Selected Topics in Mathematical Physics
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Module - 6
Lecture - 17
Multivalued Functions: Integral Representations (Part IV)

We coming to the end of our short excursion into complex analyses, and then I will come back to it over and over again, but complex variables per say we modulus going to be done with it today. And then we will come back to it as and when we need it, but this one more small digration that I would like to make before we move on to the next topic, and that has to do with integral representations for another class of function special functions which are familiar with namely the Legendre functions.

Now, I already mentioned and we saw examples of integral representations of the gamma function, the beta function, the remand zeta function and so on, but there are whole host of functions in mathematical physics used in physical problems like the Legendre functions, the Hemick functions, the Leger polynomial ((Refer Time: 01:02)) polynomial, Gagarin bowed polynomial and so on. All of which are related to something called the hyper geometric equation, and these functions have a lot of interesting special properties they are called special functions for that reason, and they have a huge number of properties which have been studied.

Now, what are like to do today is to show you that a couple of these functions the Legendre functions in particular have certain very interesting integral representations. And from those representation you can actually define these functions in a very in a much broader range of variables than originally introduced namely Legendre polynomials we introduced p l of x, for x between minus 1 to 1, and l an integer 0,1, 2, 3 etcetera, but today we like to generalize this a little bit.

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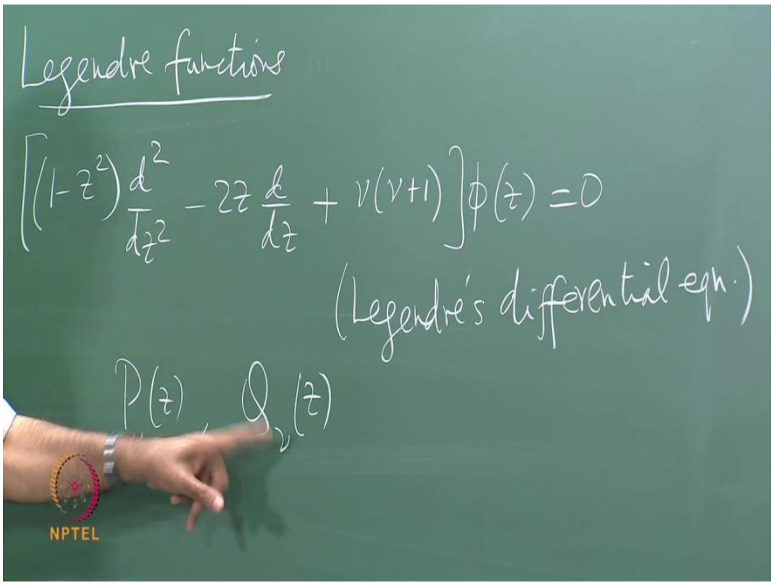

I recall to you, so let may call this Legendre functions, recall that these functions are solutions of a certain specific second order differential equation, which you would normally write down in elementary treatments as equations in a real variable x say between minus 1 and 1, but you do not have to do that the equation itself is valid for all values of the complex variables said. And it read something like this 1 minus z square d 2 over d z 2 minus 2 z d over d z plus instead of the order l restricted to an integer, let me use another symbol nu say which is going to take unrestricted values over the complex domain.

So, nu times nu plus 1 this thing operator acting on some phi of z equal to 0, this is Legendre equation. And in general this is some complex variable parameter, and this is also complex variables. This solution of this second order differential equation, the 2 linearly independent solution and they are the once which you call the Legendre functions, so the first and second kinds. The first kind is P nu z, and the second one is denoted as Q nu of z. So, this is the Legendre function of the first kind of order nu and argument z, and ditto for this is one of the second kinds here.

Now, this is the function which for integer values of nu like 0, 1, 2, 3 etcetera, and for z between minus 1 and 1 you call the Legendre polynomials, but this a much more general thing than that. The Legendre polynomials are special cases of this function here are the several thing you should observe about this equation, the first one being that this is really an Eigen value equation for the following up for the following quantities.

