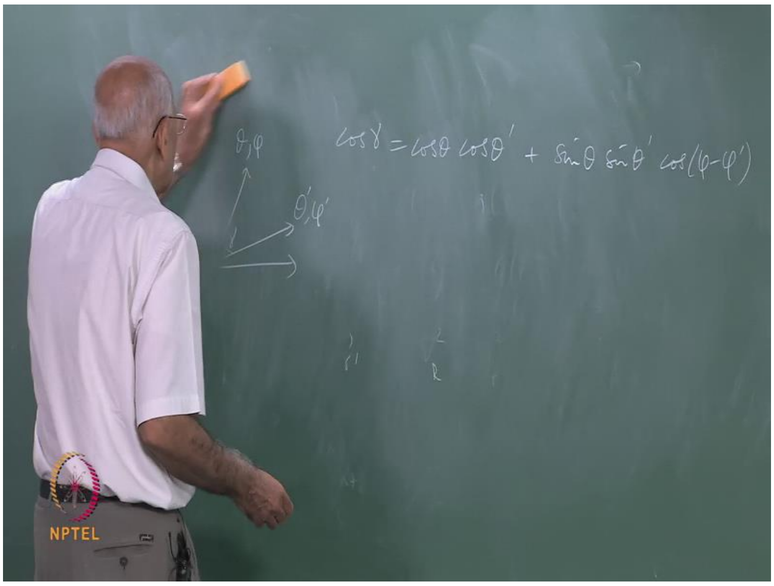

If took some vector like this and some other vector like that this is some polo co ordinates r prime five says and this is got r prime data prime f prime etc we are asking for the angle between these two guide the dia here angle between these two vectors is direction you can re right that you can right this angle in terms of the polo co ordinates of polo angles for each individual vector right which call the law of signs of it standard the problem in spherical trade on bets learn by people, but if the co ordinates here are delta phi and the polar co ordinate c r r delta phi phi prime and this angle is gamma in there is relation which is essentially they addition fear from the general function of order one. So, it says something likes cost gamma equal to cost theta cost theta prime plus sign theta sign theta prime cost phi minus phi prime. So, you have to put that in here, and then integrate of theta prime and phi prime which is a terrible job for able. So, vats not a will do it you could ask an i do it in cardigan coordinates well you could in general except that you have this factors it in here otherwise the this its self in a product. So, you could do this in Cartesian co ordinates, but doing it this case Cartesian again.

So, what should be do we should be do we should clearly look at some interference of this quantity here this here for its a scalar we fact that is it is a scalar means it is variant under rotations. So, you could actually choose your coordinate system as you piece you can choose a polar access of a co ordinate system as you pieces and given any other choice you can bring it the choice by change of variable send the base case now what is this what the vectors that is the sticking sticking out k is integrated over. So, what vector sticking out are vector. So, lets me choose the polar access in k space along this vector r we can do that you can always do that. So, once I do that then the polar angle of k is a costly there is and need to the i k r becomes it will the i care costly that is are a phi dependents know I can integrate over phi and what a say integral over phi give you gives a two part.

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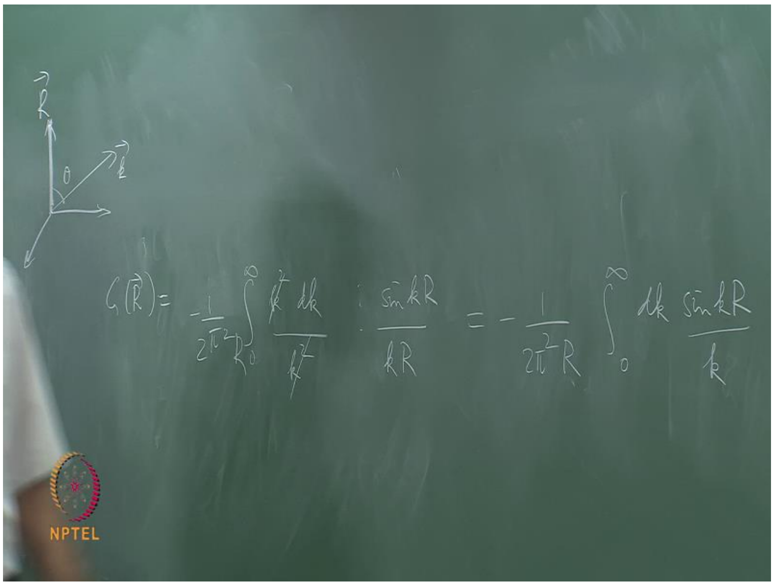

So, this immediately means that you can in case space choose this access along the direction of r, and then k is integrated over all directions, but this angle this theta here and this is a typical k. So, this integral becomes g of r equal to by the way I said the this function is a scalar. So, I really excepted not to depend on the direction of r either it should finally, end of depending on the magnitude this r right. So, I could g r would capital r without the vector sign in, but whether seen that really comes out or not. So, this is equal to minus one over two phi whole cube integrate zero to infinity k square d k, but there is a k square in the diameter which your blazing the cans out, and then there is two phi from the phi integral. So, this becomes two phi square in a done the phi integral on the phi integral know an then have integral over the theta which is minus point one d cause theta e will be i k cause theta i change variable to cause theta which are inform minus one two one. So, this thing is zero e to the i k r minus e to the minus i k r over i k r, but then numerators to i sign k r. So, if get of this in the, this is two i sign k r over i k r.

