Selected Topics in Mathematical Physics
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Module - 8
Lecture - 20
Fourier Transforms (Part I)

So, today let us take up Fourier transforms which is another kind of integral transforms, and you are little bit aware of the Fourier series.

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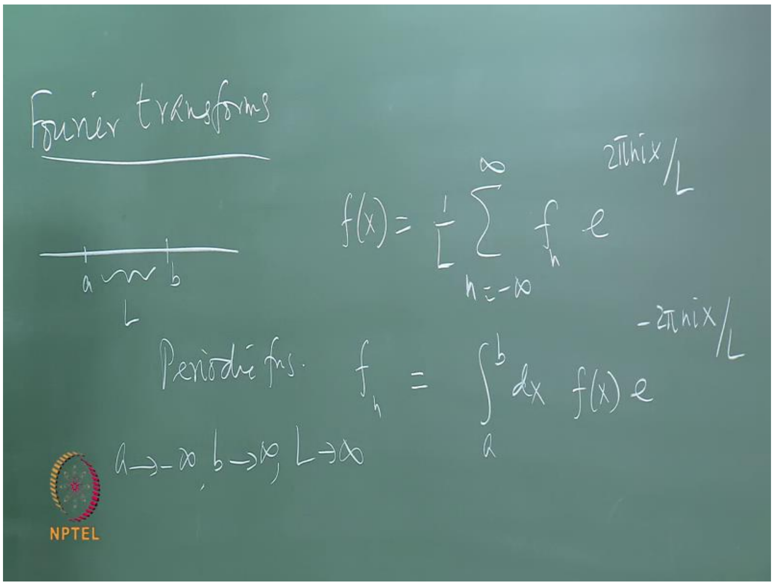

So, let us start with the Fourier transforms you are aware that if you have a periodic function of a real variable x, then you can expand this in Fourier series in terms of sin, and cosines a whole coast of sins, and cosines we can combine them into a complex notation write if you have a function that has on the x axis that has a fundamental integral a, and b some points a, and b, and this is l, and if the function is periodic with this as the fundamental period. And then the function repeats itself at intervals of l then you can write f of x equal to one over l summation n equal to minus infinity to infinity some coefficient f sub n e to the power minus 2 pi nix over l, and these are the Fourier coefficients, and if the functions is even odd etcetera you can write it in terms of sins, and co sins etcetera, but this is the general Fourier series expansion the invasion formula that expresses f sub n in terms of f of x, because this sort of functions forms a complete orthogonal set this thing here is equal to an integral function a to b d x f of x e to the power minus 2 pi nix over l. So, this true for periodic functions.

Now the idea behind the Fourier series Fourier transform is to say what happens if do not have a periodic functions at all in other words what happens? If I let a goes to minus infinity b goes to infinity. So, l comes to infinity, and you do not have a periodic function what happens to this series here now what happens is that the number of harmonics end becomes a continuous variable, and you have a Fourier integral instead of a Fourier series. So, that translation that one has is one over l summation n goes to one over 2 pi an integral over whatever you can be integrate over here minus infinity to infinity d k.

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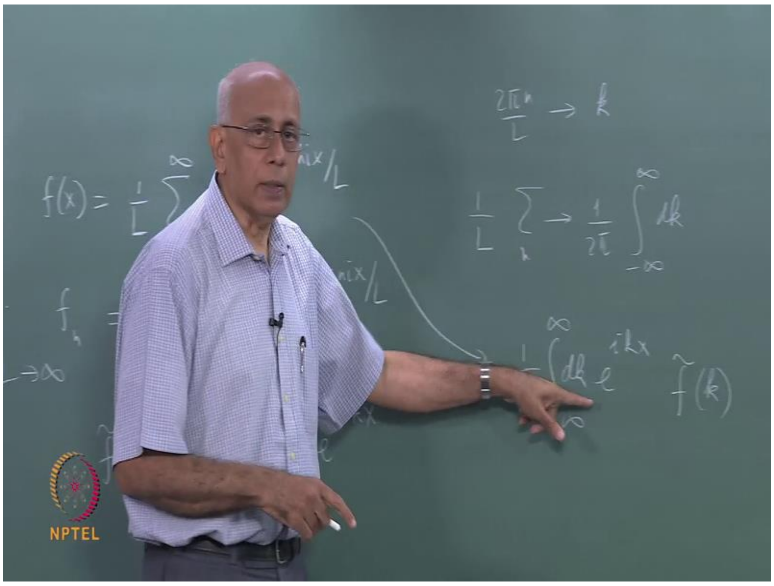

And then 2 pi n over l lets replace this by variable k, and that is it. So, now, when write f of x. Now the count of this becomes one over 2 pi integral d k e to the power i k x minus infinity to infinity times of coefficient f sub k, but k is the continuous variable, I do not want to write the f of k,, because we have already used f of x. So, let us call it f of that is my definition for the Fourier expansion for your integral expansion for f of x, and the invasion formula this becomes f root of k minus infinity to infinity d x f of x e to the minus i k x. So, notice that exactly stands for the Fourier transform of a function f of x is an integral transform. So, you take f of x, and integrated it with a this kernel which is a function of x, and k, and out comes the function of k here, and the development of f of x in terms of before you transform is this integrity you can give e to the plus i k x précising the plus transform you have in to the e to the minus s t when you went from t to s,, and then you have your plus s t in the formula which you, now there is lots, and lots of conventions in Fourier transforms that as many conventions are there this 2 pi can be put one over 2 pi could be put here instead of there.

