Selected Topics in Mathematical Physics
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Module - 8
Lecture - 21
Fourier Transforms (Part II)

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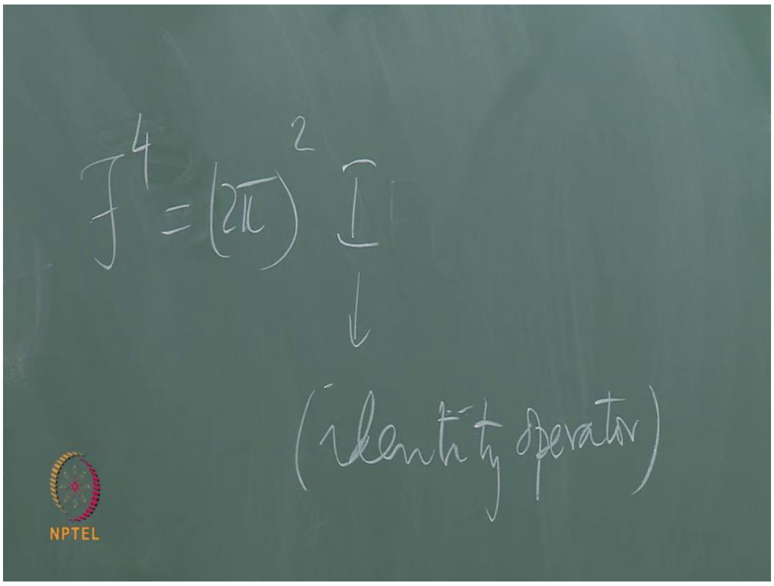

Apart from this 2 pi factor you can get rid of that 2 pi by defining 2 pi to the minus half times f as the Fourier transform operator. And then of course it is fourth power is the identity operating, now these factors are used in Fourier optics very often it is very common to use this, because as you know thin lengths essentially acts as a Fourier transform device. So, they have immediate application, but they follow from this simple mathematical property here. We also have been talking about the fact that momentum space wave functions of Fourier transformer opposition space wave functions, if one is in l 2 the other is also in l 2 square integral.

So, in some sense we would like to be able to be assured that one of them that two descriptions either in position space or momentum space are unilaterally equivalent. We would like to be assured of this, which means we need to show that we already know this from Perceval formula, but we need to show explicitly that the Fourier transform operator is a unitary operator preserves probability etcetera. So, let see what happens we can directly establish this or not.

Student: (Refer Time: 01:23)

Yeah.

Student: only transformer whose square will be the para.

No, no there is not true, yeah I mean you cannot it is like asking what is the square root of the identity matrix 2 by 2 matrix, let us take 2 by 2 matrices what is the square root give me a matrix whose square is the identity matrix. Of course, you give me the identity matrix itself you would give me minus 1 on the diagonals that 2 is true.

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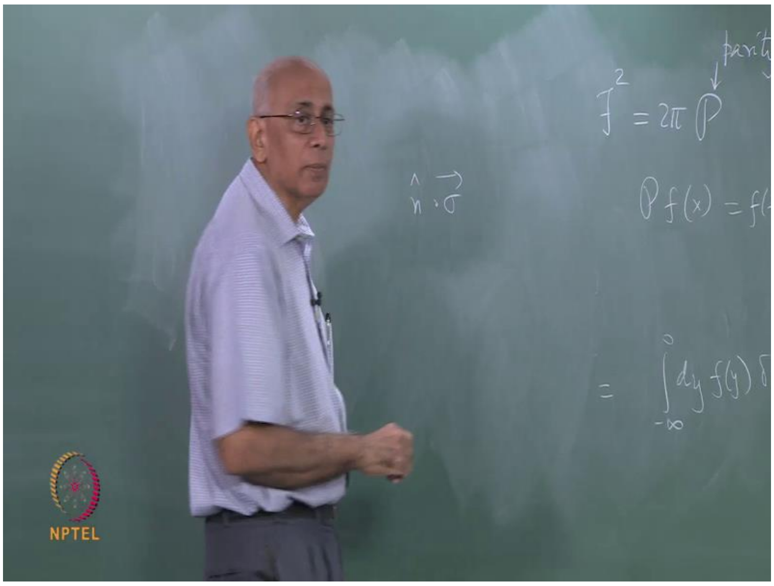

But the fact is that if you give me any matrix of the form a dot sigma make this a unit vector dot sigma n 1, n 2, n 3 such that n 1 square plus n 2 square plus n 3 square is 1, and I square this guy I am going to get the identity matrix. So, will you have continuous infinity of such square root if you like, so the question of when you have a unique square root for an operators an interesting question. Well if the operators should turn out we what is called a positive definite operator, then there exist a unique positive definite square root for this operator that is useful for many, many context, but no not the Fourier transformer is not that sort of operators.

