Selected Topics in Mathematical Physics
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Module - 8
Lecture - 22
Fourier Transforms (part III)

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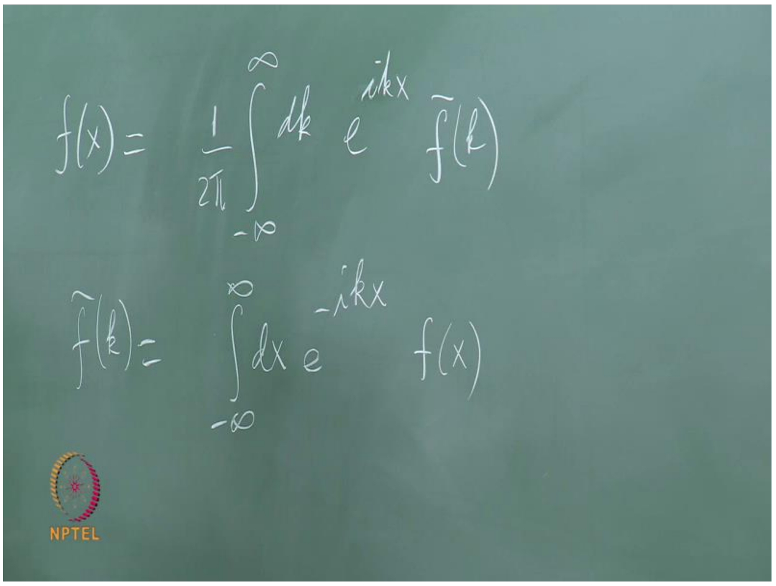

So, let us continue with our study of Fourier transforms. The first thing I want to do is to talk about higher dimensional generalization to an arbitrary number of dimensions. I recall that we had think which come like for a function f of x, we had a Fourier transforms which is 1 over 2 pi integral minus infinity to infinity d k e to the power i k x f tilde of k, and the inverse transforms was f tilde of k was integral d x minus infinity to infinity e to the minus i k x f of x. So, these two functions formed a Fourier transform pair. Now, we can do the same thing in any numebr of dimensions.

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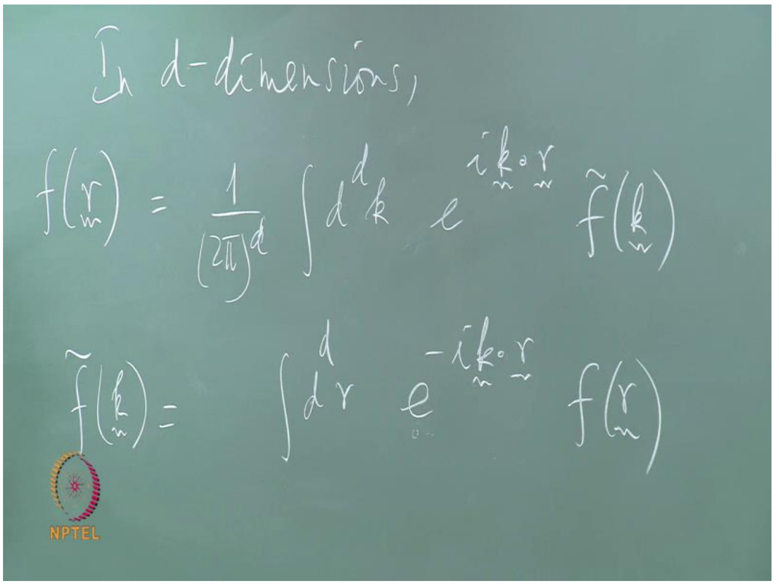

So, if you have a function of a vector r in d dimensions say d if d dimensions, then you can define 1 over 2 pi to the power d integral d d dimensional volume integral over all case space, e to the i a dot r f tilde of k, where f tilde of k is integral d dimensional integral over r e to the minus i k dot r f of x. So, this is in d dimensions, this stands for the volume integral in k space; this for the volume integral in r space.

And essentially all the theorms we prove poisson therom and so on and so forth, they are go through in this simple generalization. More interesting than this is what happens if you have a vector valued function, for instance if you have the electric field in some space or magnetic field in space and so on. You can also write Fourier expansions for vector valued function.

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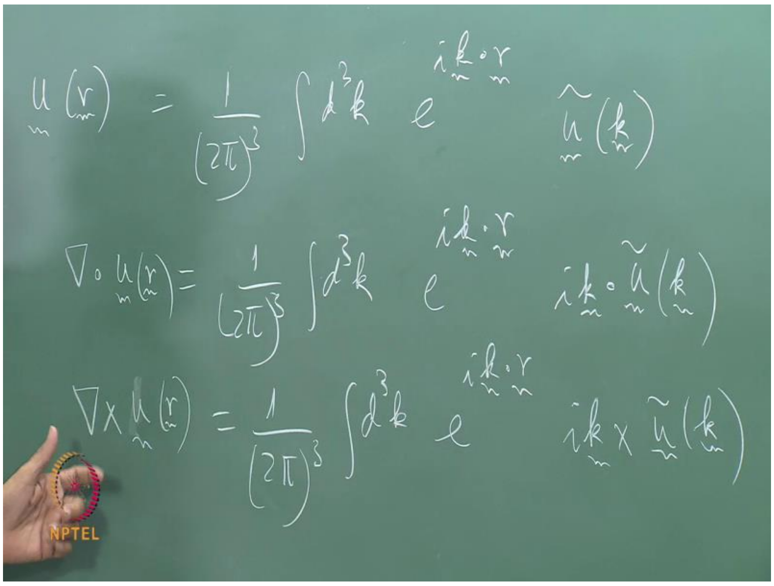

