Selected Topics in Mathematical Physics
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Module - 7
Lecture - 19
Laplace Transforms (Part Ii)

Also have a random processes were you have an integer valued variable which can either increase or decrease. So, the stack age goes up or down, that is the famous random walk problem.

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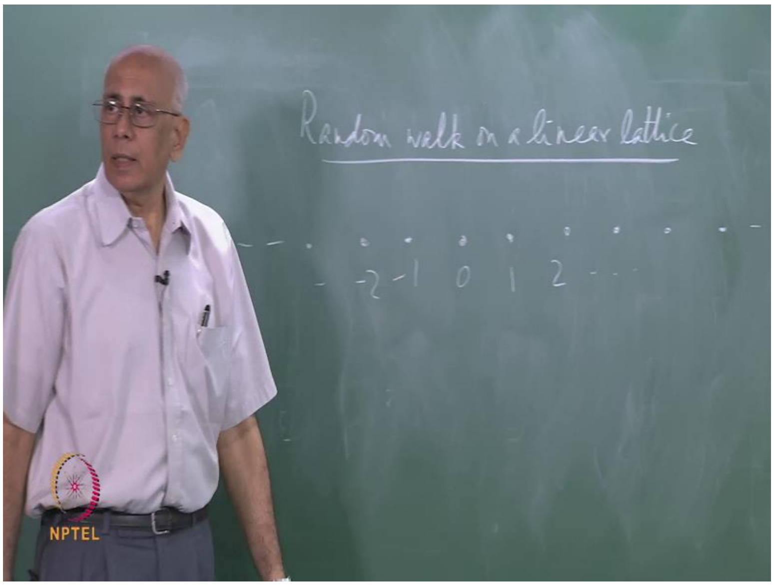

So, let me it walk down since we will come back to it over, and over again. So, let us look at random walk a simple random walk, and the linear lathice we could do this problem in decreased space, and decrease time or decrease space in continuous time, and continuous space continuous time etcetera. We will do it here for decrease space, and continuous time, then we go on to the continuous space we will talk about the diffusion equation. So, what haven mind is an infinitely in linear later as extending both is, and you have a random walker on this laterals what walk at is to toss the coin an if you heads moves to the right if you tales moves to the left once step at the end of every time step this would be a simple random walk, but now we put the little it on say that the coin toss in is done random instant of time. So, the walk away stay an some particular side from arbitrary amount of time, and then take a jump with the mean jump rate lambda once again, and lets label these guys without was a generality since its in infinite laterals of call this site zero this a site one site two these a some general site j, and similarly minus one minus two on this.

And we now seek the probability that if to walk a starts from site j equal to zero without loss of generality, we stack that the starting site to be zero what is the probability that walker is site j at time t subject to the initial condition p of j zero equal to delta of j comma zero this choice of the initial site to be we horizon is only possible if you haven infinite lattes on both sides, then the translation variations otherwise if you have boundaries is an of course it matter where you start an, then there are very interesting effects, but will start to this simplest case of a random walk on in infinitely in a lattes wit this prove is.

So, that the mean time of mean rate of jumps out of this side is lambda some lambda. So, in a given interval of time delta t the probability that a jump occurs to the right or left is lambda delta t, and the no jump occurs is one minus lambda delta t will make life little more interesting, and say that the probability of a jump to the right is some little p, and the probability of a jump to the left is some little q, but p plus q equal to one. So, this a little bias in the coin, and the random walker will move to the right with probability p, and to the left probability q at each a jump, then what is this equation going to b for d p j p of j t over d t equal to well exactly as in the previous case its clear the if any time delta t you must go to the side j if this side j, then a time t minus delta t you should e they have been here.

All in should have been there we not talking about first past it time probability, that is we much more indicate question it is also solvable in we do that some time, but we not taking about words the first time you go on it this in j at time t we could of gone very away we could of come back. So, we do not care all possibilities are included we would like to have the probability that your exactly at j at time t.