You know this business of taking square roots is very interesting, because square root of operators really profound many, many profound in follow from this. If you took the client garden operator the box plus m squared operator, and to get square root you get the Dirac operators that is what happened when Dirac linearised, this box plus m square and so on you got the get on. You could also ask what about other operators like translation operator what is it is square root? And then you end up in quantum field theory with what are called super symmetric operators, so super symmetric follows the operators something which is like effectively the square root of a translation operator. We will talk a little bit about this when we do the client or when talking about the client garden equation.

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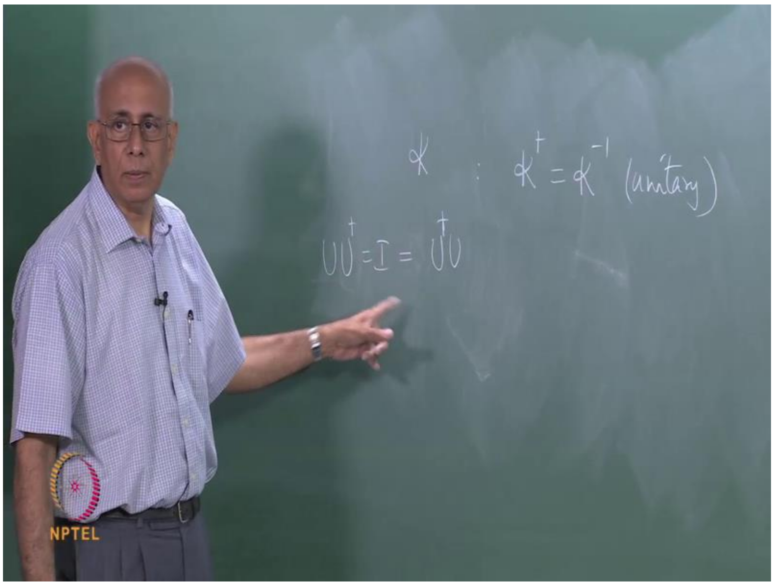

So, let us establish the unitarity of this operator this Fourier transform operator I recall to you, but if you have an operator k, then by unitary I mean k dagger should be equal to k inverse that is what I mean by unitary. So, I need a unique inverse I need k k dagger equal to k dagger k equal to the identity, we have finite matrices then there is no problem because finite dimension matrices. If you have U U dagger equal to I, this will implies that U dagger U is also equal to I, because then the right and left inverses are the same, but in infinite dimensional space this is not guaranteed unitarity requires both these.

We need both these conditions the different conditions, the right inverse and the left inverse must exist and be the same thing b equal to U dagger, the ad joint of the operator U. So, we need to establish this, but this is an integral operators first we need to know what is meant by the ad joint of this integral operator. Well if it is a matrix, then I know what is meant by the ad joint says take complex conjugate transpose that is the ad joint.

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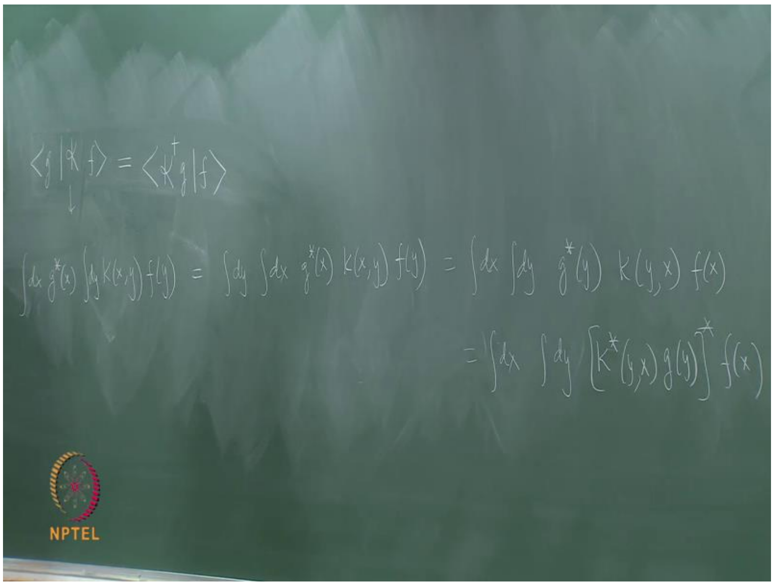

On the other hand what is it for an integral operator, let us start again and how do you find the ad joint. Well you take this k act on any vector and the function space, take the scalar product with some other vector in the function space. So, after act on this you identify k dagger by saying is there an operator k dagger, such that this is equal to that for all pair’s f and g in this space. If there is then this k dagger is identified as the ad joint of my operator k. So, let see what it does for integral operators, the translation of this says take integral k of x comma y f of y, you integrate over whatever the range is that is this vector.