So, if you have for instance a vector field u of r, let us do this in three dimensions, because that is the most common example. So, u of r is 1 over 2 pi cubed integral d 3 k e to the i k dot r, u tilde this itself a vector of k in this fashion, so that is the Fourier expansion, and correspondingly there is an inversion formula. Now, the great advantage of Fourier transforms is that just as Laplace transforms converted the operation of differentiation with respect to t to just multiplication by s. And exactly the same way the Fourier transform operation takes special derivatives with respect to the components of this factor r, and converts them essentially to multiplication by corresponding components of k.

In the following sense, for instance if you looked at the divergence of u, it would be del dot u this is equal to 1 by 2 pi whole cube integral d 3 k, and then the divergence the derivative with respect to the components of r acts on this function here, and what is the divergence of a vector times a scalar. This vector is independant of r, so it just it is there ideally. And all that happens is that it is again e to the i k dot r, and then i k dot u tilde. So, the operation of del dot is equavalent to the operation of i k dot on the Fourier component.

So, if you take the divergence of a vector in three dimentional space that corresponds to taking the dot product with i k on the Fourier components u tilde of k. In exactly the same way del cross u of r these are all functions of r is 1 over 2 pi whole cube integral 3 k, again it will be i k dot r we need the formula for the cross the curl of the vector time is scalar, the vector in this case does not involve or at all. So, just have one term, and thats i k cross u tilde of k. So, the dot product with del is converted to i k dot, and the cross product is converted to i k cross, this means that these operations these partial derivative operations are converted to algebraic operations taking scalar product dot or cross products with this vector i k.

Now, of course as you know in Maxwell equations for example, that is a set of equation in which the spacial derivatives are always see the curls or the divergences of the vector field the a and b fields. And if you Fourier transform them, then Maxwell equations get converted to the bunch of algebraic equations for the Fourier transforms. Of course, you also have partial derivatives with respect to time, and therfore you do a Fourier transform with respect to time in the conjugate is the frequency omega. So, derivative with respect to t becomes just multiplication by i omega or minus i omega depending on the convention.

So, you can convert Maxwell equations to a bunch of algebraic equations for the Fourier components, and then of course once you solve for that you can re in you can invert the transform in principle to get the original fields. So, it is extremly useful. Now, I tell me why is it just to recall to you, why is it that the differential equation of physics like those in fluid dynamics for instance or in electro magnetism, why do they specify the divergence and curles of a vector field, why not some other derivatives?

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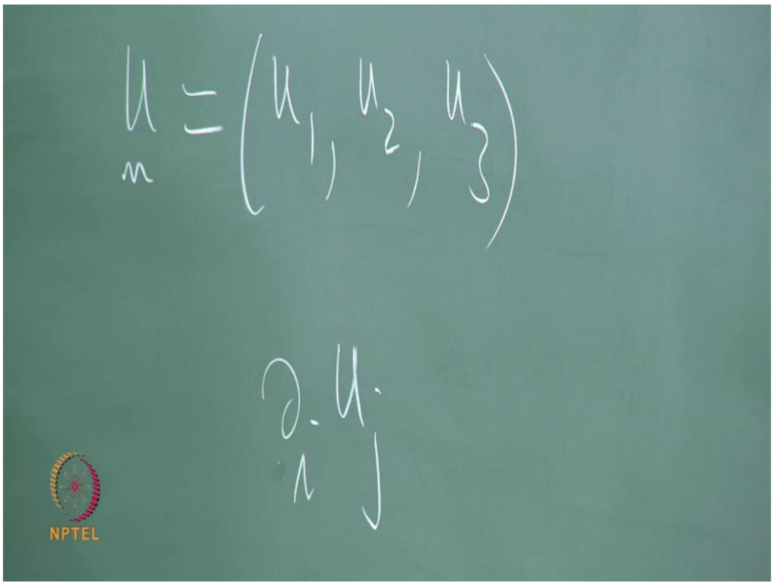

After all if you have a vector field u, and it is called components u 1, u 2, u 3 say, and del has components delta o, delta x, delta o, delta y etc you have 9 possible derivatives, but only one that seems to play a role are the curl and the divergence. So, if you wrote these derivatives down if you wrote delta i u j, this stands for delta o, delta x i where i rounds 1 to 3, this is the second rank tends it is called 9 components. And the divergence is just the sum of the diagonal components, and the curle is just the set of 3 numbers form by taking the differences of the anti symmentric components, why do those two play such a fundamnetal roles, what is the reason do you, what do you think is the reason?