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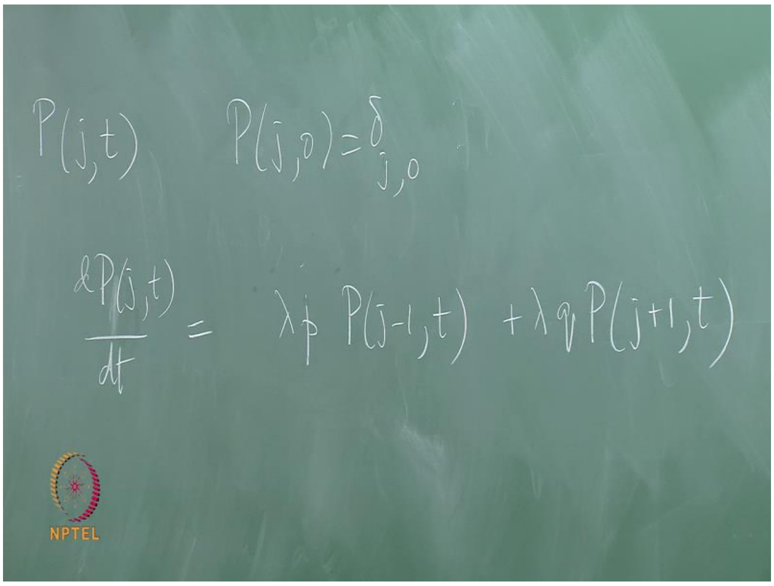

So, at time t minus delta t we should have a arrive time either side j plus one or side j minus one, if you arrive that site j minus one, then you jump with the probability lambda p p of j minus one t that is the mean rate of jump, and that is the probability that the jump is sugar right, and that is the probability that you hit the point j minus one, and I have taken a jump to the right in time delta t I move that delta t to the left divided on. Similarly, as a lambda q p of j plus one t that is the probability that your right j plus one, and at jump back with the probability lambda q, but you could also have been at the side j already at time t, and t minus delta t, and a that interval delta t jumped out. So, that probability is minus lambda p of j t lambda times p plus q, but that is unity.

So, that is the differentially equation in all their view allowed for a bias in the walk, and the question is what is a solution for the s given back when their initial condition once again we use a strength form. We solve this equation is before what is going to happen a now we not valued about p zero playing the special role anything like that, because is in infinite lattes j can increase as well as decrease. So, this equations applicable to all j what should I do for the generating function.

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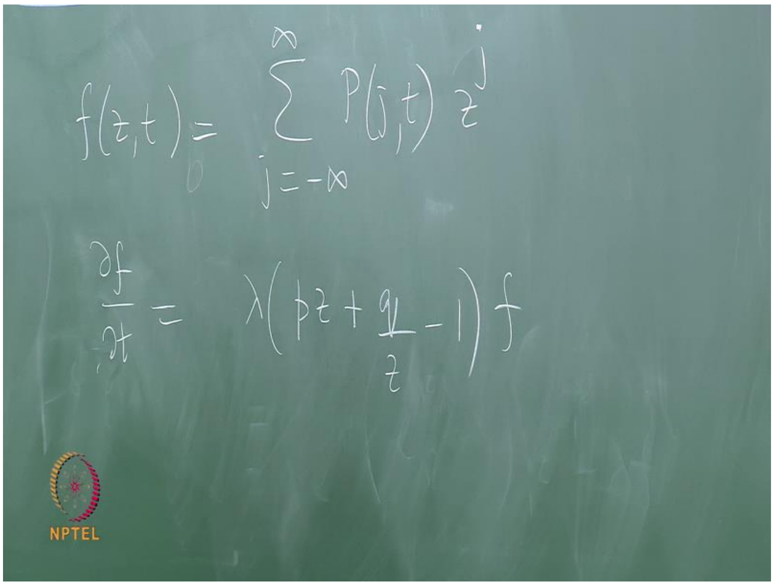

I define an f of z comma t equal to summation p j t z to the power j an pick out the coefficient of z to the power j, but j can take plus, and minus values positive, and negative integer value this summation lambda from minus infinity to infinity what is sort of series is this is not of power series it is a leonard series it is a leonard series now looks at a negative powers of z as well, but it is ok.