And now take it scalar product with this function, which means you have g star of x integral d y d x that is what the left hand side is, this is clear. So, it say's f is represented by f of y in that function space, this is represented by the kernel k of x comma y, and integrate over y I get another element of the function space as the function of x. Now, I take g star of x that is this guy here, because it is draw vector here integrate over x and that is my scale of product. I need to be able to write it in this form.

So, what do I do the first thing I do is to interchange order of integration, so I get d y integral d x g star of x k of x comma y f of y this is the first step. Next let us exchange the labels x and y, so this becomes equal to integral d x integral d y g star of y k of y comma x f of x, which I can write as equal to integral d x integral d y, and then let us call this k star of y x g of y star f of x. Well this is the complex conjugate integrated over d x and d y, but this guy here is precisely this, and when it is multiplied by f of x integrated over x I get this.

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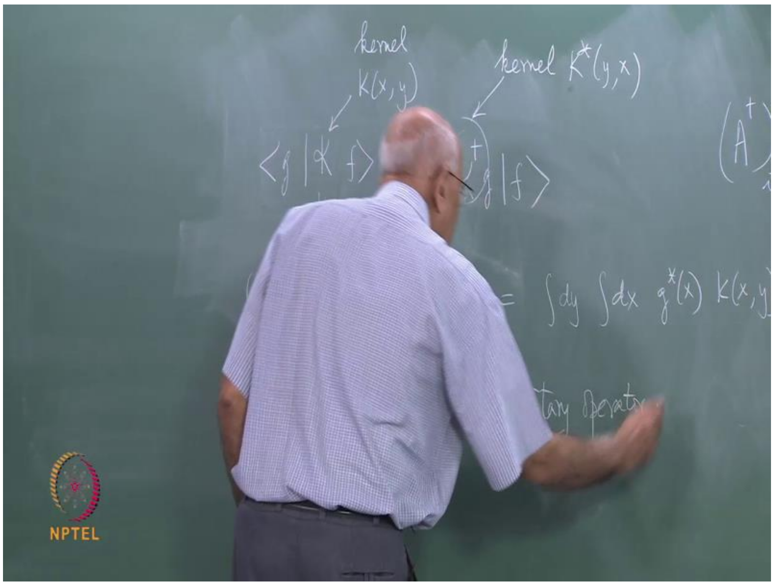

So, what is this means? It means that if this is represented by kernel k of x y this operator is represented by the kernel k star of y x, that is exactly what happens in the case of matrices. After all what is the function space is just a continuous analog of matrices in which the rows and columns essentially the labels, the row and column labels is indices become continuous. So, if I say a i j star a i a in the complex the i j th element of ad joint of a matrix a is a i j star equal to a j whatever.

That is exactly what I am saying here I interchange this x and y, and then I take a complex conjugate, which is what you would expect that will painfully gone through it in this function. Now, what about the case of Fourier transforms, well the kernel in that case remember I had f tilde of k equal to d x e to the power minus i k x f of x. So, the kernel in that case was this guy, and what about the inverse transforms? It was e to the plus i x k. So, the other kernel was apart from this 2 pi factor, you had in the other case you had d k e to the i k x or let us write it i x k, that is what has happened here.

So, I took this kernel, and I inverted it I flipped x and k that is does not that do anything, and then it is e to the x complex conjugated, but that is what has turned out to be the ad joint. So, this is immediately a proof that the Fourier transformer operation is in fact unitary, you want to make sure there is an i unit operator that appears. Then define instead of what we have here call it 2 pi to the minus half time this Fourier transformer operators. So, unitary operator, so now let us take a look now that we seen that the Fourier transformer operators is unitary. Let us ask what about the Eigen values and Eigen function of the operators if any.

Now, as you know when you have an operator, and you would like to define it is spectrum Eigen value spectrum and corresponding Eigen functions. You need to specify the function space first, first or whatever space you are on for the operator, you need to specify the vector space the linear vector space. And let us take again since that is the most common cause l 2 the l 2 space, so that the specific questions we are asking is, what are the Eigen values and Eigen functions if any of the Fourier transformer operator in l 2 space.