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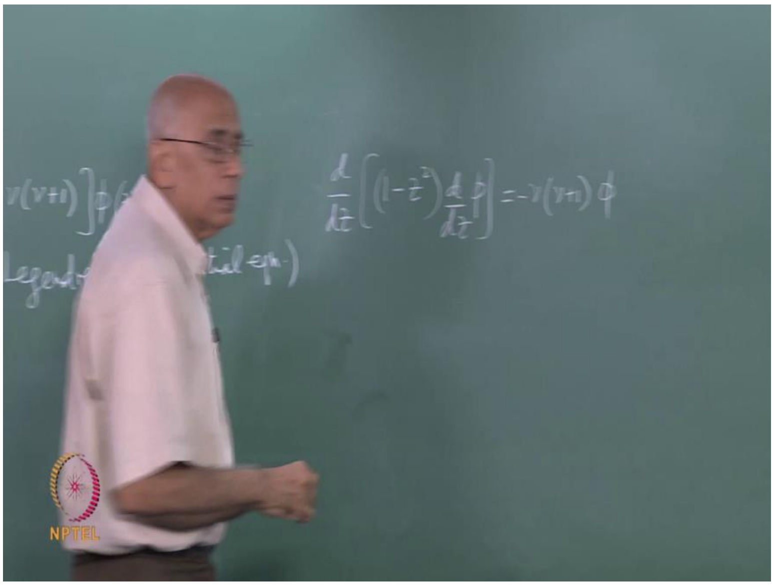

So, you could also write it as d over d z of 1 minus z square d over d z phi is equal to nu time minus nu times nu plus one phi. So, cast in this form it comes under the category of what are called ((Refer Time: 04:33)) equation problem this ((Refer Time: 04:33)) problem, and then there is a huge literature which tells you how do solves such equation and so on. No doubt you familiar with the solution to this equation by the method of series, therefore ((Refer Time: 04:46)) method of series, and then you get for special values of l, integer values of nu you get the Legendre polynomials as the solution. But, what I would like to emphasize is that this thing here these functions actually a valid have valid representations for complex values of z, and here is what the representation looks like, I want derive it I will just write this down explicitly.

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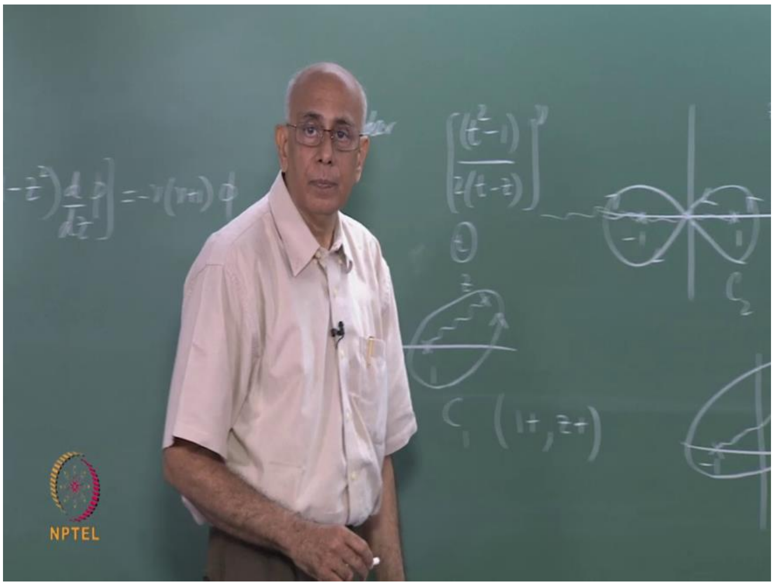

Consider the following, consider as a function of the complex variable t consider t square minus 1 over twice t minus z to the power nu as a function of nu, for some arbitrary complex z for some given z, and nu some complex number. In general this function has singularities in t, and where are the singularities? Well, when this is an arbitrary complex number you have winding points all over the place, you have a winding point at t equal to plus 1 another at t equal to minus 1, and the third one at t equal to z itself.