Student: ((Refer Time: 07:21)) there is given yes, and provided it is well behaved on infinity, then you can find out that.

You can find it you neatly this is called ((Refer Time: 07:32)) therom, it essentially says that if in a region you specify the divergence and curl of a vector field, and you specify the normal component of the curl on the boundary surface, then the field is uniquely determined by this you know. Now, of course there is one more physical reason why the divergence and curve play a special role, because they have specific transformation properties, this is the second rank tends here this thing here, but the divergence is the trace of this matrics, and is therefore invariant under rotations. So, it is scalar, transforms like a scalar, on the other hand the curl as you know transforms like a vector.

So, these combinations have well defined transformation properties that is crucial and what happend? So, infact what you normally do? If you recall if you look at the multipole expansion for the potential due to an arbitrary charged distribution for instance. Then you have a monopole term which is the scalar, you have a dipole term which is vector, then you have a quadrapole term which is a tenser of frank tool, but very special tenser of frank tool, it does not have nine components, but it has five components. The reason is it is a symentric tenser.

So, it only has six independent component and then it is trace less, because you get rid of the scalar part which is the trace of the tenser scalar get rid of it, it is already gone in the monopole term subtract it out, and then you left with 5 and so on. So, what happens in general of course is that if you have a tenser of some rank r, then you have 3 to the r components in three dimensional space, but then when you convert it you remove all the lower dimensional representation of the rotation group, and then of course you get spherical tenser, and the spherical tenser of rank r has 2 r plus 1 components.

So, 3 to the power r gets converted to 2 r, in the case of quadruple 9 components gets conveted to 5. The next stage it be 21 components to the next multipole moment, but its gets removed the earlier wants are all subtracted out and then you end up with 7 of them and so on. So, to get back to a viewer the divergence and curl play a very very special role because of this therom, and we see here that once you have this function e to i k dot r acts like some kind of eigen function for the del operator. And del dot becomes i k dot, del cross becomes i k cross, therefore life becomes extremely simple once you Fourier transform.

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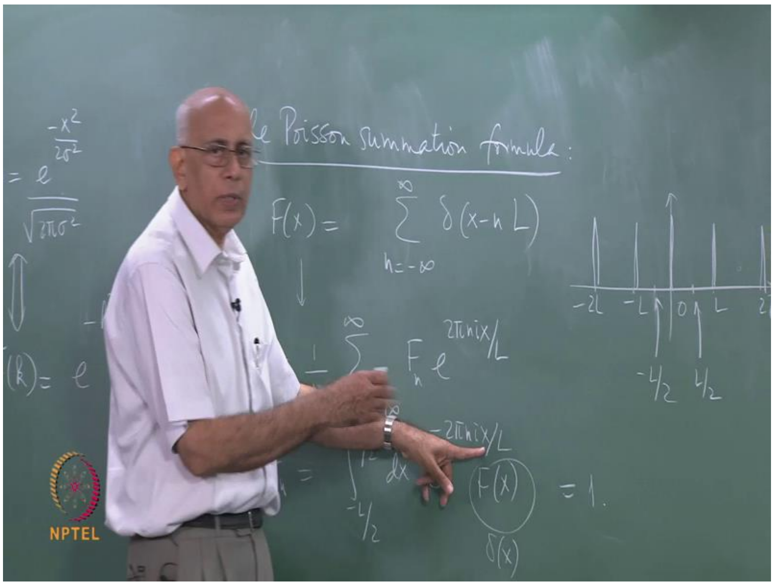

Now, let us now change topics and go on to a very very important formula in a Fourier analysis, and this is the Poisson summation formula. We come to derive this formula in it is simplest form very very elementary form, but it is a very profound result and it has very deep implications in many branches of mathematics. We cannot going to do the complex complicated parts of it, you going to see what this formula does for you, and it does something fairly non trivial. In some sense what it does has already been seen by us when we looked at the Fourier transforme of a Gaussian.

You recall that if you have a Gaussian function f of x was e to the minus x square over 2 sigma square say 0 mean thats to be simple divided by 2 pi sigma square a Gaussian centered about the origin. Then the width of this Gaussian, the full width at half maximum for instance is proportional to sigma square, the larger sigma square is the wider this Gaussian is and the narrow over it is the smaller sigma square is that steeper, the sharpen the Gaussian is. On the other hand, these thing implied that f tilde of k was equal e to the minus k square sigma square over 2.

So, very broad width in x pace let to a very small width a very in case base and vice versa. So, if sigma is very large this thing here sits here in the numerator as a positive denominator, and just does just the opposite. We saw these interms of the answer in the principle for instance, it is said that if you have a wave packet in x base for the postion of a particle, and that is very broad then you expect in momentum pace the distrubution is rather sharp and so on and vice versa. Now, the Poisson summation formula generalises this and shows you presisly where this relation comes from.

