Selected Topics in Mathematical Physics

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Module - 01

Lecture - 02

Analytic Functions of a Complex Variable (Part II)

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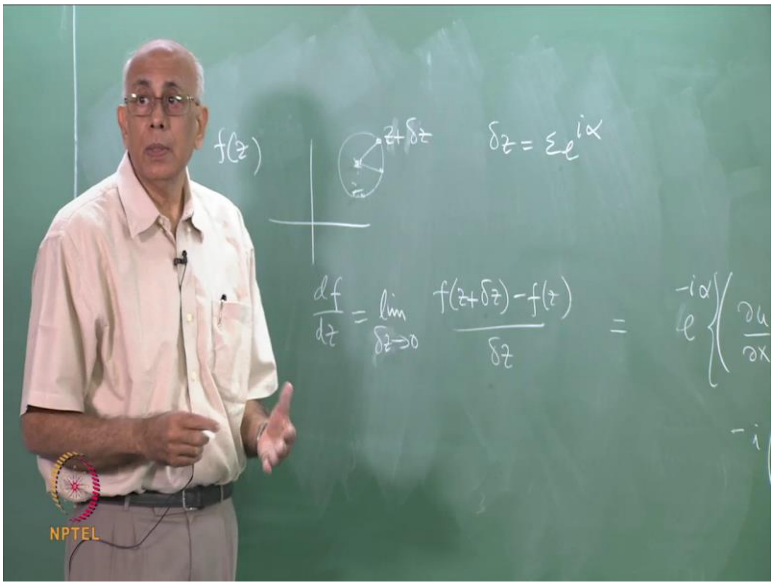

Now, before we go on to the behavior power series, the something else I wanted to mention and that is to ask what is the derivative of an analytic function that is one more way of looking at (( )) conditions. What this conditions I telling you. The derivative of a function f of z at some point z in the complex plane, so here is z and here z plus dz. So, normal way to define this would be to say that d f over d z equal to the limit as d z as delta z goes to zero of f of z plus delta minus f of z over that. This the normal way in which you write down derivative.

Now if you looking at functions of real variable that is no problem because you start with some x you go to x plus delta x, when you divide by this delta x take this difference and then you have a derivative. If you did it on the left it would be left derivate did it from the right it would be the right derivative if these two limit's are equal than you call it the derivative from both sides. Now here however, you have little bit of an ambiguity. In what direction should I take this delta z, should I take the difference between these two points or should could be anywhere on the circle and I take these two or these two what should I do.

So, let us take a very general case. Let us say that this delta z equal to epsilon e to the power i alpha. So, the magnitude epsilon and direction with respect to a fixed reference direction is alpha. Then I plug that in here and I end up with I work this out, I write z delta z as this then I am going write z as x plus iy why ditto here and then work this, out write out the derivates etcetera. And then it turns out that this becomes equals to i, it becomes equal to e to the power i alpha and then delta u over delta x cross alpha plus i, it is the minus i alpha because delta z is in the denominator. So, this guy is going to minus up there plus i delta v over delta y sign alpha minus i times delta u over delta y cross alpha minus i delta v by delta x something like that not sure about this sign here. So, this is what this thing is.

Now we go to the limit of asking now you have ask in what direction should alpha be. So, one way to say I define the derivative of complier function of a complex variable f of z is by saying that this thing here, this derivative should be the same regardless of which direction delta z goes to z goes to zero. So, no matter how I approach zero from any direction and want have the same answer that is a reasonable thing to percolate. Otherwise you do not have any uniqueness in the derivative at all, infinite number of directions in the pen and you want say does not matter. Can I ever have it independent of this alpha?

So, the next question is when does this fellow become independent of alpha. And it is easy to check the only time the if and only if condition it necessary and sufficient that this whole thing be independent of alpha is if the (( )) conditions are satisfied. If this is equal to that you take it out to the bracket you get e to the alpha which cancel this and ditto here. So, this think is independent of alpha if and only if c r conditions are satisfied, so that is another way to understand what is meant by an analytic of a function to say that it is got derivative and this derivative does not depend on the direction in which you take this limit epsilon goes to zero.

There is a bonus to this whole business, and the bonus says you have infinite number of pieces of information here, unlike start contrast to what happens in the case of real variables. You can have in the case of real variables you know that you could have a function which is differentiable at some point has a first derivative, but does not have a second derivative, has a second derivative first two derivative does not have with a derivative and so on.