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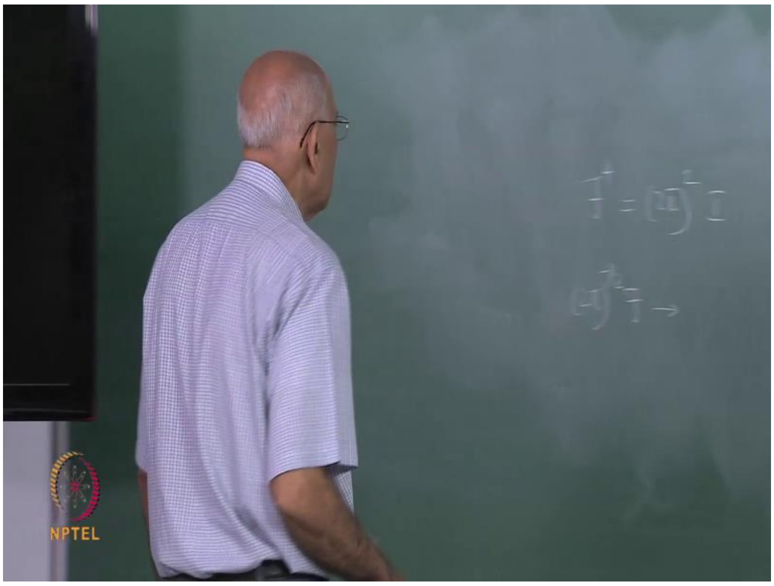

Well, we can prove this regresily, but I am going to do this heuristically we can kind of guess the answer, because we just discovered that f to the power 4 was 2 pi square times identity. So, it is kind of telling you that 2 pi to the minus half times f 4th power is equal to the identity, this guy here it is 4th power is the identity. So, I would expect quotation marks I would except the Eigen value is to be the 4th root of unity, denotes a unitary operator this operator on the left 2 pi to the minus half f.

And for unitary matrices we know that the Eigen values have unit modules they lie on the unit circle. So, I kind of expect that this here two, if the space is respectable one like l 2 the Eigen function Eigen values would be the 4th roots of unity, this is what I expect. And what are those they1 minus 1 plus i minus i I expect this is going to happened, I need to find the Eigen functions, but here I am going to cheat a little bit and say look. We already know the answer; we know the answer by studying the quantum mechanical problem of the simple harmonic oscillator.

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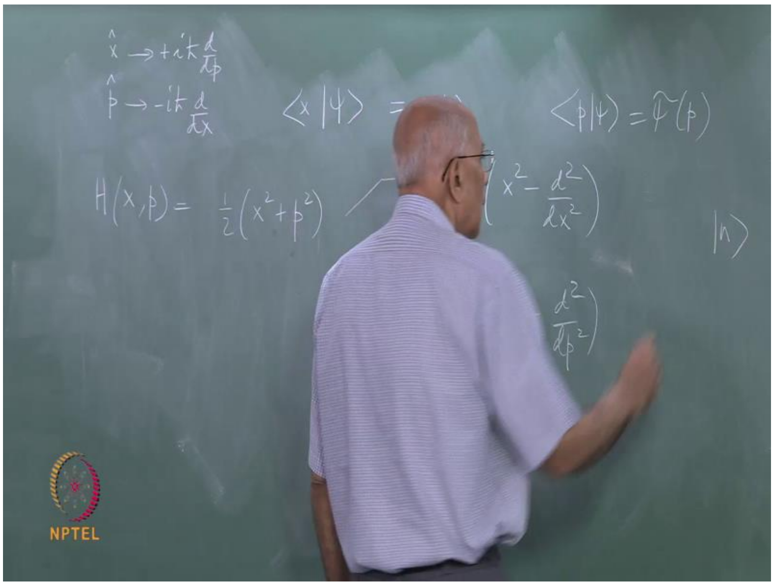

In the simple harmonic oscillator what happened was the Hamiltonian of x and p, I set the mass equal to 1 plank’s constant equal to 1 the frequency equal to 1 etcetera, this is equal to one half x square plus p square in ((Refer Time: 13:31)) on x. Now, as far as wave function are concern, if I take an abstract state side and look at in the position bases, this is what I call the position space wave function psi of x in the position bases, what is p represented by in the position bases minus i h cross gradient whatever, and h cross is z in equal to 1. So, this saying is represented by the operator 1 half x square minus d 2 over d x, when it acts on function of x which are in l 2.