So, in the t plane there is a singularity at plus 1, this is singularity at minus 1 and these are in general winding points a branch points, and there is a singularity at the point z which could be anywhere, so let say z is here at this point. So, here is the function with three branch points, and in general there is also singularity at infinity, because this thing goes like t to the power nu at infinity and definitely this is a singularity at infinity. So, there all these branch points in this plane.

Now, a contour integral representation if it is a close contour, I have already pointed out that when you have branch points you can have the close contour provide it the contour returns to its original that then integral returns to its original value once you come to the end of the contour to the starting point once again. Now, what when is that would happen? Well, if I in circle the branch point at t equal to plus 1 once in the positive sense I pick up a phase e to the 2 phi i nu, if I do it in a negative sense e to the minus 2 phi i nu similarly for the branch point at t equal to minus 1.

And if I encircle the branch point at t equal to nu at t equal to z, once in the positive sense I pick up e to the minus 2 phi i nu, because it is in the denominator here. So, this immediately suggest that if I go round this circle this singularity at plus 1 once in the positive sense, and at t equal to z once in the positive sense I am go in to come back to the starting point. In other words, I can choose cuts in such a way that encircling both these things and nothing else once in the positive sense brings me back. So, this means that there exist if z is here for example just for illustration. This contour would be a close contour which is equivalent to sink that I can choose the branch cut go like this, and the cut from here I can choose to go after infinity in this direction.

So, with suitable choice of cuts and phases this is the close contour, so let us call that C 1, and this is it says you end circle 1 once in the positive sense z once in the positive sense, similarly you could do that with minus 1. So, you could also do it minus 1, but this one more thing you can do in that is here is 1, here is minus 1, here z you could encircle this once in the positive sense, and this guy here once in the negative sense, so you could do this. This contour let us call it C 2, this is 1 plus minus 1 minus, because the first one picks up e to the plus 2 phi i nu, and the second one picks up e to the minus 2 phi i nu and they cancel, and you back to the starting point here.

So, regardless of how I choose cuts and so on, I can arrange it such that this contour is a close contour once again and it is clearly different from the contour C 1. Now, the cut from z it other branch cut from here goes off to infinity in that direction. There is of course a branch cut from here to infinity, another one from here to infinity etcetera, but it does not matter we went sure that by doing this your back on the first sheet here starting from the principle sheet your back to that original value, the phases are cancelled out. Well there is one more possibility, and that is in the t plane you have z here you could have encircle these 2 both in the positive sense this is 1, this is minus 1 and that z.

So, the cut is like this and this cut can be arranged go off to infinity in that direction. This two will bring you back to the starting point and this contour is C 3. So, I want you to convince yourself by playing around with this that these are really the only independent once there are any more. And more over that this contour is relate whatever you get here is going to be related to what about you get here by change a variables t to minus t are something like that. So, really it is just C 1 and C 2 that we have to deal with as independent contours.

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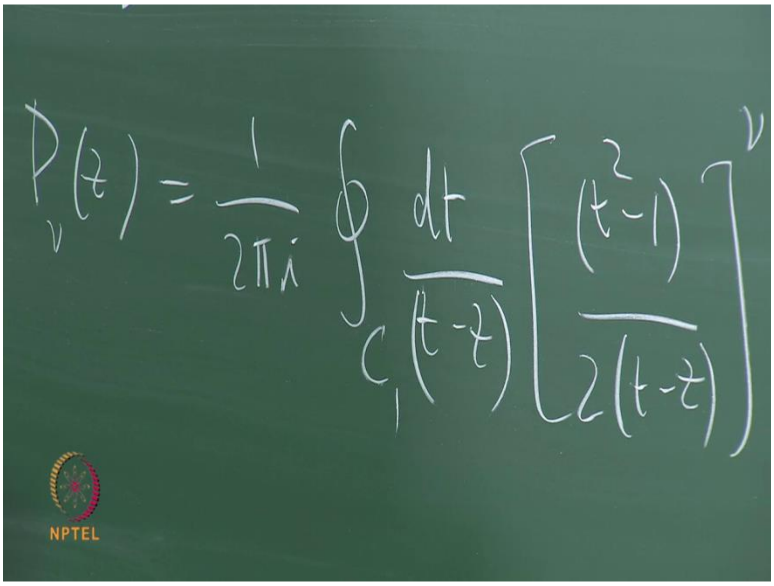