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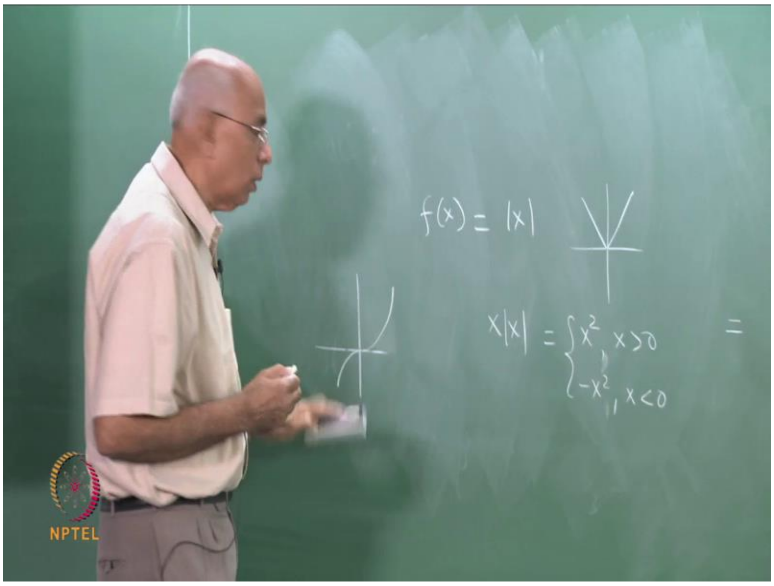

For instance if you look at this function, if you look at f of x equal to mod x, this function does not have a derivative at the origin, so v shape curve, no derivative at the origin not no unique derivate, but the origin looks like this. How about this x, mod x, how about that function, what would that do, does it have a derivative at the origin, first derivative?

Student: (( ))

Professor: Yes, does it have second derivative?

Student: No

Professor: No, does not have a unique second derivative at the origin, because this thing here is equal to x square, if x greater than zero minus x square if x less than zero. So, at the origin the slope is zero from both sides no problem, but this immediately tells you that what you really done is to take a parabola here, and you joined it with a parabola here this direction. So, the derivative is fine the function is zero at the origin, it is first derivative is also zero from both sides, but moment you have go to the second derivative we have a problem because a curvature is different on both sides. So, that is an example of function which is once differentiable, but not twice differentiable.

Now you can construct more and more such functions, you can add more powers and then multiple by mod x and you will get functions three times differentiable, but not four times and so on and so forth. In the case of complex variables, that complication does not appear at all; the moment you say a function is analytical at a certain point, it means it is got a derivative this derivative is itself an analytic function, therefore, it has a derivative which is an analytic function and so on. So, it means that every analytic function is infinitely differentiable at every point where it is analytic. All derivatives exist as a great simplification, enormous simplification.

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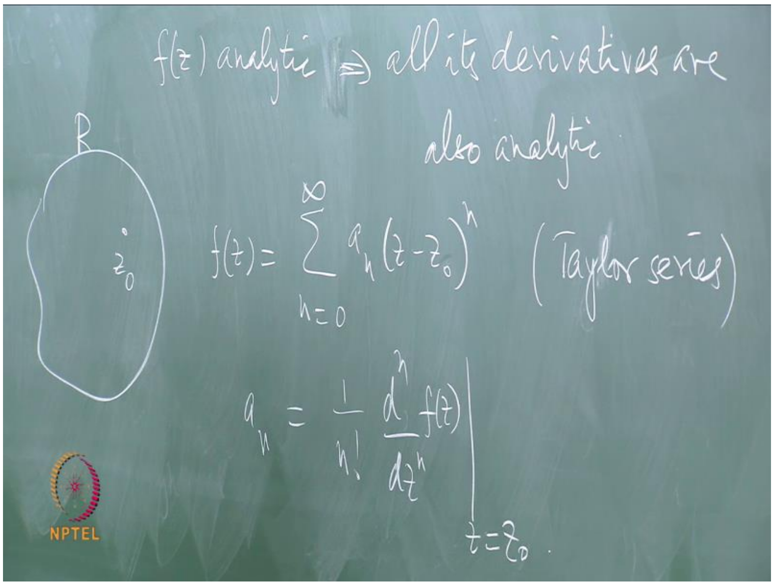

If f of z is analytic implies that all it is derivatives also in that reached. So, you do not have the complication that you have in the case of real variable, functions of the real variables this is an extremely useful property. What it means, what it implies immediately, is that you a very useful representation for analytic functions in some region where it is analytic. You can always if a function is analytic in some region like this, let us say it is r, this region here, when in general some arbitrary point in this r let me call it z naught.