And what is it represented by in momentum space. Well in momentum space the represent I am looking for is this guy p psi, and this what I called psi tilde of p and these two fellows of Fourier Transform of each other. And the reason I know the Fourier transform of each other is, because I know that this guy x p is essentially e to the power minus i p x over h cross of whatever it is apart from square root of 2 pi h cross. This guy here is the kernel that will take you from one basis to another either this or it is complex condiments, what is x represented by in the momentum space.

Student: ((Refer Time: 15:04))

Well, p is represented by minus i h cross d over d x in this operators represented by back minute axe in positions space, but x operators represented by plus i h cross d over d plus, only then can you maintain the commutation relations x commutate of e is i h cross time the unit operators. Well for the square it does not matter, but the x plus i or minus i, so this becomes 1 half p square minus d 2 over d p.

In other words they become exactly the same the operators identical, and you want the Eigen functions to be normalizable, and we know the complete set of Eigen functions they are Gaussians multiplied by hermit polynomials. And they must be the same, apart from constant multiple factors multiplying factors. And those factors will be the Eigen values, so if call the Eigen functions here remember that this operator this a h has got Eigen states label by some non negative integer n 0, 1, 2, 3 etcetera.

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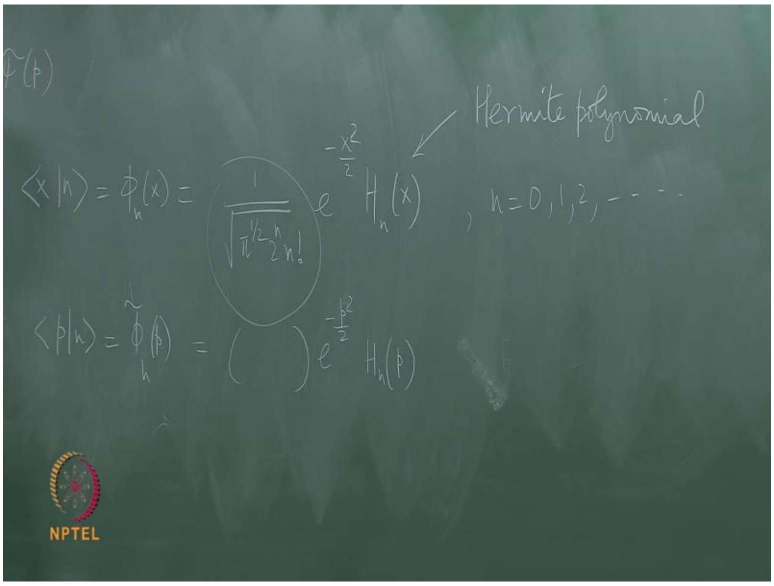

And this guy here the position space thing let us call that phi n of x began write this down, this normalized I want to normalize it. Although that one matter for this problem 1 over square root of pi to the half 2 to the n n factorial e to the minus x square over 2 the a meet polynomial H n of x, those are the normalize Eigen states Eigen functions in position space for the simple harmonic oscillator.

And what is this guy here is the hermit polynomial, it is satisfies certain differential equation second order differential equation, and H n of x has the property that when n is even it is an even function, when n is odd it is an odd function. Because this Hamiltonian is in variant turn an x goes to minus x under parity, it commutes with a parity operator. Therefore, this Hamiltonian and the parity operator share Eigen states, in other words since the levels are not de generate you can assert that each level each wave function must either be a odd function of x or in even function of x, and similarly in p (s) space.

And these follows here reflect that parity at h 0 is 1, h 1 is 2 x, h 2 is some x squared and constant and so on and so forth, what is about this? Well p n equal to phi n tilde of p this is equal to apart from this factor, which may not interested at the movement, because it some independent normalization factor it is e to the minus x squared p squared over 2 h of p times some fact. It discover what this factor it is, it is a normalization factor of some kind, but this is the Fourier transform of that and it looks exactly the same as that function. So, this means these are Eigen functions of the Fourier transform operator, what we need to do is find the Eigen values, and explicitly see this true or not.

In other words is the Fourier transform of this operator equal to this or not that is what we got to check, and if that is true we found what the Eigen value function are, and of course once you check that out the Eigen value are also drop out. So, let us do that and the simplest were to do this inside do not have a table of integrals where I can actually write down the full transform of this guy. Let us do the follow on, let us use a famous generating function.