Sometimes we put one over square root of 2 pi here one over square root of 2 pi here sometimes you make this a minus sign, and this a plus sin, and all possible combinations at this, but I stick to this, because this follows directly from here to that translation. So, the notations I mean the conventions the circular sin is not changing, but we have to stick to one convention otherwise you have to miss fact as of the two point. So, the crucial thing that you reg is that you regard the Fourier transform as an integral transform of this kind no one could immediately ask what are the conditions on f of x as such that the Fourier transform exists, when would this integral exists well clearly this is just an oscillator factor for real values of k. So, aspect that this will certainly exists if f of x is absolutely integral that is for sure. So, if this quantity f of x the x mod f of x integral of minus infinity to infinity exists then this integral certainly exists. So, now, that a sufficient condition it is not always essential, but it is a sufficient condition.

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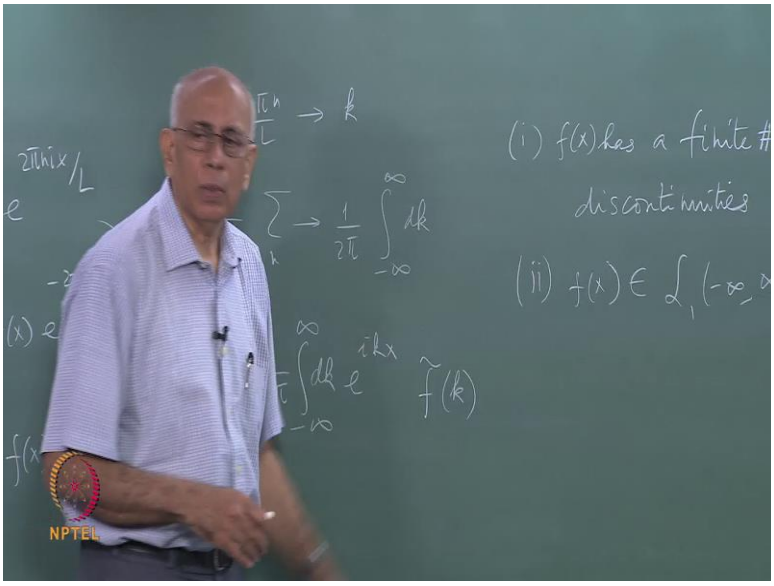

And the two conditions are to need for this to exists are similar to the conditions, you need for a Fourier series expansion to exists you write this f of x to have only a finite number at most finite number of finite jumps, and it should be l one, and it should be absolutely integrated. So, let us put those down as some kind of necessary sufficiency conditions one f of x has a finite number of finite discontinuities, and two f of x over all amount of l one now it exists integrity absolutely integrity we do not go to write down Fourier transforms for things like delta functions and so on it certainly do not satisfy these conditions, but these are sufficiency conditions, and there is a theory that tells you under what conditions these integrals exits what are the minimal conditions and so on.

So, we are not get in to this right now, but what we need to understand is to specify a given space of functions such that the Fourier function, and the function itself are both in the same space that turns out. If you look at the space in the square integral functions then this is. So, the Fourier transform of a square integral function is also square integral, and the most important application of this of course you know in quanta mechanics, where if you look at the position space spare function of the particle, you want it to be square integral, because that is the total probability you like it to be normalized in between, and when you do a Fourier transform you where can momentum space in terms of momentum space ware functions that is another physical observed with the momentum itself.

And therefore, it would again like the way functions to be normalized to the, and it convenient that since these two are related by Fourier transformation, then the momentum space ware function is Fourier transform of the position space ware function, it is good to have the fact that if one is an element of the, l is to source the other an element of l two well write that down.

Now having said this about the Fourier transforms having defined it in this fashion, we should also ask what are the analytic properties where would such a thing be analytic as function of k well, as it stands here the question is can I move off the k axis to the complex k plane, and would this till make sense the answer in general is no, because if we move off the k axis, and make it imagine, and have an imaginary part, and even upper or lower half plane which ever half plane it is in x runs from minus infinity to infinity. So, in one of the two half planes this thing is going to be exploding exponential, and that is not good news right this immediately means that unless this function of f of x cuts off very rapidly of very large values of mod x, such a thing has no chance of existing at all if this were a like, for example, then yes you could do it, but otherwise if this thing here decreased infinity like some polynomial like some power of k x then this is not going to exist at complex k. So, in general you only have the real axis unlike the case of the that you had the analyticity in the whole half. So, let us write down simple Fourier transform of functions, and see what this whole thing looks like. So, let us start with the functions like very familiar functions.

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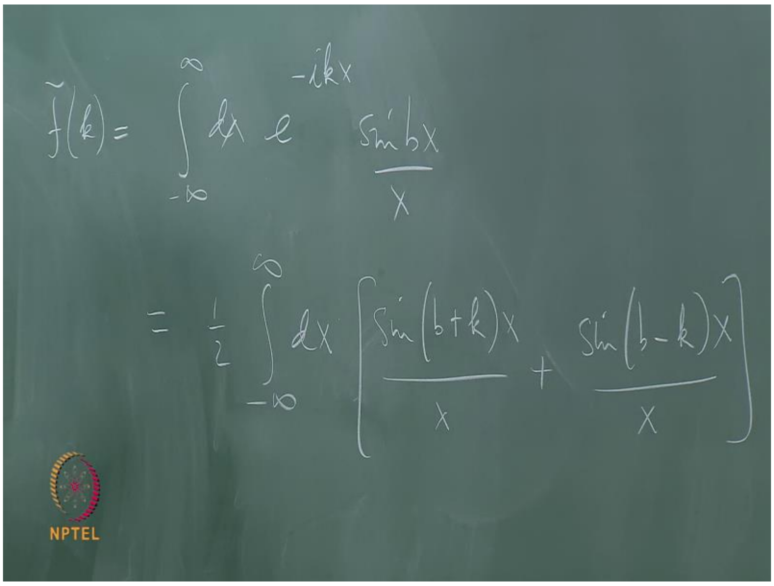