We know how to invert to the strength series as well we do do that. So, we need to right the this down like the equation down in this case for the generate in function, and then ask what happens to this guy. So, let us see says delta f over delta t equal to well this term here is just lambda times f in this side. So, there is on minus lambda f on that side, and then what are these terms called give you were this is z to the power j is multiplying it, but this is j minus one here. So, we should convert that j minus one, and then shift j to j minus one this summation, and you get f once again. So, its immediately clear that you want to get lambda times p time z, because you have a z to the power j, and you want make it j minus one. So, you z…

And here you got a z to the j, but you need as a z to the plus one therefore, you divide by z right. So, p z plus q over z minus one times f an, that is it that is in easy equation to solve its x an exponential.

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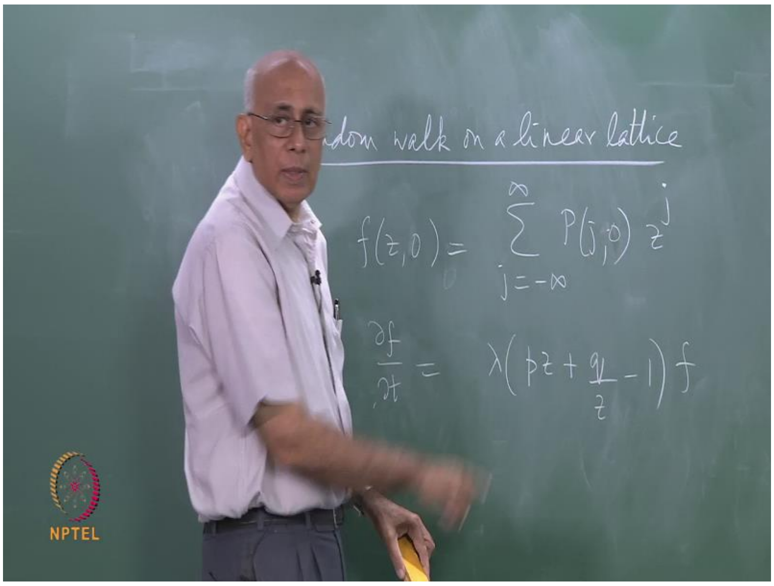

So, its series f of z comma t equal to e to the power minus lambda t e to the power lambda t times p z plus q over z this fashion multiplied by f of z comma zero, but what is f of n's comma zero what is the initial condition.

It is not zero this time, because f of z comma zero is p of j comma zero, but that is a delta function a delta j equal to zero equal to one, because you said j equal to zero z to the j becomes one. So, I do not need this this is one. So, I haven a explicit solution, and I need to the z to the j power in this plus or minus we do not care from this in clear, but that is leonard series now it not a power series. So, unit use the formula for the leonard series do the integral and. So, on, but actually this is a known function it terms out.

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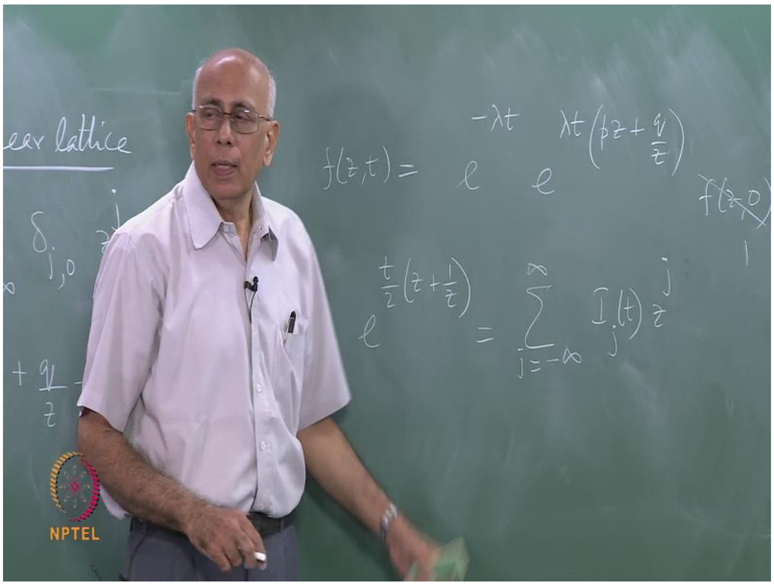

This quantity summation z equal to minus infinity to infinity z to the power z, and then a bessel function I j of. So, let me right this out the generating function for t over two z plus one over z this is equal to summation j equal to minus infinity to infinity I j of t said to the power j this thing here this exponential form is a generating function not of the bessel function of the first kind, but the modified bessel function of the first kind defined in this fashion.