In fact, it is this case is so important that this is occasionally call the Poisson summation formula, but in the sense of summation which I am going to talk about, but that is this is not it in more general form. So, we start with the following let us start with the function of x which is just a sum of delta functions periodic array of delta function. So, this summation n equal to minus infinity to infinity delta x minus n times L, where L is some lenth scale some constant positive constant.

Now, what is this function look like? It is just an array of delta function. So, as a function of x there is one at the origin, there is one at L, there is one at 2 L, here is one at minus L, minus 2 L and so on is just an array of delta function it sometimes called the deract comb looks like a comb, and this is just a bunch of deract delta functions. Now, we could regard this as a periodic function with the fundamental period that is that lies between minus L over 2 and plus L over 2, and this delta function is repeated with the period L, and the fundamental interval is minus L over 2 to plus L over 2.

Then of course I can also write this same f of x as equal to 1 over L summation over n equal to minus infinity to infinity f of n e to the 2 pi n i x, you write it in a Fourier series and the period is L. So, that the Fourier series expansion of this periodic function which is just an array of delta function. This immediately implies that the inversion f of n is equal to an integral from minus L over 2 to L over 2 dx e to the minus 2 pi n i x times f of x, x over L that is the inversion formula for the Fourier coefficients of this function f of x, but in this range this f of x is a single delta function at the origin. So, this guy here is essentially delta of x that is the only one which contributes in the fundamental period that will immediately of course we give 1, you put in this delta function and do the integral in a set x is equal to 0 and that is it, it is equal to 1.

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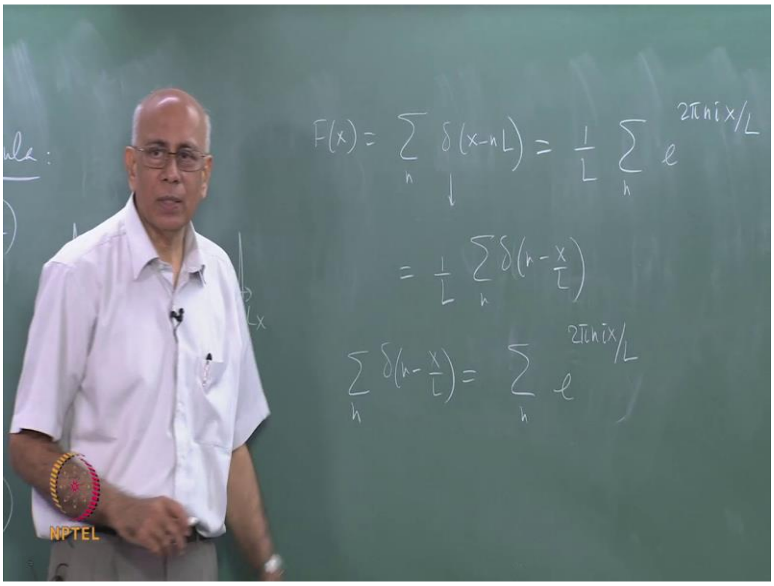

So, going back here this say that f of x is summation over n, if I do not write limits it means from minus infinity to infinity this for short delta of x minus n L is also equal to one over L summation over n e to the 2 pi n i x over L. I would like to get read over these 1 over L. So, let us write this as equal to take out the L, so there is a 1 over L summation over n delta n minus x over L, because we call that the delta function is a symmentric function of its argument delta of minus x is same as delta of x.

So, this gives us a very very interesting results. It says a summation over delta functions of the form n minus x over L over n is the same as the summation over n of exponentials. So, some of these exponentials is equal to this, and of course you could ask how is going to happened this delta function are all extremely sharp, the answer is of course the distractive interface between these different core science and science are that they cancel each other, and give you essentially peaks from the scale. So, that is the first part or first result that we need.

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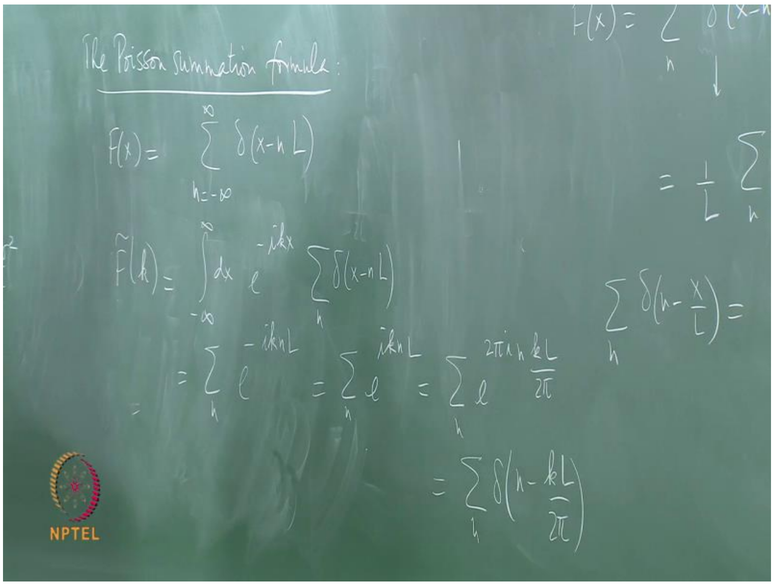