And turns out that this function P nu of z equal to 1 over 2 pi i over the close contour C 1 d t over t minus z times precisely that integral there t square minus 1 over twice t minus z in power nu with this extra factor here, this turns out to obey exactly Legendre equation and reduce to the Legendre polynomials for integer values of nu as we will see.

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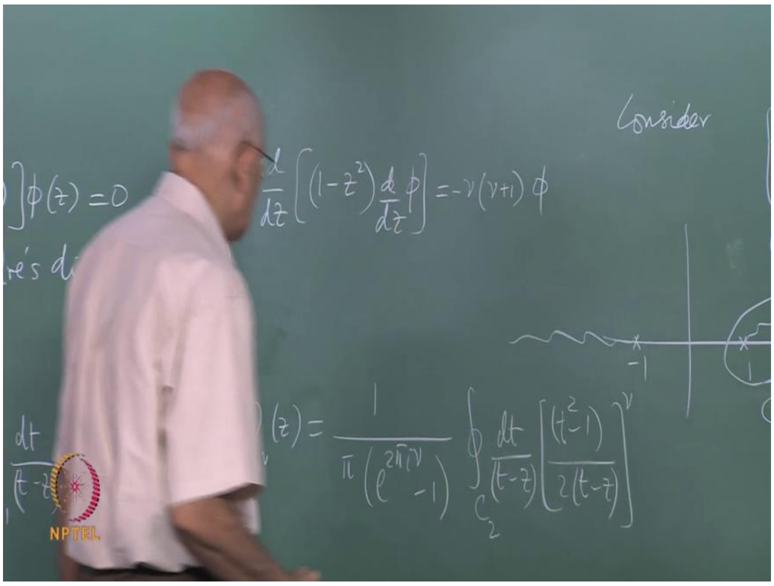

And Q nu similarly turns out to be 1 over I am not sure about the factor here, but pi times I know for sure that there is this factor that is the sign pi nu and then over C 2 d t over t minus z t square minus 1 twice. So, the 2 solution of Legendre's equation the linear independent solution, and given by suitable contour integrals of this integral of this implies which as branch point z plus 1 minus 1 and the point z itself and from this you can read of all the properties. Incidentally you can verify that these functions satisfy this equation is not such a trivial job, but that is definitely doable as well as all the other properties which we going to examine. So, to start with let us look at what happens when you have non integer values of nu in general. Then it is clear that you expect the following things to happen.

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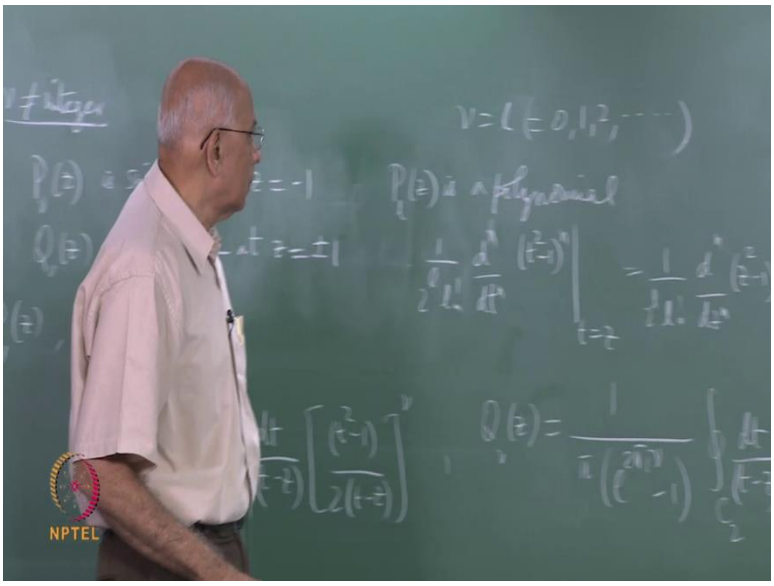