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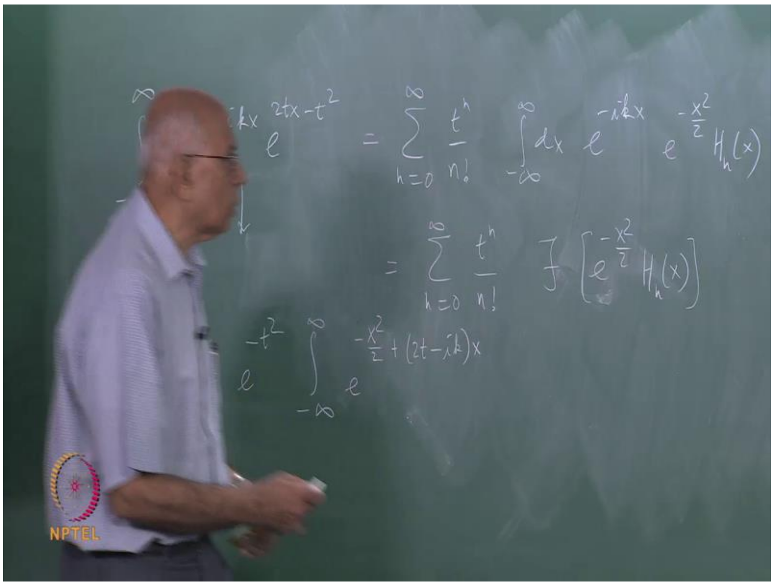

So, we need a generating function for the hermit polynomials t to the n over n factorial H n of x is given by e to the power 2 t x minus t square this is the generating function. So, I do not remember the deferential equalization, I do not really remember all the normalization factor, but I remember all the generating function, because from that you can always derive the everything else.

Now, relation like this you should really we are interested in real values of x course, but a relation like this should always we looked at ((Refer Time: 20:31)) power series, it should think of it as an analytic function of this variable in which the power series expressed, what is the radius of convergence of this in t, what you think it is?. Well look at the left hand side what sort of function of t is it, it is an entire function it is an entire function it is called e to the t square d to the t n and so on.

So, no finite singularity is at all, so what is the radius sub convergence of this series, so this is a convert, so we can play with this series, otherwise you got worry. So, I need the Fourier transform not of this guy, but of this fellow. So, let us multiply both sides by e to the power minus x squared over 2, and then when I take the Fourier transform I have d x e to the minus i k x s o, and integrate minus infinity. So, I multiply this minus infinity to infinity d x e to the minus x squared over 2, e to the minus i k x e to the minus x square over 2 H of x, this is go to be equal to this.

But what is this equal to this is summation n equal to 0 to infinity t to the n over n factorial, this is the Fourier transform of what we want of that wave function. So, it is the Fourier transform of e to the minus x square over 2 H n of x and x equal to this, this is a Gaussian integer. So, let us write it out this write here is equal to d x this is equal to e to the minus t squared, and then an integral minus infinity to infinity e to the minus x squared over 2 plus 2 t minus i k x this integral convergence there is no problem, because a this even though this is imaginary or complex it does not matter convergence this takes care of it, how do you do such an integral complete squares.

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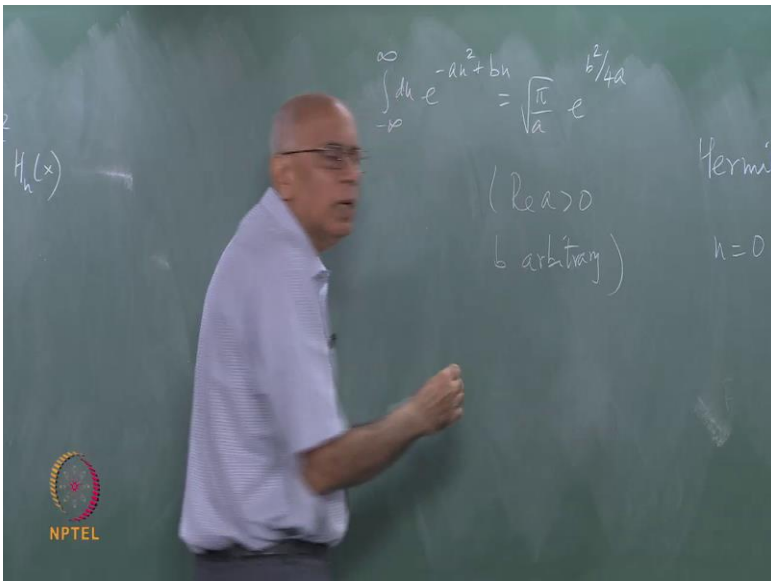

We know the formula already, so let us again I useful formula to remember the formula is an integral d u e to the minus a u square plus b u minus infinity to infinity is square root of pi over a e to the b square over 4 a what condition we need on a for this. These are better provided damping factor, yeah real part of a greater than 0, what condition do you need on b do you need any condition on b at all, no arbitrary including complex value we do not care.