In general, you can represent f of z by a power series about the point z naught in general an infinite series. So, you can represented in the form n equal to zero to infinity a sub n z minus z naught to the power of n some constant times that. So, every analytic function in general at any point of analyticity sub perhaps near boundary point or something has neighborhood in which you can represent this function by a power series in powers of z minus z naught. The question arises as to the convergence of the series where it convergence and so on, but before that we need to write down what is a n is. What do you call it is series by the way it is a Taylor series this is a Taylors series and what is the inverse formula for this Taylor series, what is a n in terms of invert this, how do you write a n in terms f? In any Taylor series what is n eth coefficient?

Student: It is…

Professor: It is one over n d n over d z to the n f of z at the point z equal to z naught it is the n the derivate at that point. Now where would the series typically be infinite where would it converge was now I has to define what kind of convergence I need there could convergence I mean there could be convergence absolute convergence there could be uniform convergence and so on and so forth conditional convergence etcetera. But let us talk about absolute convergence in other words if I take this power series and take the modulus of each term and at that I want that to be finite that is obviously, the worst case in a scenario because we take mod list of each term, there are no negative terms. Everything is getting added up if that convergence than off course if there oscillation, it will be even better it will definitely converge. So, such a series typically whenever you have such a power series has region of absolute convergence called it is circle of convergence. So, thus typically some circle.

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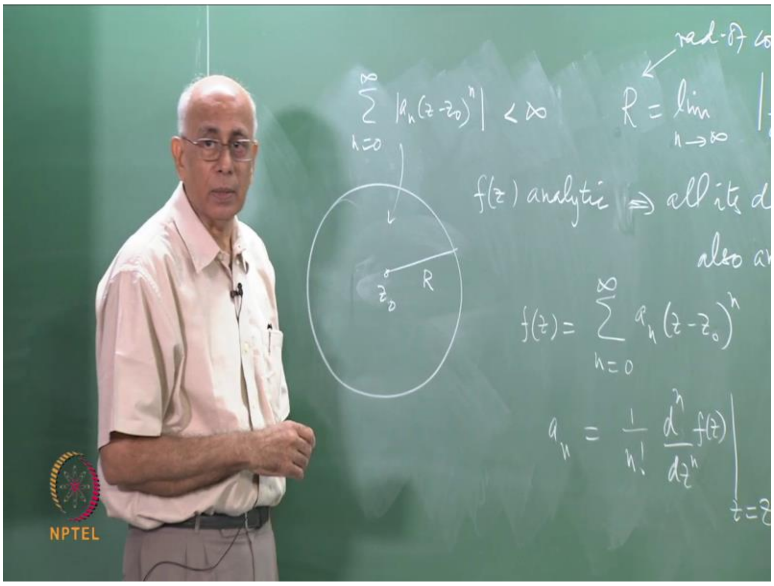

So for every power series, you give me a power series about a point z naught. A series of this kind and some circle of convergence about this point z naught with a radius r inside which inside this region summation n equal to zero to infinity mode a n z minus z naught to the power of n is less than infinity convergence absolutely. This is called circle of convergence this r is called radius of convergence can we find r how do we find r well clearly depends on how this series how the n the term goes to zero you need this thing to get smaller and smaller otherwise it not go to converge.

So, an necessary condition is that this thing must get smaller and smaller and absolute negative as n becomes larger and larger. So, that the whole answer as a chance of being finite in a some right now what is the formula for a n. For this radius r where the simple ratio test for absolute convergence would suggest that this is limit n tends to infinite modulus a n over an plus one and these r is call the radius of convergence. So, all that we rise is the ratio test which you know from elementary algebra you know that if these things if the ratio here is n plus one over a n that is less than one than you have convergence otherwise you do not have convergence etc you got test further.

So, now, they would guaranty than all cases this limit may exist may not exist this thing here we need a better definition and better one is r equal to limit superior mode a n to the power of one over n that is the least of a boundary. So, it has a maximum or not limit or not we do not care this is the least of a boundary of this mode a n to the power one over n that always exist it may be infinite may be infinite, but if it finite I have a power series which is absolutely convergent inside here. What happens if this r is infinite at some finite point z naught is a function for which I find a Taylor series and the radius of convergence, this is infinite.

What sort of function is it than it is an entire function the immediate conclusion is that it is an entire function because every where there is a representation a power series representation which i convert absolutely convergence. So, I can differentiate term by term integrate term by term you can do whatever you like and rearrange terms and so on. So, if r is infinite how have an entire function if r is not infinite you have function singularity various kinds what about this case the exponential.