So, nu not equal to integer in this case. Well we will say a little later, we talk about singularities of integral today itself that this function t nu will have a singularity then the following happens, how can such a contour integral ever have a singularity. Well remember that the whole thing is function of z. So, if I moves z this pole moves around, this branch point moves around, one thing that could happen is z tends to 1 of course it is collapses that could happen.

The other thing that could happen is at z moves around and hits minus 1 in the z plane, when that happens you cannot distort this contour away, because it is close contour is trapped between the singularity at minus 1, and the singularity of integrant at z itself and it is pinched between them and you get a singularity in which variable in the variable z. So, we expect that this thing here P nu of z is singular at z equal to minus 1and we need that so. We can show explicitly that there is a singularity a certain singularity the branch point in general at the point z equal to minus 1. Keep in mind the fact that the contour is in the t plane, but the function whose analytic properties we are discussing is the function of z.

So, you move z around in the z plane, and this movable singularity in the t plane moves around in it does thing to the contour that is how singularities would arise. What about this guy, what kind of singularities can you expect here? Well, here the only way I can have a singularity is it this branch point here attacks the e the plus 1 or minus 1, and then I cannot move the contour away. If it comes towards this point I can distort it a little bits etcetera, but if it hits plus 1 if it is directly toward plus 1 it is abstract. So, I immediately expect that Q nu is singular at z equal to plus or minus 1 at both points for general values of nu that is my initial expectation, this is what is going to happen.

Now, let us look at what happens when you have integer values, and then a whole lot of cases would arise immediately. So, let us look at a nu equal to l equal to 0,1, 2 and so on. So, let us look at the non negative integers first, and ask what happens to P l. Well, in that case this branch cut disappears completely, and then what happens to this. So, I said nu equal to l where l is an integer thus already a t minus z here, and then what happens. There is no branch point at plus 1 or minus 1 is gone is to the power l here some non negative integer power, and you got a t minus z to the power l here.

So, there is the pole of order l plus 1 at the point t equal to z, and what is this contour integral reduced to? This integral reduces here plus 1 here is minus 1, and there are no singularities at these points now which is gone, and all you have is the pole and the integral encircle it in the positive sense, and you just have to evaluate the residue at this point in order to find out the value of the integral. So, it immediately says P l of z what can it P. Well, it immediately tells you that you have a pole t minus z to the power l plus 1, you need the l plus 1 derivative or l's derivative of this function to find the residue.

And t square minus 1 to the power l positive integer l is some polynomial and you need the derivative of that at t equal to z. So, we end up with the polynomial. So, this says is a polynomial and you can even write a formula it is 1 over 2 to the l factorial d n over d n over d t to the power n t square minus 1 to the power n at t equal to z, I might as well write this as equal to 1 over 2 to the l l factorial d n over d z to the power n z square minus 1 that is what P l of z is do you recognize this formula? What is it called?