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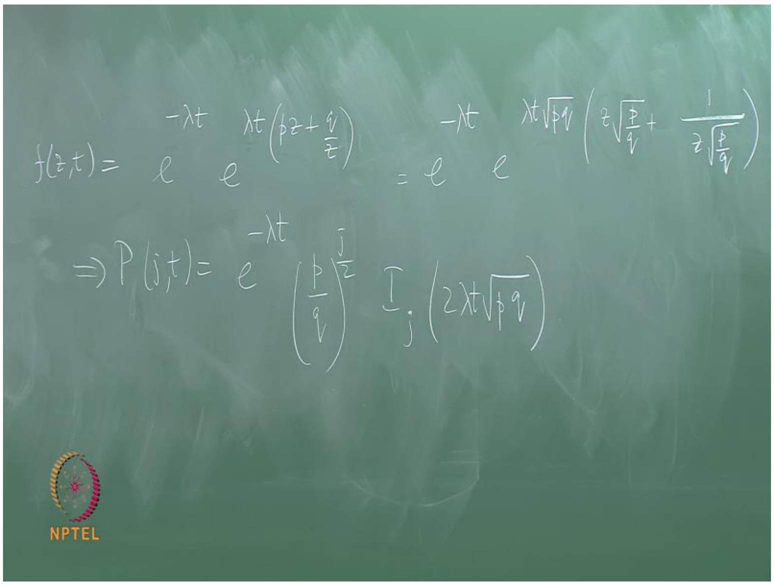

Let me defined in this quantity I mu of z is exact I j mu it is any equal to zero to infinity z over two to the power mu plus two n over n factorial gamma of mu plus n plus one its exactly the same as the bessel function j mu except that this factor minus one to the power n is missing in the some end this is also an entire function. In fact, is you can except except there is a close relation between i, and j if in the original definition of j you made z equal to I z there is a z square to the power z to the power to n. So, that are bean factor I square to the power n which is minus one to the n. So, it is immediately a that j new of z, and I knew of z connected by equation of form z new of I z I time z is proportional to I new z it is also an entire functions satisfied this modified bessel equation and so on.

But we have that at end, and we know all the properties of this I knew. So, can be use that here can be read it of can read of the coefficient of z to the power z not quite right, because you got as z plus one over z, but you got a p z, and a q over z if p equal to q equal to half, then you no problem, because it half comes out an you have precisely this half here, and then off course you can read of z plus one over z, but now you got this unsymmetrical thing your probability of a jump to the right, and to the left are different from each other with like to make that unsymmetrical thing what should to do very simple trick off course is to immediately say this thing here is equal to e to the minus lambda t is the cheap trick a lambda t square root of p q take out square root of p q, and then you have inside z times square root of p over q plus one over z time square root of p over q isn't it you just take out square of p q one, then off course it becomes some. So, this immediately implies right away that implies that p j p of j comma t equal to e to the minus lambda t that is called sit here all the time multiplied by we want coefficient of z to the power j. So, this this factor always appearing along with z to the power j. So, we have p over q to the power j over to multiplied by I j.

There is a to here in the generating functions we got to put at to back in that is the solution. So, the expressed it solution to this probability is some exponential factor e to the minus lambda t multiplied by this bias factor which disappear when p equal to q, and then this modified by bessel function, which is an entire function of its argument in we no it we can defined by extra of power series. So, we have an express it solution to the random walk problem bias random walk problem in what a mention what we need to do to extracting interesting information from if you do not have a bias, then it is clear that the probability of being at plus j must be equal to the probability of being minus j at any instead of time by symmetric in this problem.