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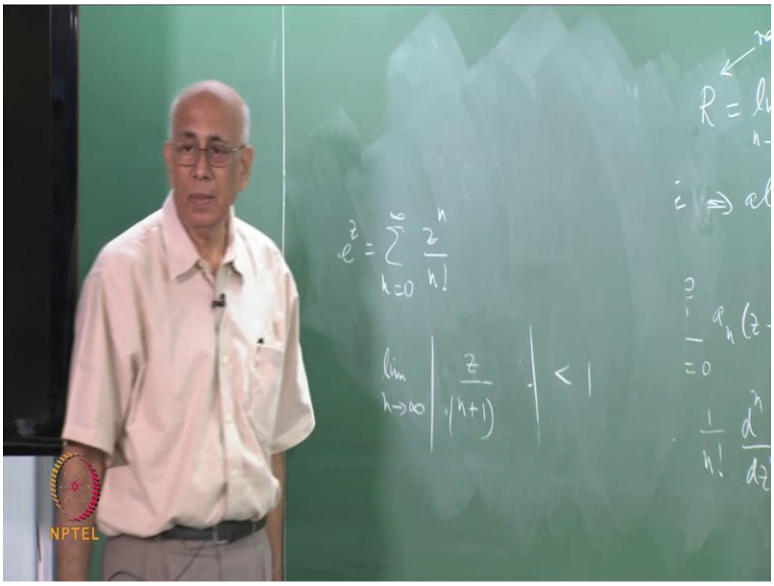

So, you have z to the power n factorial summation n equal to zero infinite that was e to the z only you have to do is to take the do the ratio test in this case. So, a n plus one over a n is z to the n plus one over n plus one factorial n factorial over z to the power n, take the modulus of this n plus one term divided by n term modulus. If that is less than one in the limit has tends to infinite than at say an entire convergent. Now this set to the n cancels against this and gives you z and this one over n plus one for all finite values of z as n tends to infinite this tends to zero which is early less than one, because there is a one over n denominator. So, the conclusion is that the exponential series is an entire function represents in entire function a to the z is an entire function same thing is for sign co sign etcetera.

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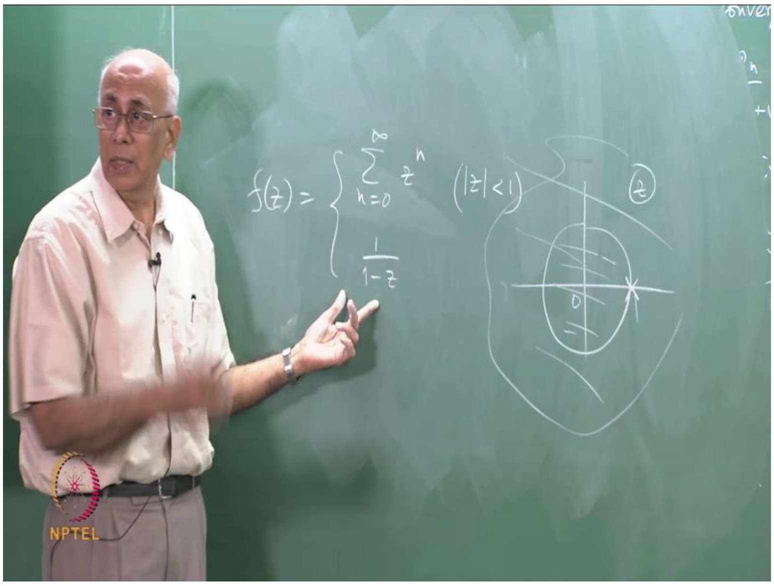

Just by trivial test of the kind what about this series f of z equal to one plus z plus z square summation n equal to zero to infinite. What about this series is an entire function, it is not an infinite series is this an entire represented entire function where is these series converge mode z less than one definitely it converges only for modes less than one. So, it is clear that for mode z less than one. So, radius of convergence r equal to one in this case mode z must be less than one is greater than one it diverges definitely. So, here is a function.

Here this function in the z plane this is one this the unit circle represented by this infinite series inside this unit circle and by guarantee that this function is infinite outside this infinite series diverges outside. You plug in any value of z whose modeless is greater than one and then this series is guaranty to give you divergent answer, but inside this series inside this region here we can actually identify what these series is in same we can same in close form it is binomial series. So, what is that equal to we have f of z equal to summation n equal to zero to infinite z to the power n mod z less than one.