Now, let us do the follwoing let us go back here, and say look I am going to regard this function f of x as defined from minus infinity to infinity, and let us do it Fourier transform when what do I get, f tilde of k is equal to summation is equal to an integral minus infinity to infinity dx e to the minus i k x times summation over n delta of x minus n L, and the this is equal to summation of over n, and I pick up the contribution from the delta function everywhere. So, it is equal to e to the minus i k n L, but since the sum is from minus infinity to infinity, I can change n to minus n on the sum does not change.

So, this is also equal to summation of n e to the i k n L just by changing n to minus n does not change the summation. And now, I would like to use this formula, I like to use this. So, what I do is to write this as equal to summation over n e to the 2 pi i. So, let me write this, so that it is transparent 2 pi i n x over L, so e to the 2 pi i and then there is an n x over L and that becomes delta of n minus x over L here. So, let us do the same thing here, you have 2 pi i n k L over 2 pi and that exactly of this form. So, this becomes equal to where does it go, it is equal to summation over n, a delta function of n minus k L over 2 pi. So, f of x is this array of delta functions, it is Fourier transform Fourier integral Fourier transform is also an array of delta functions from this kind.

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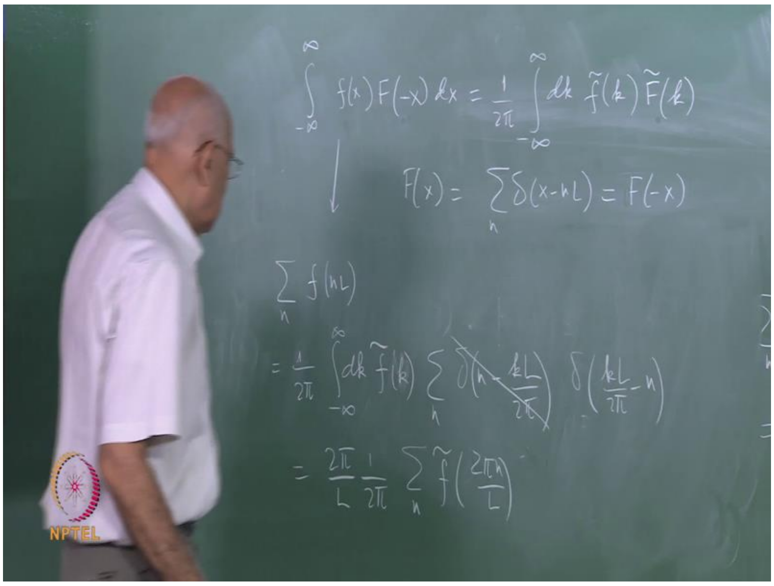

Now, we go back and look at the generalization of the ((Refer time: 19:45)) therom that we already did used, and what that result said was the following said that if you have two functions f of x and capital f of x, then the integral from minus infinity to infinity dx was equal to 1 over 2 pi integral minus infinity to infinity in case pace f tilde of k F tilde of k that was the generalization of ((Refer time: 20:24)) theorm. And if you recall I said if you said this F of minus x capital F of minus x equal to f star of this function, then you got the ((Refer time: 20:34)) theorm which said that mod little f of x whole square integrated is equal to 1 over 2 pi mod f tilde of k whole square that is the norms were essentially equated in the k basis and x basis, but this is the generalization of that result.

And now, what I am going to do is to substitute in this equation here, I am going to substitute for F of x from here, and I am going to substitute for f tilde of k from here. The F of x is this delta function array, and incidently notice also that in this instance F of x is equal to summation over n delta of x minus n L it is also equal to F of minus x, because all I have to do is to change n to minus n, once I put minus x here I change n to minus n then I back to the original formula. So, it is again symmentric and as we saw it is an array symmetrically distributed about x is equal to 0. So, if I plug that in this immediately says summation over n f of now x is replaced by n L.

So, n L is equal to 1 over 2 pi integral minus infinity to infinity dk f tilde of k, and then sumation over n and this thing here delta of n minus k L over 2 pi, and I like to do the k integration. So, I need a delta of k minus something or the other to do that and thats easily obtain, because I can also write this as delta of k L over 2 pi minus n, and I can take out this constant L over 2 pi. Now, what is delta of a x is equal to where a is a constant, 1 over mod a times there is and in this case a is L over 2 pi. So, this becomes 2 pi over L and then there is 1 over 2 pi summation over n, and I pick this up so summation over n f tilde the of 2 pi n over n.