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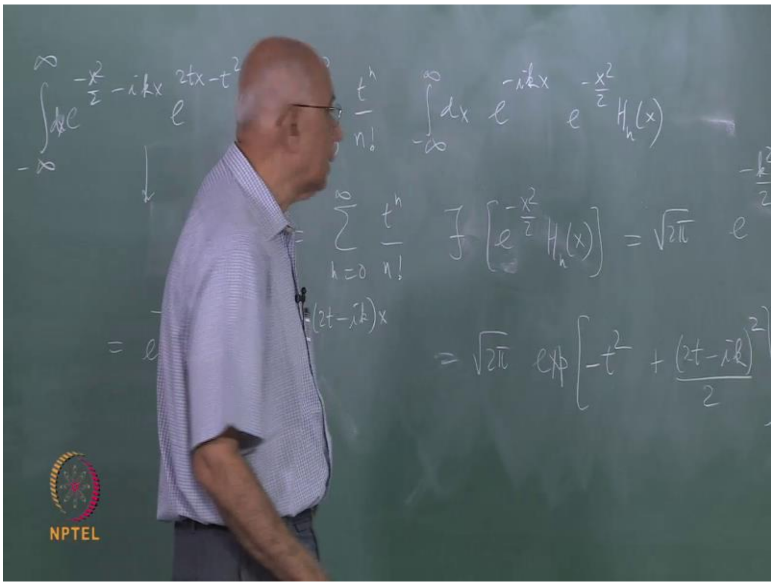

So, this thing here this integral is equal to e to the minus t square, so let us write this as exponential that is what makes it easy. By the way in this case a is equal to a half, so you got square root of pi over a that is square root of 2 pi (s). So, let us keep that factor 2 pi square root of 2 pi, and then you have exponential of minus t squared plus b squared over 4 times a, so b square is 2 t minus i k whole square over 4 times a is half, so is it 2 here which is equal to square root of 2 pi exponential of what happens now?

This is minus t square, and then you get 4 t square over 2 so that is a plus 2 t squared, and then minus 4 t i k divided by 2, so minus 2 t i k, and then a k square minus k squared over 2. So, we end of to the fact that this thing here is equal to square root of 2 pi e to the minus k square over 2 that factor, and then e to the power minus 2 t i k that is this factor plus t square. So, let us eraser with else this is a better work out, because I am not that goes All right.

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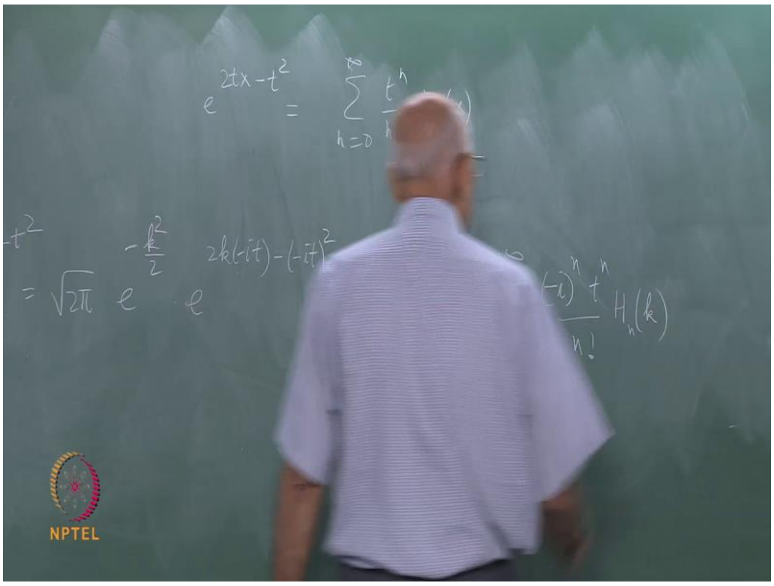

But now what happen? Well remember I said that is summation n equal to 0 to infinity e to the power 2 t x minus t squared, this guy here was equal to a summation n equal to 0 to infinity t to the n over n factorial H n of x. I need to write this in that form. So, let us write it as 2 k times minus i t, so this is equal to root 2 pi e to the minus k square over 2 e to the power 2 k times minus i t plus t squared, which is minus minus i t whole squared.

And that in turn is equal to square root of 2 pi e to the minus k square over 2. And now let us use this formula once again the fact that t is complex does not matter, we already seen it is an entire function. So, I replace t by minus i t and these becomes summation n equal to 0 to infinity minus i to the power n t to the power n over n factorial H n of k almost there.