So, write that because otherwise series does not make sense, but I can sum the series expressively what this series equal to the geometric series, you know the answer. This is equal to one over one minus z what that means is, if you take any point with mod z less than one here like and you plug that in to this series find the sum plug it into this formula. Find the sum these two are guaranty to much point by point, but does this makes sense even for mod z greater than one this this thing here why not if I said z equal to six for example, do not take it value which says one over one minus six which is minus one fifth it make sense outside.

So, this form of the function this from on the function make sense even outside and inside this series it is guaranty to much the Taylor series inside the radius of convergence. It is guarantee to match the Taylor series where is the point where this (( )) up only at z equal to one only at this point only at that point it (( )) up otherwise it make sense everywhere all over. And you can check out that this satisfies the conditions everywhere except at z equal to one. So, it is an analytical function, we will write different power series if we writes lots of power series for it, but the fact is this function has a singularity at the point one you cannot avoid it, but everywhere else.

It make sense and inside this region of convergence where unit circle it matches this series which itself does not make sense outside. So, what would you call this function with respect to this series I call it the analytic continuation of the function defined by this series. So, I think gives you the idea of analytical continuation very clearly you found a representation for a function a power series which is valid in some region analytic in that region. If found another representation which matches this point by point value by value, but the second representation is valid in a different region there is an overlap region in the second region bigger and it is the analytic continuation of the same function just same function here now.

Suppose I decided not to do this, but I decided to write this in a different way, suppose I wrote this as one over I want say find the series about this point these minus half.

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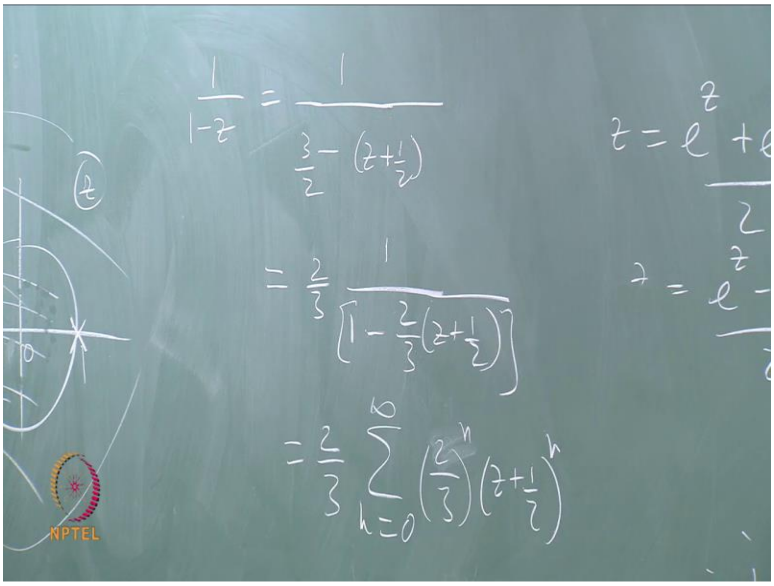

Let see I want a power series about the point minus half, so what will I do, I work backwards. Write this following formula write one over one minus z equal to one over one minus z plus half. So, what I have done subtract half I add that half back against. So, this is three half same as here and then pull out this three half's. So, it is two thirds of one over one minus two thirds z plus half and this I can write back as a binomial series as long as the argument is less than one.

So, I could write this as two thirds summation n equal to zero to infinite two thirds to the power n z plus half to the power of n it is a same function I started with one over one minus z. So, it is a same function, but now it is look like a totally different power series, because the center of this power series is the point minus half and not the point zero. As earlier when does where does this converge this is the binomial series two where does it converge well by the ratio test it is go into converge when module f of two thirds times z plus half is less than one or this module as z plus half is less than three half's it converges in that region.

Now what is the look's of the points it look like not z plus one equal to constant it is a circle with center at minus one and radius equal to that constant right now where is the center of this kind at minus half and what is the radius three over two. So, this converges in a circle of this kind, but it still struck at this point and that is because this master function one over one minus z from where all this power series flow that is got a singularity you cannot escape that. So, you see the boundary of this region can made bigger and bigger, but it will hit the singularity in this point, I can write infinite number of series for the same function with overlap regions. In fact, I could start with this as the center in which case it would converge.

In a semi circle like that and then I exploit that to go here it would converge something like this. So, I can actually go round this point all over I can go anywhere I like by writing different power series such that each time there is an overlap region there all analytic continuation of each other valid in different regions. So, the lesson you have to learn is that a given function analytic function may have an infinite finite or infinite number. So, representations valid in different regions y h function. Of course, we can y h, we will see that suppose you have one over z minus one time z minus two let us call a singularity at one as well as two. So, if you wrote a power series you would end of with two singularities would have a power series which is valid about a point for example, one and half three over two which will have singularity is an both heads.