So, let us put f of x equal to sin b x over x b real this is going to integral. So, I am going to use that by the way this function has a name in engineering it is called the sink function, and you know what it looks like it is got an maximum value its b here, and for positive b for intense, and then it goes down like this, and this aptitude here decrease like a one over x the magnitude there go the modulus decrease like a one over x. So, what does the Fourier transform do or we can write it down immediately f of k equal to integral minus infinity to infinity d x e to the power of minus i k x, and then sin b x over x.

And what does that give us well this is the cosine k x, and then a sin k x, but the sin k x part is going to be an even function on the numerator, and an odd function here, and this thing is going to be zero by symmetric. So, the only thing that contributes to a non zero extent is the real part. So, this is equal to cos k x, and the x. So, let us write this as the one half integral minus infinity to infinity d x sin b x cos k x is sin b plus k x over x minus plus sin b minus k x over x, and that is the directly integral we know how to do this integral that is equal to pi over two times zero to pi, but now regarded to minus infinity to infinity it is like zero to infinity. So, its equal to pi itself. So, it is a half,, and then a pi the sin of b plus k plus the sin of b minus k by this i mean plus 1 b plus k is positive minus 1 in plus k would be negative, and what does this thing look like let me just plot this.

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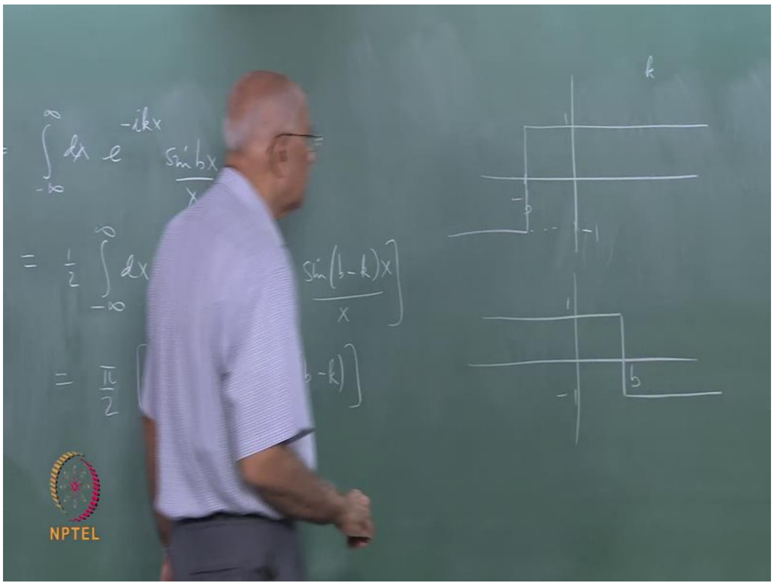

So, here is k here is minus b as it says b plus k is positive. So, b is greater than k is greater than minus b it is like this, and then a this is one, and this is minus 1, and the second factor the second term here is sin b minus k. So, its equal to one as long as its sufficient to negative,, and then it is this is a b, and what we got to do is to add up these to guys now I add this to this, and the minus 1, and plus 1 cancel, and out here minus 1, and plus 1 cancel. So, this when you transform is non zero only between minus b, and plus b. So, what does it look like f root of k equal to, and in between the two it adds up to two there, and it cancels the half here. So, its equal to pi times the pulse which is zero all the from minus infinity, and it is like this, this goes on this is unity this is the minus b, and that is it.

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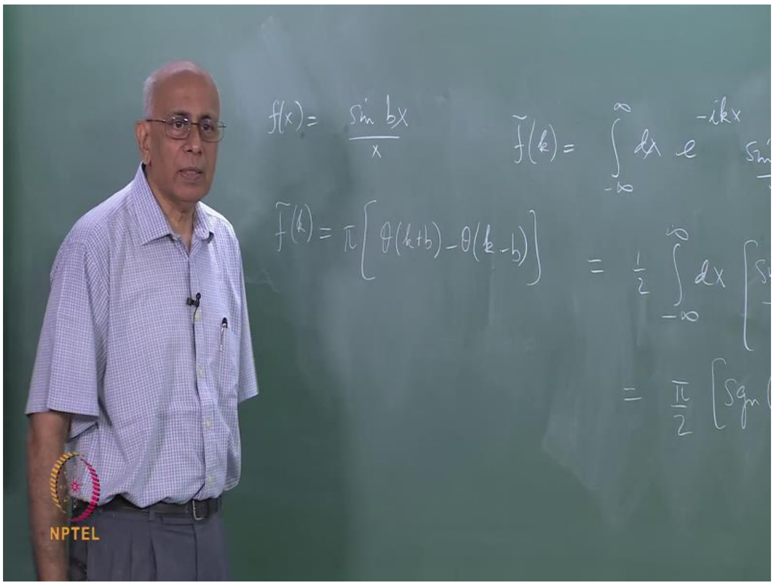