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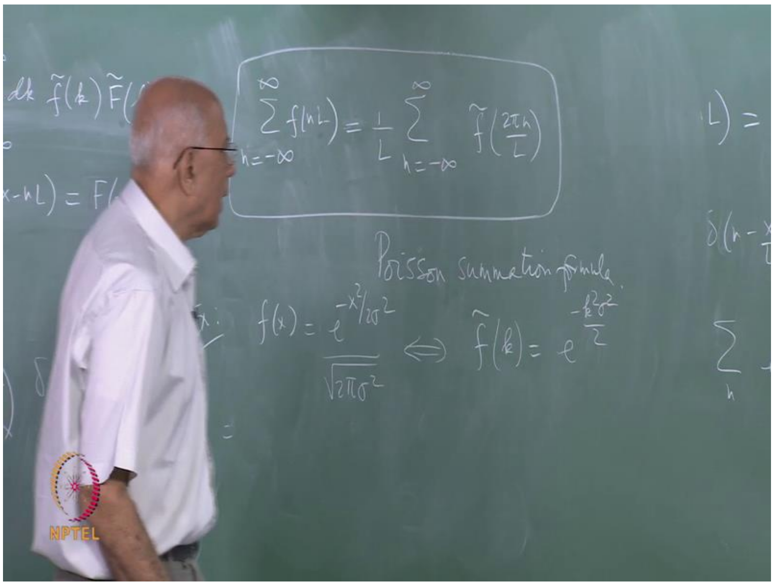

So, we have a final result which is the Poisson summation formula, and it is says that the f of x is a nice function, when summation over n f of n L where n is L is any arbitrary positive constant, this is guaranteed to be equal to 1 over L summation over n. And now let us write this out minus infinity to infinity f tilde, this is the famous Poisson summation formula. So, it is says if you sample of function at all the integers some integer multiple of some constant L, you get the same thing as what you get if you sample its Fourier transform at integer multiples of 2 pi over L.

So, L of course is very large, then the sampling in case space has to be very small sample many very much more frequently. So, notice this is the crucial point that L appears in the numerator here, and it appears in the denominator here this is a conjugate fashion if you like. So, this entire business with delta function and so on and so an artifact, I put L is an arbitrary positive number, for any positive number this is true this relation, and that is the famous this is the summation formula in this instance.

Let us applied it now to the case of the Gaussian which is what I said I would do? And in the case of Gaussian I know what f tilde I so I know that f of x is equal to the e to the minus x square over 2 sigma square over square root of 2 pi sigma square, so that is an example. This implies that f tilde of k is e to the minus k square sigma square over 2.

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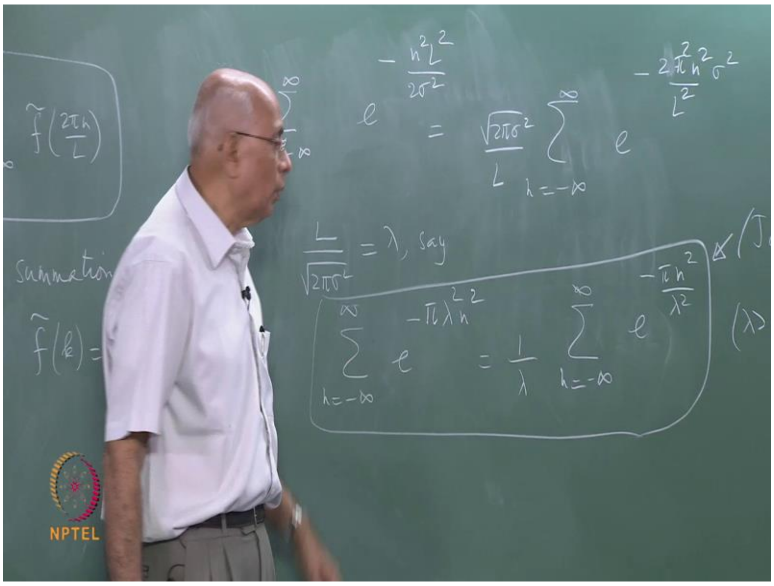

Let us put that in, and then you get this very remarkable formula which says summation n equal to minus infinity to infinity e to the power minus instead of f x you put n. So, it is n square L square over 2 sigma square over square root of 2 pi sigma square, this is guaranteed to be equal to1 over L minus infinity to infinity e to the power minus instead of k you have 4 pi square n square over L square sigma square over 2. So, let us move this up here, and you have equal to root 2 pi sigma square over L where too many constants here there is L and sigma we can combine these guys.

So, let us combine them in the following way let us put L over root 2 pi sigma square equal to lamda say, and then what does this give you? L square over 2 sigma square is pi lamda square, so summation n equal to minus infinity to infinity e to the minus pi lamda square n square that side is equal to 1 over lambda summation is equal to minus infinity to infintiy e to the power minus pi, and then 2 pi sigma square lamda positive. So, this is an extremely remarkable formula, because it saying something about a sum over the integer of not e to n it is not linear in n, but it is a quadratic in n such sums are called Gaussian sums, they have a large number of weared intricate properties, and the whole theory of theta functions and so on involves such sums.