So, I am going to since that taken e right to the point where I want this is something I want to emphasize that these are all analytic continuation. We are going to spend a lot of time in this course in analytic continuation trying to find these master representation for functions by various tricks. But they take home lesson is that a given analytic function may have many representations. These are all representations when each time you write representation you have to say where it is valid you may or may not be able to find the master representation like this from which you can find all the others may not be possible. In general, not possible, but it still does not stop you from finding analytic continuations in this form. Now on final point is the following and going to state this as a result and will come back it and that is the following in this instance the circle of convergence is unit the unit circle in any power series you can show regularly.

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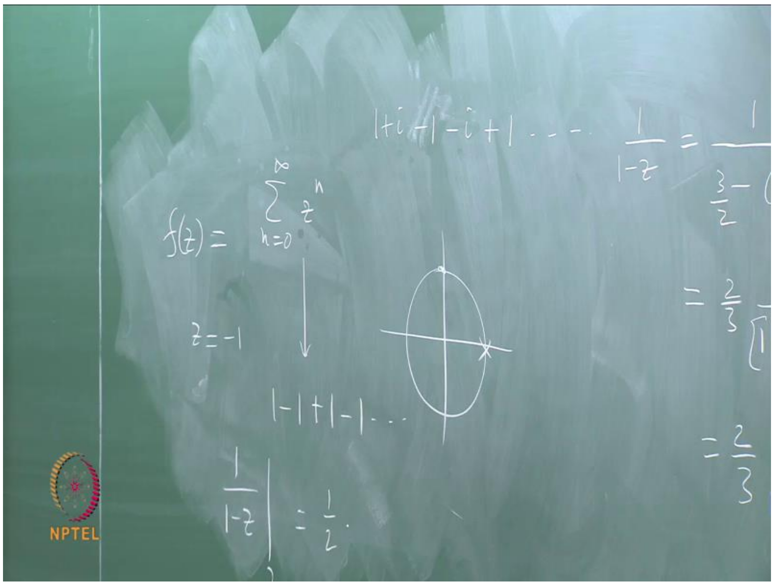

There exist a circle of convergence, which may be infinite for an entire function, but there exist a circle of convergence. And the function the power series is absolutely convergent inside the circle of convergence divergent outside the circle of convergence and not much can be said expect on a case by case bases about what happens on the circle of convergence on that circle you can do all kinds of crazy things, in fact, series will do crazy things. Let us take a look at this for instance we know in this case it is infinite here that is very clear, I put z equal to one and one plus one plus one etc goes to infinite suppose you put z equal to minus one what happens y h one minus one plus one minus one etcetera. So, equal to minus one than this guy here one minus one plus one minus one.

And the series oscillates its value will depend on whether you keep even number of terms or an odd number of terms, but it still has some use even this oscillating series has some use. If I take just the first term what is the value of the series i retain only the first term one I add the first two I add first three. So, I get partial s m s which are either one or zero that is it no other thing two partial sums two distinct particle things one of them is zero. And other is one what is the arithmetic average of those two half right and what is the value of one over one minus z at z equal to minus one it is a half. So, the point is this oscillating series this.

Series does not make sense in this unit circle, but it still has use because at this point where it oscillates if you add all the partial sums it actually gives you the value of the analytic continuation it actually gives you the right value if you take the arithmetic average. Let us try to this suppose you put the point i than what was the partial sums. Now you have one plus i plus i square which is minus one plus i cube which is minus i plus i four which i once again one and so on. So, you see the partial sums one one plus i i and then when you add of all four you get zero. So, what does what is the arithmetic average of those kinds one plus i over two and what happens when you put z equal to i here you get one minus i which is one plus i over two.

And let be complete this sort of thing is called (( )), sum you take all the partial sums and take the arithmetic average them and you guaranteed that the analytic continuation of this function has that value at that point. So, even divergence even non absolutely convergence series will have some use. Now we saw that there was singularity here at this point and thus a theorem which says that every power series must have the function which this power series represents must have at least one singularity on the circle of convergence at least one singularity could be mode, I could be at many points. We look at more examples of this, but it must have a at least one singularity. So, with that let me stop here we come back to power series because they are so crucially important than all that we want to do because all the time.

Our effort will be to try to find representation in terms of power series, because that will tell us guarantee that we have an analytic function typically in its region of analyticity, any function. Analytic function is represent able at each point by an infinite series in general by a power series about the point z naught for instance in a power series in non negative powers of z minus z naught which converges absolutely inside some circle of convergence with some radius which can you can determine from the co efficient. And in that region, this power series inside the circle of convergence is a representation of this analytic function and because it converges absolutely you can take liberty this series you can actually integrate term by term differentiate and so on and so forth.