So, this is four phi square let us make it a minus one over four phi in this is a two sorry two phi square two cancel the i cancels by the by k square cancel we r comes out and you have d k s sign k r over k. So, this is the equal to minus one over two phi square capital r integral zero to infinity d k sign d r over k that is out, but this is the d k integral capital r is positive. So, what is the value of this integrate phi over two sorry as gives here.

So, you have the coulombs potential back again if the four phi is an sign and. So, what. So, in the sense what is dell square is telling you is that the dell square phi in electro statistic potential giving you the charge density on the right hand side, this is not an accident coulombs why directly consequence for the factor they operator dell squad. So, very off in you if you see the statement that green function for dell square is just the coulomb potential. In fact you could define the coulomb potential in a r bitted number dimensional as the inwards for a transfer from a one our case squared or as the green function free green function after dell square operator will do that very shortly.

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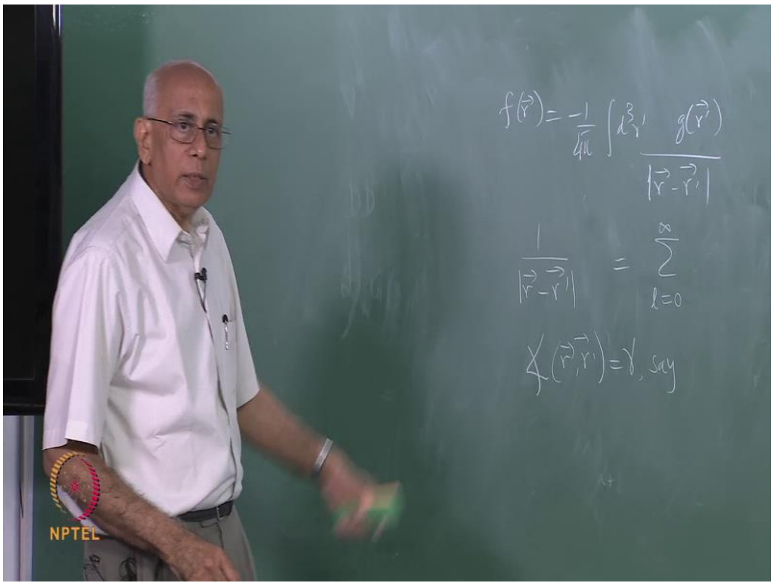

But I want to see how this one over r minus r prime model lessees image and rest of course the rest of course follows, now you can right mark f of r equal to minus one over four phi integrate d three r prime g of r prime over one r minus r prime plus of course the (( )) function, but I took the boundary condition that this went to zero at infinity an then this is the solution. Now of course you can various special cases you could ask what happens if this is a function only of magnitude if it is a charge density. For example, what happens? If it just spherically symmetrical charge density is depends only on the vector in the magnitude on the vector r prime.

And so in all this cases can bellow that and all of you sure of familiar fact this that quantity here one over mode r minus r prime is called the coulomb for; obviously, reasons and you namely when you do this multiple expansion give expansion this crave in powers of r over r prime or something like that here and brought us series look like for is a famous c d, which close like summation l equal to zero to infinity if the angle between r in the r prime for the angle r r prime equal to gamma say if the angle between them is some gamma then this expansion here says it depends on by the r is bigger than r prime or r is less than r prime and the expansion generally goes like this is equal to one over r the greater of the two, and then r prime over r to the power L, and there is a p l of cost r, but if I want to right to in terms of the polar co ordinates of a of a r and r prime separately than I get spherical harmonic expansion of the coulomb which again I am sure well known to from electro magnetism you have to really right this cause gamma back in terms of cause theta cause delta prime plus sign delta a sign delta a etc and use this addition for the general for nomins and this expansion would look something like summation I am not sure about the factors, but l equal to zero to infinity this part of the same r greater r smaller over r greater to the power l, and then they would be pieces which would like as also summation from m equal to minus l to l that be y L m of greater phi y l m star o d delta prime phi prime, and so what some as them from two plus one factors on. So, on depending on how this is normalize, these are a spherical harmonics. So, that is the multiple this gives your multiple expansion in general for this say coulombs potential, but this is to in general form any one over mode over r minus r prime.