So, that of course we know how to write that is equal to theta of k plus b minus theta of k minus b. So, if k is large positive its zero, because if k exceeds b this gives you a minus sign, and k is this out here, and if k becomes less than minus b this kills that. So, it is just a pulse function unit pulse function between minus k minus b, and plus b. So, what is the lesson its telling us that if this function f of x dies down like one over x roughly then the figure transform is quite compact where one over x is a slow d k barely converges the integrate barely converges, and this whole thing where. In fact, where this integral from minus infinity to infinity is only conditionally convergent no its not absolutely convergent. So, if in x space the function is quite spread out then ion case space the function is the figure transformation is very compact now let us look at another example, and see what suppose in x space the function dies on into faster say it dies down like one over x square, and see then what happens well by the way I should point out something here.

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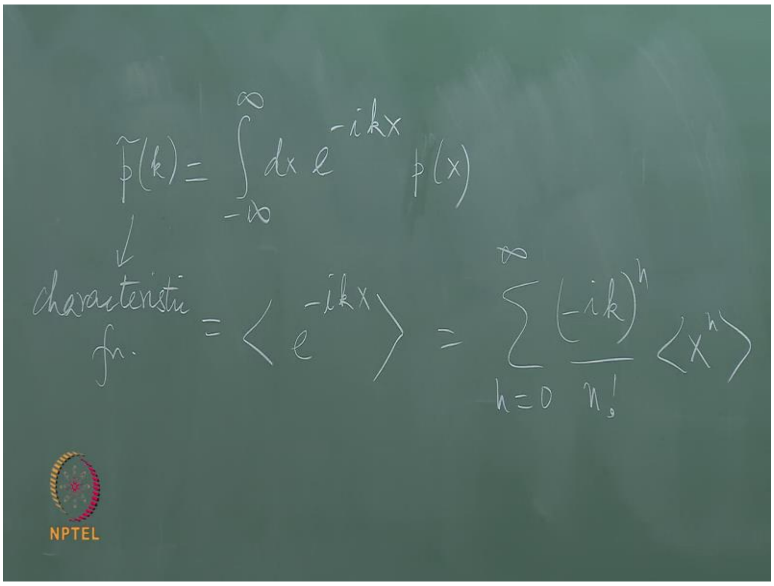

Before I do the this remember that the f of k equal to minus infinity of infinity d x e to the minus i k x f of x very often, we are going to be concerned with the cases of f of x is the probability distribution a density function by itself. In other words its non negative, and its normalized to unity we are gonna look at that case in particular lets this equal to [vocalized-noise is if this a probability density, and lets for a moment let us call it p of x then the Fourier transform of that probability density p k is given by this, but that is also equal to the average of e to the minus i k x, because it is the weighted average weighted with the probability density in just the statistical average whatever this probability density of e to the minus i k x, but I can expand that, and write this as equal to summation, and equal to zero to infinity minus i k to the power n over n factorial,, and then the average value of x to the n. So, it is essentially generating all the moments of this distribution. So, the knowledge of this function the extract the co efficient to k to the n in a power series of this function that co efficient is apart from this i to the n over n factorial factor it is the average value of x to the n the n the moment of x.

So the Fourier transform of probability distribution is called as characteristic function, and the characteristic function generates all the moments for you once you expand it in a power series. So, useful piece of information you have we are going to use this, in fact later on. So, this thing here is characteristic function. So, this example here this is not a good probability distribution density function why is that yea it is not positive definite it is no not non negative it is like plus, and minus values and so on, but for the Fourier transforms as the uniform distribution between some minus b, and b, and which one you call the Fourier transformation, and which you call the original function it is a kind of arbitrary right. So, if you look at a uniform distribution its characteristic function is this. So, imagine that I started with the pulse in x,, and then in k space it would look like this since b k over here or something or sin k over here k, and form here you can generate all the moments of this uniform distribution in this case let us look at a function of f of x which dives down a little faster than one over x, and let us look at the probability distribution itself by the way.

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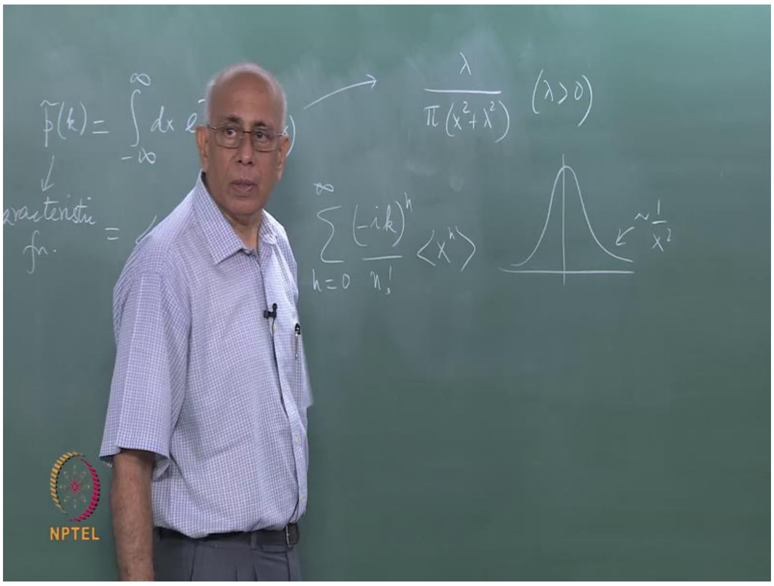

If I choose this to be one over x square plus lambda square il normalize it. So, it is like this lambda over pi times x square plus lambda square now lambda is some possible constant it is the lorrensian shape, it is called the couchy distribution the symmetric couchy distribution, and what does it look like line, and this line, and this looks like a bell shaped thing based on one over x square, and the width ids related to this lambda if i integrate this from minus infinity to minus infinity its twice the integral of zero to infinity zero to infinity is one over lambda which is cancel this ten members x over lambda, and only infinity part it will be contributes a five oat to the five is cancel. So, normalized now what is the for a times from of that. So, given that what is now he can do this fails easily.