Now, notice what is happen it says that sum over such exponentials e to minus pi lamda square n square is equal to the lamda appears in the denominator here, and it is an identity. This identity by the way is due to Jacobi and it is so important that is somtimes call the Poisson summation formula itself, but actually the summation formula is this, and that is the special case of it here. Now, lots of implications here, in physical problems you might end up with lamda is suitable as a prime meter, and you would like to study it is small lamda behavior, large lamda behavior and so on.

And now, these two things will give you complementary views for the same thing, for instance in the diffusion problem which will talk about little later, this lamda will turn out to be some proportional to the time itself in a difussion problem turned out to be diffusion constant times t little t. Then we would like to see what the solution of a diffusion equation does for small values of time, and for a large volme by long times. And this identity here will give us a handle on that, this will tell us how which formula is better at small times and at a long times.

It is clear that if lamda is very large this dies down as an increases, so you could approximated it with the first few terms, and if lamda is small then you would do the same approximtion on that sum here, and a different representations all together. Now, one can play this game with any function. So, we already looked at one more example of a probability distribution, and that was a Gaussian distribution let us see what it says in that case.

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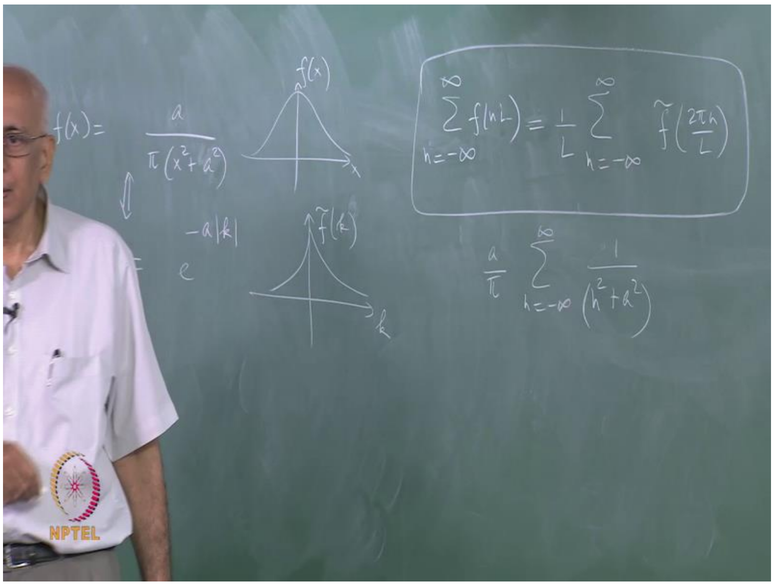

So, in case of the Gaussian distribution by f of x was equal to some lamda over pi times x square plus lamda square that is not the ((Refer Time: 30:37)) lamda appearance. So, let me call this a for the moment x square plus a square, and what was the Fourier transform of this function f tilde of k was equal to this is a normalised probability density, normalised unity from minus infinity to infinity. And it be it behave like this, this thing or some bell shape curve of this kind, where this width was proportional to a, and f tilde of k in this case was e to the minus lamda modulus k a moduls k, it is a characteristic function being the Fourier transform of a probability density.

So, f tilde of 0 must be equal to 1 for normalisation, and what is this function look like? It is a exponentially damped and it is ((Refer Time: 31:34)) per the origin in case x. So, let us plug that in there, let us put that let us put n equal to 1 for instance, and then this says a over pi summation n equal to minus infinity to infinity f of n, so that is 1 over n square plus a square recall we already found the sum here, this gaves you the lines of function it was pi over 2 a caught hyperbolic pi a minus 1 over pi a or something like that. So, here is another way of doing it, and that is use the Poisson summation formula, and this was equal to summation n equal to minus infinity to infinity f tilde of 2 pi n in this case e to the minus 2 pi a modulus n, we could simplify that.

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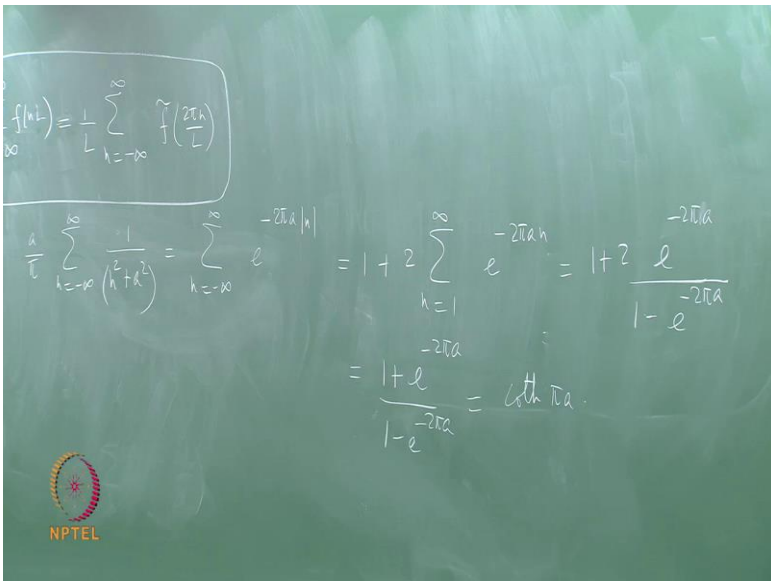