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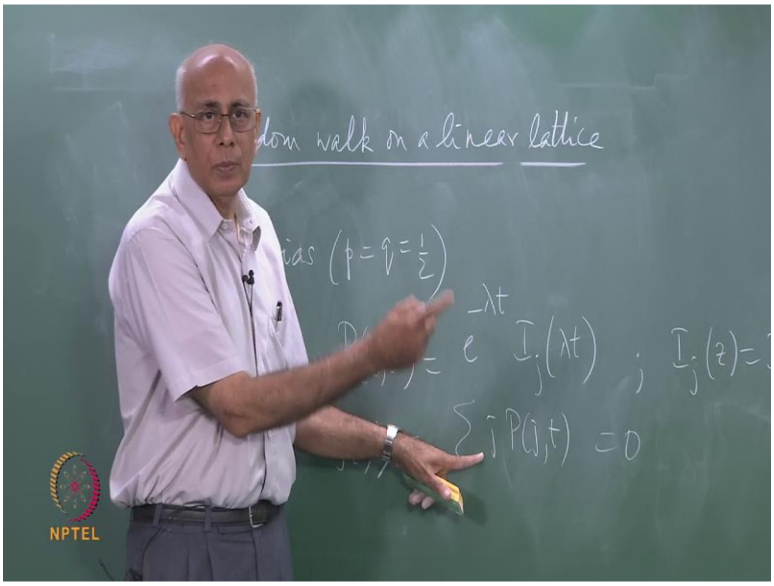

So, no bias p equal to q equal to half this implies that p of j comma t equal to e to the minus lambda t I j of lambda t oh by the by if got to mention that I j of anything is equal to I minus j of z I already mention that of the bessel function j when mu was equal to an integer, then j minus n an j, and not linear depended of each other translator for the modified bessel function they actually happen to be exactly equal to each other in the bessel function k is your have a minus an factor, but your just exactly this same is to. So, this immediately says that p j equal to p minus j as usual except, then you do not have any bias.

They other point is what happens for long times very nice you started of at they all j, then you did this right left etcetera what happens for very long times not surprisingly you can show that be average value of j is actually zero that all times in that is not surprising, because how be define the average value of j this function is normalize to unity, because we started with the differential equation in which we looked at all the possibilities probabilities conserved completely. So, at any time t the walk as to be an some site j from between minus infinitely, and infinite. So, that order probability is unity is some this over j from minus infinite to infinite of guaranty to that one now question is what this equal to since this is normalized it is clear that this is equal to j times p of j comma t.

Some overall j, and what is the value of that what to do yes, yes definitely, because you except that the walker might a taken at any instant of time the mean this placement of the walker from the origin is going to be zero that us as much probability of the walker moving to the left as there is refer moving to the right. So, this is identically zero, because this is an even function of j, and that is on to function. So, the sum is the; obviously, zero what is not zero is a means square this placement that is not going to be zero on that is going to be very interesting as j square of t equal to summation j square p of j comma t, and what you thing that is going to be value you can see that you have a heading to as the problem of diffusion right. So, it is clear that is the random walk going on when the step size is do not to become zero, then it is going to go away in to un unbiased diffusion of some kind now diffusion off course you know that the root mean square placement is famously proportional to the square root of the time. So, the means square this placement. So, is the mean this placement is zero this is the means square this placement equal to this terms out the exactly equal to lambda t at like you proof that this enough material here to able to. So, that very easily. In fact, we should be able to do this using the generating function what would you do go back an writing.

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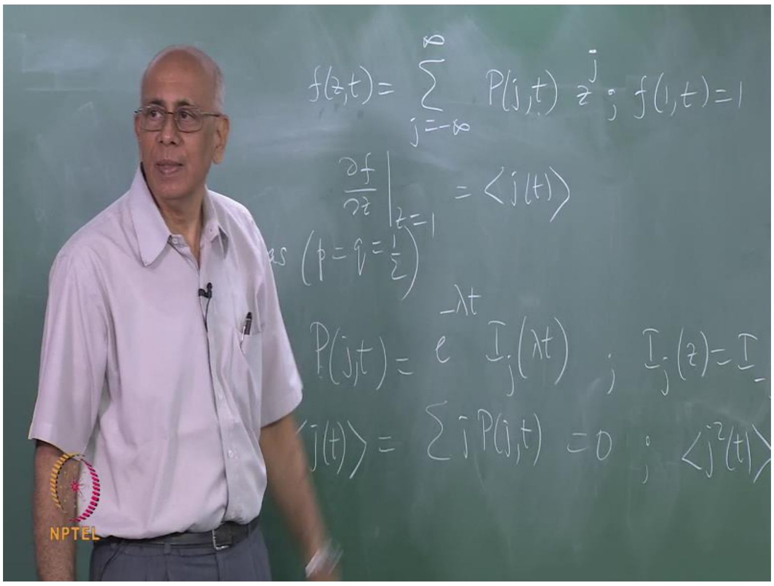