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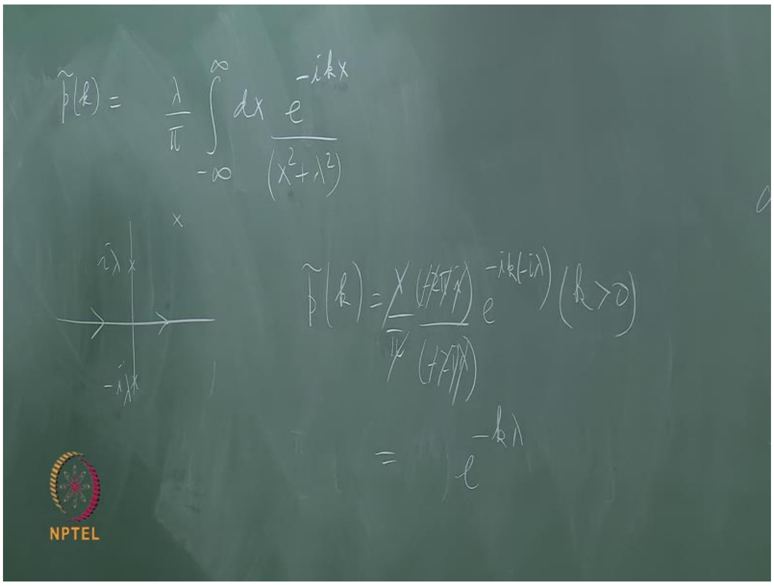

So, given that lets look at use that notation this is equal to lambda of five, and in integral from minus infinity to infinity d x e in to the minus i k x over x square plus lambda square, and it stale make for using contributions just solved of think, we can do it contributions there are singularly is of the integrated in x this one at i lambda, and another one is minus i lambda, and the contributions. Now when case positive this possibility number, then I need to close to can to in the lower half plan. So, that i will get when case posited if I could x is plus i something of other it blows up in the halfer of play if as x is possibility imagine a part, and it goes to zero in the lower half. So, in need to closed it then the lower half play. And then and what happens. So, first let us look at petaled of k for k posited this is equal to lambda or pi, and then and gone to close this to the control of this faction, and pickup at contributions from this pole minus i lambda write. So, its contributes he I am going to this in the around sense in the negative sense. So, this is a minus 2 pi i, and this is a x pi lambda x minus pi lambda x minus pi lambda. So, this is divided by minus two i lambda he to the power minus i k into minus i k lambda t is equal to lambda cancels the minus cancels the two cancels, then pi cancels everything cancels out to get e to the fast minus k.

If I closed it, if k one negative I am forced closed other have play, and then the same thing would happen it get a lambda of pi, but now I have plus to pi i, and plus i lambda hear it, because of close of the plus i lambda for this partner contribute plus i lambda, and to get is the k lambda, but k is negative. So, this means there are both cases at could write this as equal to e is the minus not k and. So, he have it careful which have if close things if have very careful, because in these case it is clear that not analytic function of k it is called a modular there even a function of there in real variable this function is not differential to k is equal to zero god a cast k is equal to zero, but ten percentage important lesson we solve that if this function if the function in x men top write down like on over x then the for a times of value compact now its function is tangle like one over x square, and what is happens surplussed very compact full data, and you got something that d k exponential So, that spread out, because to the function of the little more compact, let us look out the doubt if see what at have lets k situation need all the time.

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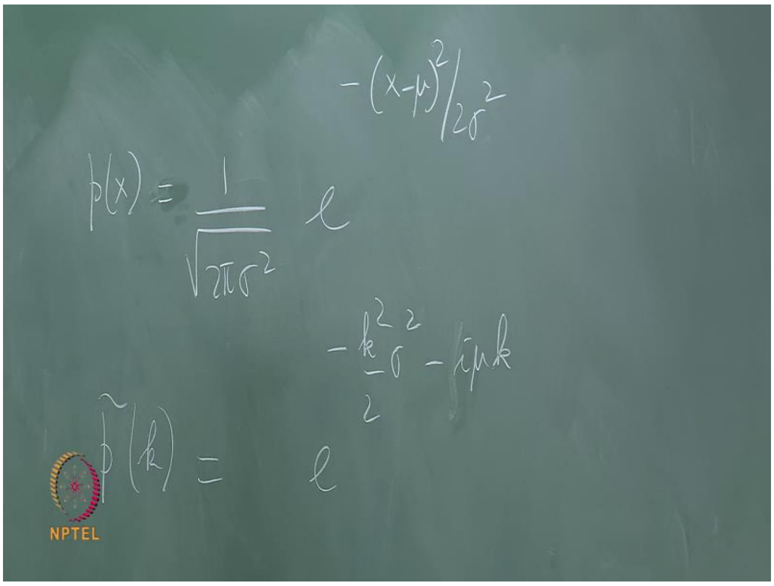

So, that look at f x which could also b f probability distribution may be usually take it to a normal distribution one over square root of tau pi sigma square e to the bar minus x minus all square two sigma square this is a normalized doubt in distribution which mean by value, and various is equal two sigma square, and what is the petal of k in this k by the what should petal zero b if it should be one in this properties distribution is normalized it should be one. So, that no fact sitting outside all that will get canceled doubt, and as no that for a time of form of doubts itself standard in do to, and this thing term e to the minus k square two sigma square minus i knew k checks out.