We need to simplify this, so this is equal to, so let us take the term which is n equal to 0 that just a 1 plus twice the summation n equal to 1 to infinity e to the minus 2 pi a, what is this equal to a is some positive number, what is that equal to that is it geometric series. So, it is trivial to sum and this is equal to 1 plus 2 times e to the minus 2 pi a over 1 minus e to the minus 2 pi. So, that gives you 1, so that is equal to 1 plus e to the minus 2 pi a over 1 minus e to the minus 2 pi a multiplied numerator and denominator by e to the pi a, and this a gives you cot hyperbolic pi a.

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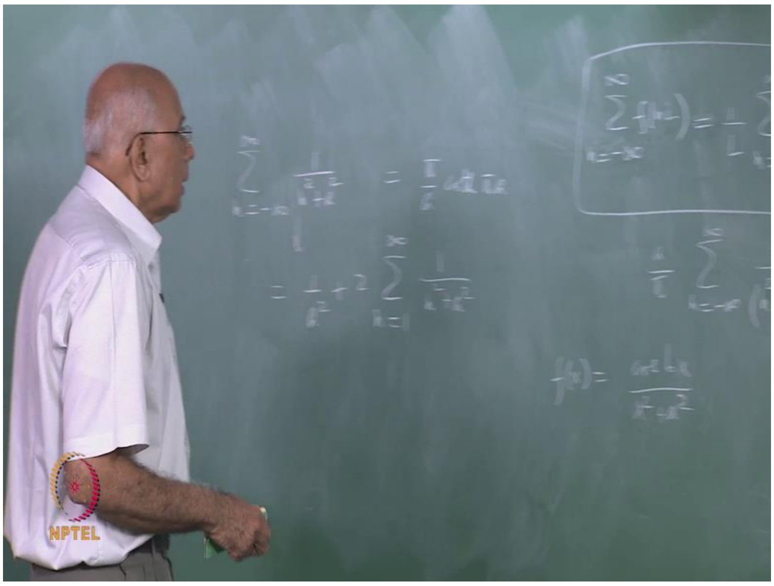

So, we back to this result we already know, which essentially says that summation n equal to minus infinity to infinity 1 over n square plus a square is equal to pi over a cot hyperbolic pi a. Now, of course what we should do is to separate out n is equal to 0 term rewrite this, so you can rewrite this also as equal to 1 over a square plus twice summation n equal to 1 to infinity 1 by n square plus a square equal to that, and then you get the usual expression for this sum. And if you recall in this sum, if I said a equal to 0 what do I get? You get the zeta function for two argument two, so you get pi square over 6 and so on.

You can do this little more generally, you could actually start with f of x, and I will leave that as an excercise to you, start with f of x equal to some cos b x over x square plus a square, this whole thing is for b equal to 0, but you can do the same thing the cos b x here, what you have to do is to find it is Fourier transform, and when you do that write this as e to i b x plus e to the minus i b x over 2 combine it to the k, and then you get once again you get Laurence ((Refer Time: 36:03)) whose Fourier transform you already know. So, I leave this is an excercise little generalization of this formula.

Now, once again as you would expect this summation formula also is generalizable to high dimension, you can do this in any number of dimensions. So, these things here this n instead of that you have r to pulls of numbers of integers, and you can now to do Fourier transforms and higher dimensions is in given analog of this summation formula here, and there are further generalisations. So, there are lots and lots of very interesting results which starts with this very elementry Poisson summation formula, and then go on to do go on to fairly intricate things about the heat kernel it is as an entotic behavior and so on.

We will come back to the use of this formula then we saw the diffusion equation which going to write down the fundamental green function for the diffusion equation, and when we do that we will see the use of this the actual practical use of this formula. In addition of course to the fact that when you have a relation like this, it is useful for summation of series, because there cases when you can sum this and this is harder to do. In this instance for example, this side here was just in geometric series where this side here was much more complicated here. On the other hand you see immediately how this formula is written immediately helps you to find one or more, the left hand side given the right hand side sometimes it happens the other way.

In the Gaussian let us equal in both sides the both Gaussian sums and what we have done is to show that two guys in sums are equal in this specific formula. Many, many other things you can unuse yourself by doing this various other probability distributions in this case, you can see what happens to this formula. So, this is about all I wanted to say about Fourier transforms, we will return to the use of Fourier transforms when we do scattering theory, we want talk about integral equations, and at that time we will come back to this to some of these formulas here and use them in its.