Suppose you do not know the property is of the best function remember that f of z comma t was equal to summation j equal to minus infinite to infinite p of j comma t, and then z to the power j there a way to with respect to with respect to z. So, it is clear that delta f over delta z, and said equal to one differentiate, and then said z equal to one f of z comma one comma t is equal to one, because that is normalization, and if you differentiate is you full a j here, and then said j equal to one. So, this is exactly equal to j of t. So, to establish this result which we saw, then by all you have to do should take this quantity differentiated to once we know the express it expression for at even in the bias case differentiated you end of with mean this placement what to do except is the mean this placement.

In this in terms in the bias case what to do except would it be zero no something well if p is greater than q its clear that the bias to the right. So, you would except the positive value of j of t if p one less than z e q, then you would except negative value right. So, you would except very speaking that the mean value j of t is proportional to p minus q which is acting some kind of draft all you have to do is to differentiate once him an verify that yourself you need the value. So, once you put this in here in computed expressly what the various means are many case from that exponential expression that the easiest place do it this.

Thing here terms out to be lambda t once again in the case one you do not have a bias when you have a bias what do you except this is value what to do except. Now you see what is could happened is that this walker if she is completely unbiased random walk, then she moves to the right hand left to equal probabilities assaults, and on the average does not get any were this placement, but the means square this placement goes like lambda t which means the root means square this placement goes like this square root of time in this problem for all time this happens, but in general when have a diffusion process only in the limit of what is call the diffusion limit good this happen, but this random walk is essentially free diffusion in the presents of a bias you would except systematic drift to one side which means that you essentially have a velocity a drift velocity to one side.

Then the means square this placement will become proportional to the square of the time, because there velocity a drift velocity available, but when p equal to q that is not present, and you have this pure diffusion behavior I want to keep this in might that in unbiased diffusion the means square this placement is proportional to the first power of the time does not it also diffusion equation next continuous place, and time this is want to become of a very crucial feature can be generalize this to high dimension suppose you have a letters in high dimension square letters cubic letters some high of cubic letters in any number of dimensions can be generalize this were now you have to tell me what are the probabilities of moving in various directions. So, suppose you are in three dimension or two say, then there is an up down, and a left right, and each site on a square letters as four nearest neighbor we could say with probability one fourth person moves up or down or right or left in three dimensions as back in front as well. So, one six and. So, what? So, it immediately clear that if you did this in any number of dimensions you right an equation.

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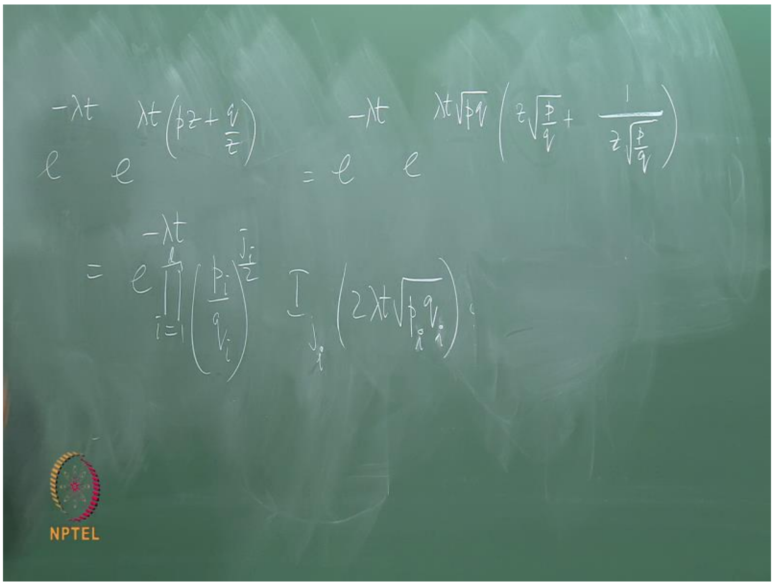

Like d p over d t for the vector j at time t the vector j is specified by a lat set up last points each j component of j is an integer from minus infinite to infinite, and let say you working d dimensions. So, j equal to j one j two j d each of the dimension integer j I is element of set of integers this is equal to now what is the master equation above on to right down.