All unit do this use the limited formula for the doubts limited doing formula thing is that this hear will stoled what the, with was in the x space appears in numerator is. So, this is telling new that today transform of doubts in doubts in, but it is narrowness it is very ford in k, and wise person, and it be very neatly moves it up hear this is do not have lot of sanctification latter on begun to look at words called to process summation formula very casual thing, and this factor will be to casual root also by the by cary to next stream against to talking about mechanic all the time clarified next stream this thing here in doubts in if a make out thinner, and thinner forfeiture like something like delta function which means position i gains state one point you respect the moment of one’s certain become infant.

And that exactly what is happening, because this thing hear about to zero this with become broad broaden broader about opportunity diffidently ha will infect this thing is exactly clear minimum certain steam we packet, but will come back to these. So, little late. So, we see now that com partners in month variable means broadness in available, and wise person contribute variable, and by person this is a this is a very hart very consider different in contribute mechanics now what i dint mention was that.

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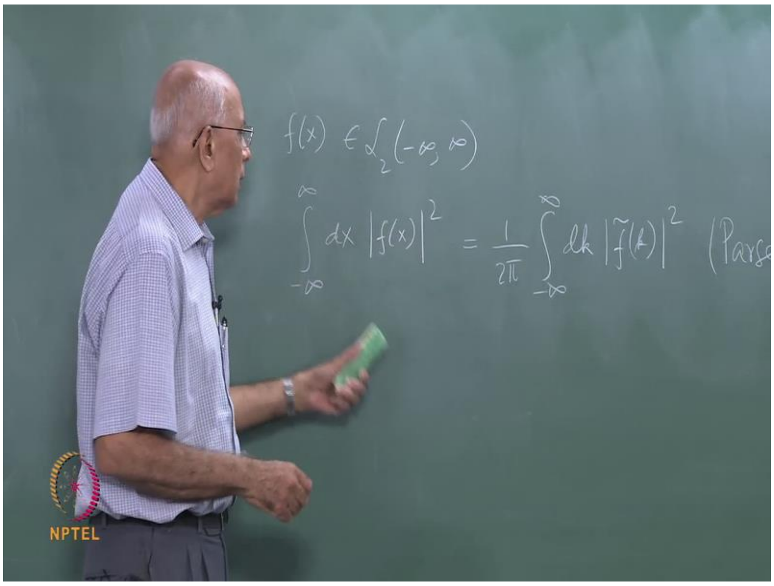

If you have f of x, and aloe man l two minus infinity to infinity another words, if this is two minus infinity to infinity d x more f x all square is find out normalized some number there it terms of I mentioned that f tilled of k is also allotment l two its infected trans out this is exactly equal to to the normalized without can mentioned of to price, and in this is example carried about for hear taken from, but you already no from for is serious this called par sables.

It is essentially says one function normal is way other function it also normalized, but parcel formula is to true general true for any function in l 2, before a transport mast quiet integrated this is equal to this what else well also have the consumption.

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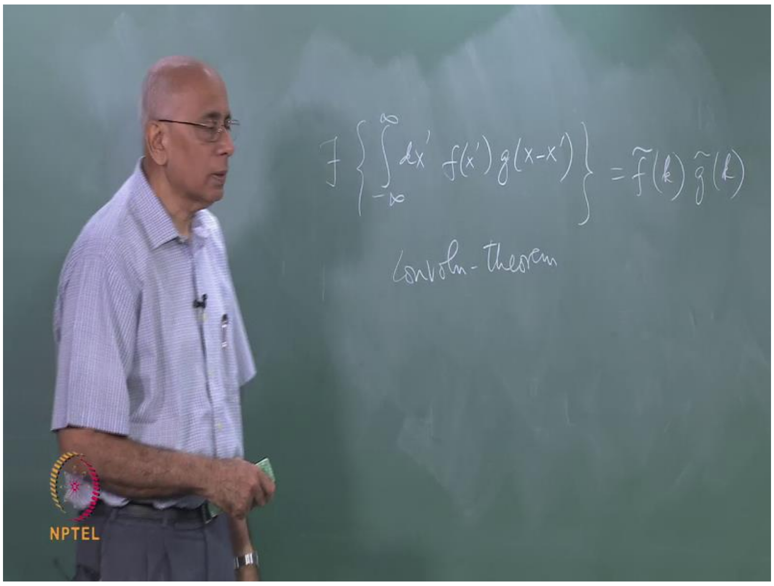

So, if have minus infinity to infinity d x fine f x fine g x minus x fine is a convolution of the functions f, and g then for a times of transaction for this is called that f in this faction e is equal to the for a trans form of that is convolution period con-volume transaction these are the similarly period for pastern from there entangle one zero, and t, but this is now minus it to by the way it is also true covers thing. So, if have a convolution in case pays that menu ember the transfer about to get product express apart from some to by factors this relation perusal theme formula is a itself a special case specialictly more general formula.