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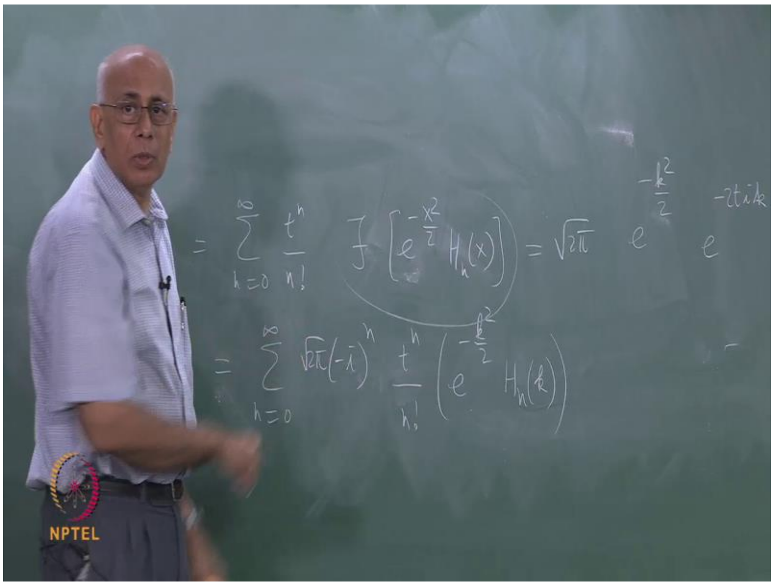

So, this thing finally is equal to summation n equal to 0 to infinity square root of 2 pi into minus i to the power n times t to the power n over n factorial e to the minus k square over 2 H n of k, both are absolutely convergent power series in t you can equate coefficients. So, it tells you that the Fourier transform of this function is this function itself same function multiplied by minus i to the power n times square root of 2 pi.

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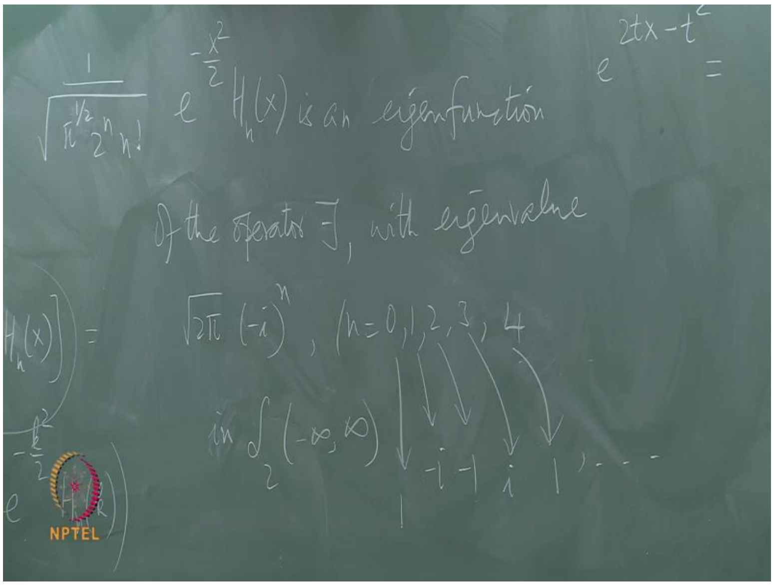

So, it is says now, therefore putting in the normalization factors you have 1 over square root of pi to the half 2 to the power n n factorial e to the minus x square over 2 H n of x is an Eigen function of the operator f, with Eigen value square root of 2 pi times minus i to the power n. So, those are the Eigen functions in l 2, just as the parity operator had Eigen functions in the even and odd Eigen function of the harmonic oscillator were Eigen functions of the parity operator also. And of course highly de generate, you have infinite number of Eigen functions in l 2 for each Eigen value every 4th one.

So, n equal to 0 as Eigen value 1, 1 has Eigen value minus i, 2 has Eigen value minus 1, 3 has Eigen value plus i, and the 4th one again has Eigen value 4 this again has Eigen value 1 and so on. So, we use the knowledge of quantum mechanics, we use the fact that you have a Hamiltonian which look exactly the same in x and p. And in this case it was also function of x squared and p squared, so you did not given have complex conjugation involved, but you had exactly the same differential operator.

And it would be known it is Eigen values in l 2 and Eigen functions, and they immediately tell you this directly proves that you have these are the Eigen value of the Fourier transform operator. So, you have an integral operator, and we found it is exact Eigen functions and Eigen values ((Refer Time: 31:56)) this does not this happy circumstances does not always happened, but this particular case it take.