How do you right this term first this again term we should a eastern nearest neighbor site of this site j, and then you jump in to the site j how many nearest neighbors are there in d dimensions in one dimension there are two two dimension there are four in three dimension there are six. So, there are two three in d dimension. So, this should be; obviously, lambda over two d, because as completely unbiased in this case times p of j plus an nearest neighbor delta some nearest neighbor vector at time t some over these nearest neighbors delta. So, up down front back left right etcetera is a some a down minus lambda times p of j comma t. So, this kind here is a nearest neighbor vector. That is master equation, and you could exactly the same trick as before. In fact, be could co one better be could given say look an go to have a bias. So, in particular direction as have a bias p i, and q I in the component i. So, its p one, and q one for left right p two, and q two are front back p three, and q three four up down an. So, q want I right exactly the same equation down and. So, immediately obvious that this factors completely each time a square root of p I in the over thing factors. So, we can in factorize the exact solution down its equal to I j not the single j, but it is going to be I j one p one q one I order j two two lambda t square root p two q two dot dot dot all the way to the end, and this is going to be a product I equal to one to d p I over q I j I over two in this fashion. So, we could just right this I j I p I q i, and about these that is going to be might p of j that is the exactly solution to the random walk problem in high latest of d d dimension with bias factors p i, and q I for each correction the question is what happens of very long times remember this probability as to be normalized.

So, it is clear that this fact here cannot increase to rapidly not fast, and then this any wit. So, provided jumping factor here and. In fact, is a very simple as entotic formula which says I mu of z goes as mod z tends to infinite like e to the power z over square root of two pi z independent of mu. So, in unbiased case this factor e to the lambda t minus lambda t cancel the exactly by I mu of lambda t, and you have a pure power lot d k, but when you have a bias thus also exponential d k, because a whole probability mass a shift in will come back to the s will look at this is some are detail, and a understand by this random walk problem goes over in to the solution of the diffusion equation in particular will see how this very strange looking thing this modified by bessel functions.

Then you make the lattice constant go to zero an you go to continuous special continuous it go were in to gaussian function, and that will be the solution to the diffusion equation provide some interesting connection between in this, and the gaussian which call the central limited theorem, and we will do this in some detail with later. So, here is a place where used a place terms form in a simple way, but let us think just for a second about the physics behind this we got first order differential equations in time, and they a fairly easy to solve we do not need the place terms form for that at why did we get first order equations in time what was the physical assumption that let to this first order equation I already mention the these are examples of what a called mark of process the point is we wrote in equation the first order, because what happens to p of n references of j at time t plus delta t depends on what happened at time t they immediate previous instants not on things before that in other words the feature is determinate by the present date. So, all conditional probabilities depend on just one times step before, and that is the mark of assumption at be had any history dependent saying how you got to the state, then off course would have higher order differential equations, and then the mato become a little more complicated. So, there are, then some memory in this problem if there random walker as she taking this random walk a says it is true that that is site I coin, and move to the right or left some probability, but if those probability also depended on a memory of in which direction at the previous step in taken had it been from the left this point are had it been from the right point it is like remember in not just the position, but also the moment up an, then of course you have second rather in higher order differentially.

That's call persistence diffusion an it has its own interesting, and then mathematical solutions. So, it is a one step process if remembers both position, and velocity, and then mention this, because when we talk about with diffusion in equation we will see that what happening there is that only talking about to position of the diffusion particle not its moment, but if you want it do the same thing in face space, then you have to worry about the position, and the moment in remember than the equation skate higher order differentially equation in time in. So, its order is worth remembering that first order equation there a plus term form method practically not necessary, but for higher order equations it gets more, and more useful.