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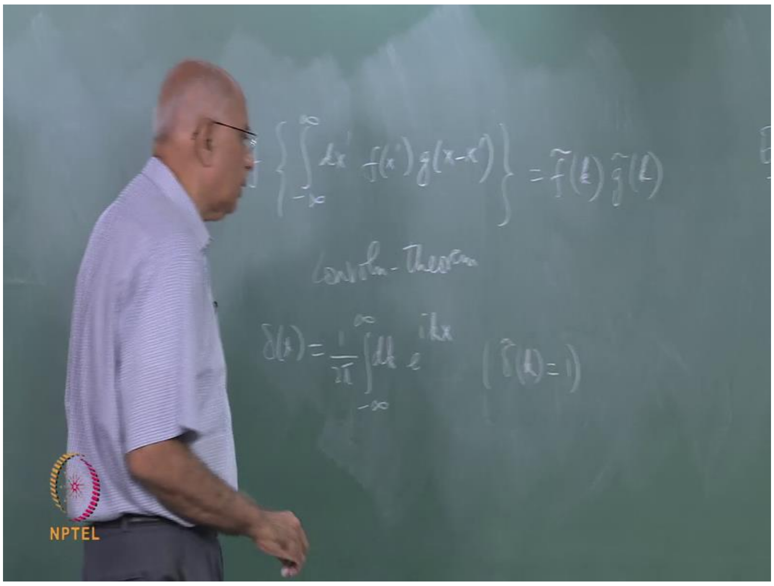

And that is the following is a established d x. So, if have a function f of x, and another function capital f of x which as a for a time of format etcetera, etcetera. And you considered this little of x, and capital minus x then these is equal to minus infinity to infinity d k for a 2 pi f delta k f delta k. So, excise show that this is to it fuel is straight forwarded to you jest make use fact that the delta function has a for the representative you needs to fact that delta of x equal to one more 2 pi d k infinity to infinity e k x which up course implies that delta till of the k equal to one that for a transfer of the delta function is one with the normalization.

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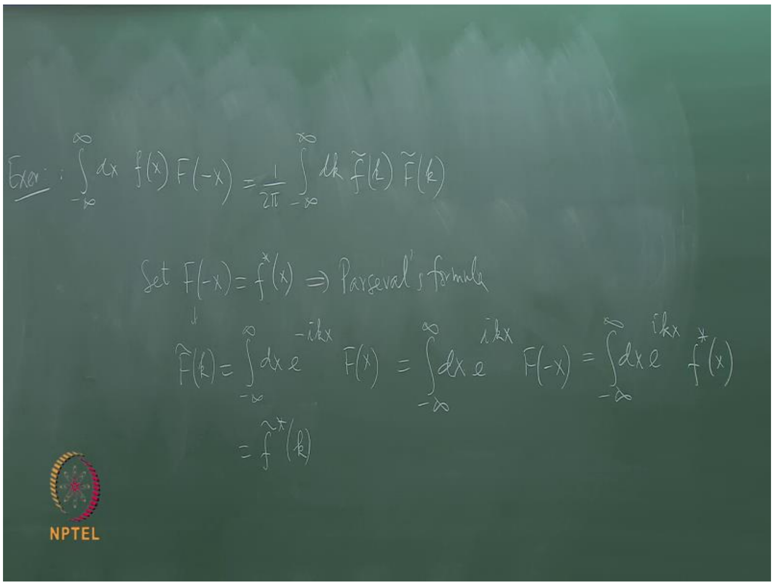

So, which need established possible same itself formula from in these, it is very straight forward all you have one do it to use the is to set f of minus x equal to f star of x. This usually may be implied parse val formula from this infinity to infinity, but this is an easily established show it is should be first established this, and then look at special case more foot implies for today transport f tilled of k what to till employ. So, as lets write down it should say f delta is k equal to his integrated minus infinity to infinity d x it is the minus i k x f of x agree, but exchange k to minus k next change x to minus x in the range of integration. So, this is also equal to integrated minus infinity to infinity d x is the i k x f of minus x exchange into minus x, but I have now say f of x minus x is equal to star of x. So, this is equal to minus infinity to infinity d x it is i k x f star x. So, that complex it from of f x. So, this is the, and they got perusal formula.

So just simple tips of this kinds to mind it reminds etcetera, etcetera. Now done this much lets use something interest let us ask what happens applied to fore eta any to I want to, and little more about for a time from operator its can integrated law operator that much is scale, but i this write know what is lots of integrated low operator some properties profit, and menu is thing about, but operator things it your ask what first thing you ask its what happen applied it second time second time in. So what? So, let us applied for a transformation operate it tries. So, its take for a time from, and certain I take a for today transport once again then what happens well f f.