Now the question is what happens on the circle of convergence outside. The circle of convergence, of course the power series diverges, but what happens on the circle of convergence and I very briefly pointed out that if are appointed on the circle of convergence the series oscillates. When if you compute all the partial sums of the series and take the arithmetic mean that turns out to be that is called the (( )) sum turns out to be the value of this function which represent is represented by this power series. At that point namely it turns out to be if you had a master representation for this function and you substituted this value of z in that master representation than you get exactly which is our sum now off course the it is possible that this series diverges at any point on the circle of convergence and does. So, at more than one point it could diverge almost everywhere on the circle of convergence it could also converge on the circle of convergence need not even oscillate it could even converge. In fact, could even converge absolutely at all points let me give one example

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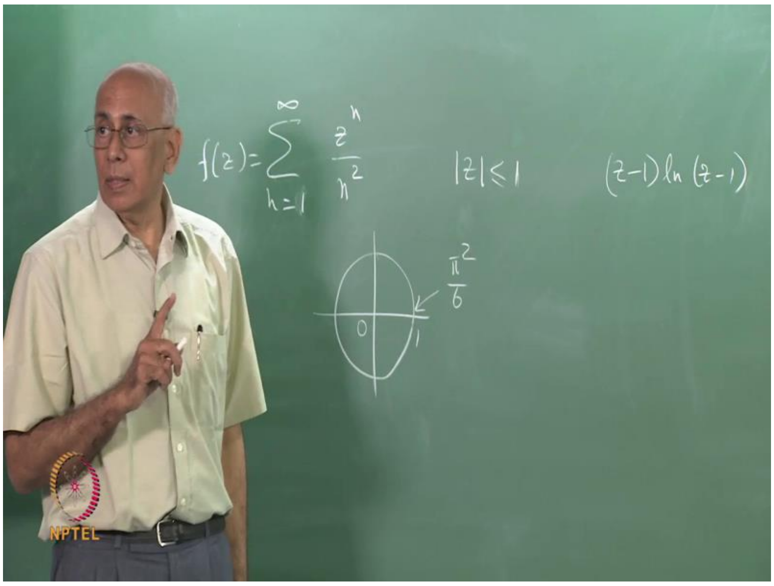

If you look at this series, summation n equal to zero to infinite z to the power n over n. Let us put n equal to one to infinite in square this thing here. This series converges, it converges absolutely for all points even on mod z equal to one, of course it converges absolutely for mod z less than one radius of convergence is unity. And it converges even on the radius circle of convergence at all points everywhere. How do we see this well you can sort of see that the largest value that this series can take on the circle of convergence is one z equal to one itself, because then there are no, factors, no cancelation nothing of that kind is term is as long as it can get z equal to one. And then you have just series one over n square, but one over n to infinite is finite it is z of two.

So, this thing here says even at these point the value of the series is six and everywhere else off course it cannot see this value and therefore, here is an instance where this function represented by this power series inside the unit circle also converges this series converges absolutely at all points in the unit circle. You might then ask we already made a statement I made statement that if a function is represented by a power series inside some circle of convergence than that function has at least one singularity on the circle of convergence at least one singularity I have to define what singularity is our do.

So, today a statement was that at these side one point that has to be singularity of some kind, but then this looks like it is completely regular everywhere at all points. It converges absolutely, but that is deceptive because it turns out that if you call this equal to an analytic function f of z than you can show that f of z has. In fact, a singularity at z equal to one, but settle sort of singularity and will see an example of this little later more carefully for instance suppose the series behave like z minus one log z minus one. Then in the sense of analytic function theory, this log has a singularity whenever it is argument vanishes z minus one off course nothing happens to it its regular. But log is logarithmic branch point as we will see later on, but what is the limit of this function as z tends one what is the limit of z minus one log z minus.