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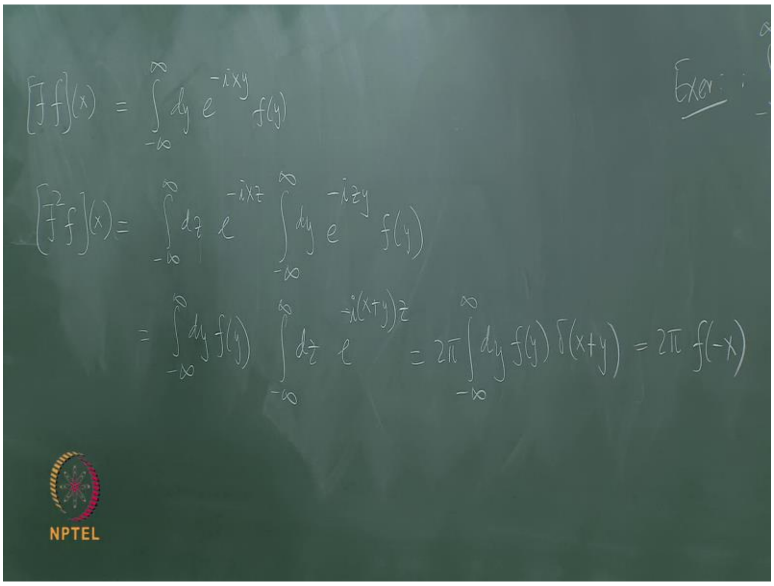

So, I take some function f in this space I am go in work health to all the time in are apply the for today transform, and escarole the result some function of x. This thing is equal to by the definition minus minus infinity to infinity d y e is to the minus i x y f of y y is integrated over, and again the function f x. So, this is f x acting on f acting on the function f, and extra size of the function x is, that is equal to this write it. So, i want ats once again one is f square f of x equal to, and integrated minus infinity to infinity d z is minus i x z no f y, but tilled updated.

So, d y e is to the minus i z y minus infinity to infinity f of y which is equal to, and integrated minus infinity to infinity d y f of y,, and then integrated minus infinity to infinity d z e is to the minus x plus y, and z i can that, but you know what is this it is just to pi time of delta function write. So, this is equal to 2 pi, and minus infinity to infinity d y f of y delta function of x plus y notice the plus in there, which is equal tau pi times f of minus x. So, what is happen is that if you take some function, and applied for a times of transformation operator tries all your doing is to change x minus x apartment for this factors 2 pi, but you know if you apply the party operator to a function its change a minus x in that.

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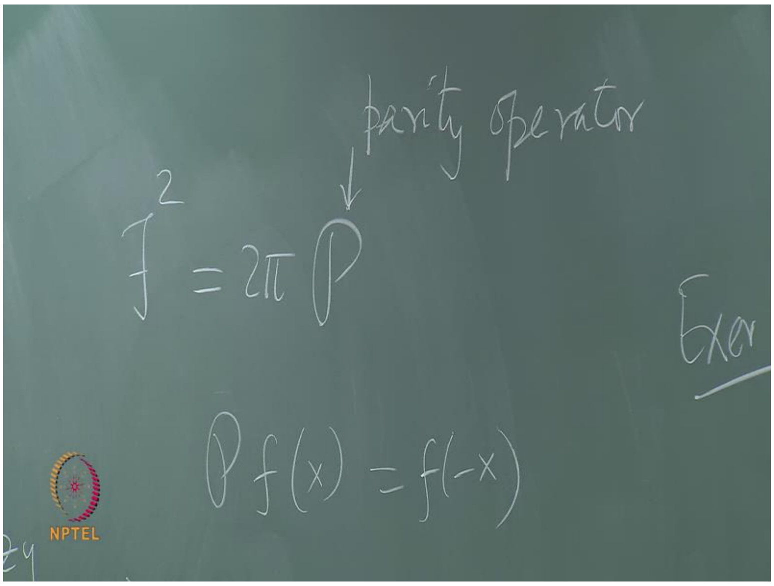

So, this tells you that x square is essentially tau pi times of party operator by that I name p f on x equal to f of minus x. So, remark to be enough apart in this tau pi factors which is insistently a rows, because of to a rate of convention, and it shows on one over square root of tau pi for defining in the transform in this way is this 2 pi on away, and you find out that square of the for a time of from out operator its parity operators some sense, if you ask what is this square root of parity of operators its parity operator change is the x to minus x in this f unction.

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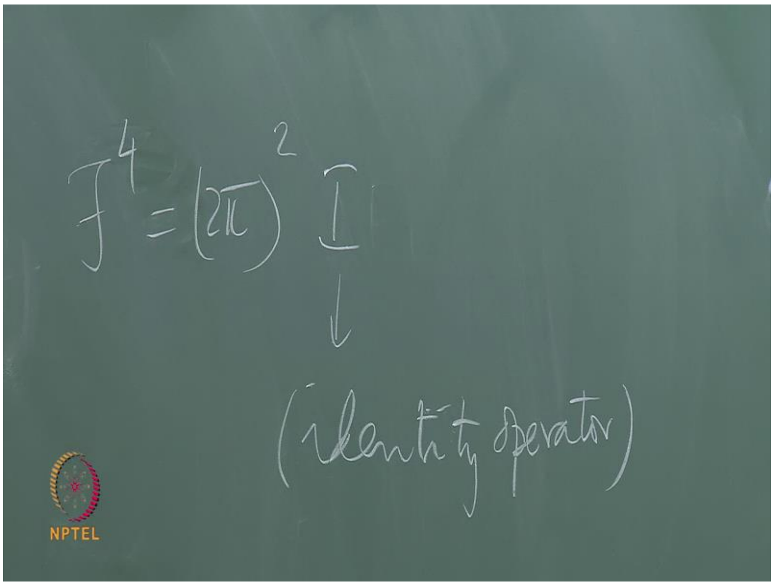

But today transform of some integrated transform, and at as square is equal to parity operator it is what happens by take which square of this operator, once again what is square is in the parity of operator they identify operator. So, this is telling use is really very result that f four equal tau pi all square kinds the identity operator.