One as z tends to one the log tends infinity and z minus one tends to zero, but which is stronger you thing the power or the log the power is stronger. So, the limit of this as z tends one actually is zero. So, the singular part of this function vanishes happens to be zero, but never the less it is singular at that point. So, you have to understand the singularity is do not necessarily mean infinite do not necessary mean something comes infinite it singular in the sense of an analytic function theory as will explain what this means in this case it has a logarithm singularity at this point this is related to a. So, called data function, but eventually the fact is that this theorem is still valid and you have this strange behavior that a power series converges and converges absolutely even at at all points on the circle of convergence even then the function representing represented by the series does have at least one singularity on the circle of convergence. It could have two three many singularities it could have an infinity of singularities for instance

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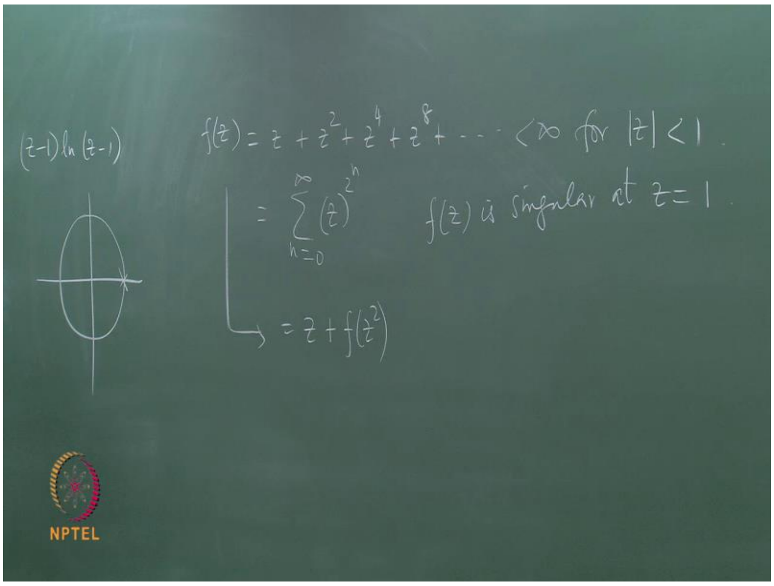

Suppose you took this series f of z equal to z plus z square plus z to the power four plus z to the power eight etc. So, the powers of z are two to the power n as you can see and now what is going to happen. So, these equal to summation n equal to zero to infinite z to the power two to the power n now this function this infinite series has radius of convergence equal to one that is very easy to determine by the ratio test. So, it make sense is less than infinite for mode z less than one on the other hand we know that it definitely diverges if you put z equal to one.

Than off course you have one plus one plus one all the way and this function diverges this series diverges in the function is singular at that point for sure. So, f of one is a singularity of this function f at one is singular f z is singular at z equal to one. So, we see that on the unit circle yes definitely a singularity at that point, but you can also write this as the same series can also be written as z plus a function of z square the same function of z square instead of z you replace it by z square and the remaining terms just give you a f of z square right this quantity must be singular at z equal to one well this part is fine, but this quantity.

Is singular when it is argument is plus one, but when it is argument is plus one z could be either plus one or minus one. So, we did use that this singularity here as well as there for this function, but you could also write this. Now as z plus z square plus f of z to the power four these are terms these are harmless terms they do not have any singularity at any finite z right on the other hand this portion is singular, so this at z four equal to one. So, you have these singularities as well and then off course the next stage you find that you these singularity and. So, on with a result that you going have ten set of singularities on the unit circle and there is no way in which these series this function represented by this series can be analytically continued outside this unit circle this circle here form what is called natural boundary.

You cannot get out of that circle at all. So, this thing here f of z has natural boundary mode z equal to one and you cannot do this earlier trick I mentioned that if had isolated singular somewhere here than you can find the series here about this point we get analytic there and then you can find the series which is analytic there and so on. You can go out you cannot do that there is no analytic continuation possible outside beyond this natural boundary this unity y h.

Now that will never come in one of those thing y h is the theorem that there is an arbitrarily close to that yes there are singularities, you cannot you need a reach you need a finite you need a segment. So, it is not possible to continue this and this sort of series which does not have a continuation outside this kind of thing called lacunaria series constrict many many other such example of lacunaria series. Typically in a (( )) way what happens is, if there are lots of gaps in the powers as you can see here the next term is z to the power sixteen. So, everything between eight and sixteen is absent and then everything between sixteen and thirty two is absent and so on.

So, there lots of gaps in between in this powers and you have very spares powers of z and whenever that happens you get a lacunaria you have the makings of lacunaria series for instance here some other example f of z given by summation n equal to zero to infinite z to the power n factorial. Again this is caught unit radius of convergence and it has no one analytic continuation outside this circle of convergence.

So, this is in fact, the only representation that you have inside valid in the entire this circle circle of convergence, but that is it you cannot do anything better than that. Now these things rarely occur in physical applications, but they do work once in a while natural boundary do appear once in a very rare while is good to recognize this possibility does exist. We say more about power series as we go long will see much more about power series by example.