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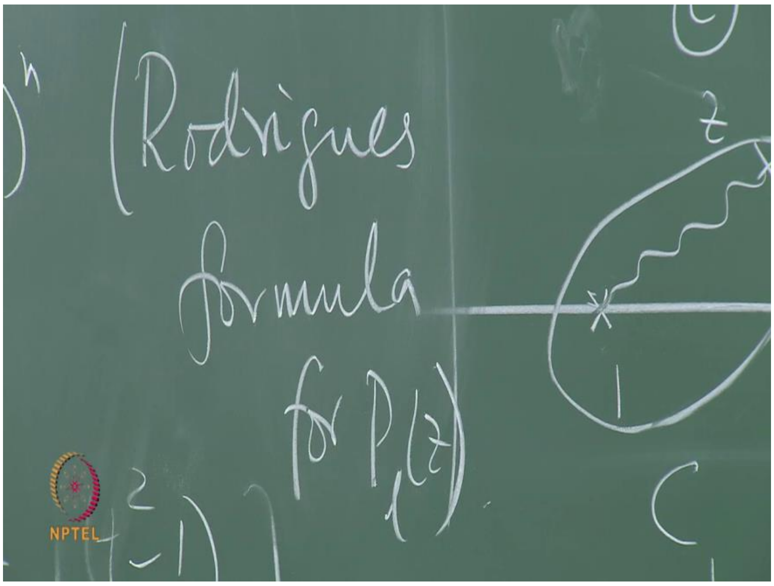

It is the Rodriguez formula for this Legendre polynomial for P l (z), and it is true for complex z is true for arbitrary complex z not just for z equal to x equal lying between minus 1 and 1. So, this function p l of cos theta which where use to cos theta is a number between minus 1 and 1 it is a real numbers real variable, you do not need that it is true in general this Rodriguez formula is true for complex values at z as well. P l of z for arbitrary complex z is a polynomial of order n, what happens if l is an negative integer say minus l minus 1 where again l run 0, 1, 2, 3 etcetera what would happen then.

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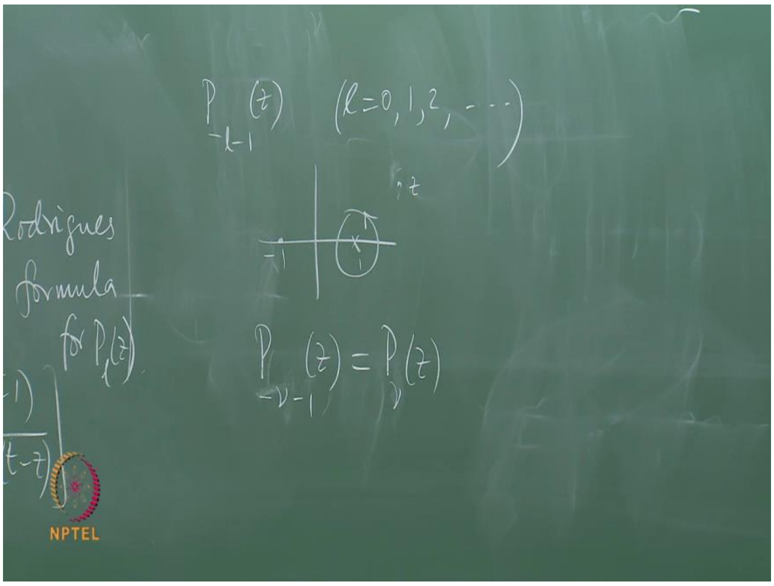

So, what about p minus l minus 1 of z where l equal to 0, 1, 2, etcetera what would happen then. Well go back to the general formula and put nu equal to minus l minus 1, and this time you end up with the pole at t equal to 1, because this gets in the denominator it is some integer power, and you have pole at t equal to 1 and there is no singularity at t equal to z that goes up in numerator.

So, once again the contour collapses, but it collapses to this is the pole at 1 and there is this and there is z is sitting somewhere here and here is minus 1 from this point. And again you get some derivative or something like that, and I leave you to verify that you end up with P minus l minus 1 of z P equal to P l. So, there is a symmetry property of the Legendre polynomial about the point minus half in the order l, you should it be 2 surprise to see the kind of symmetry, because if you go back to the differential equation notice that l times l plus 1 appeared.

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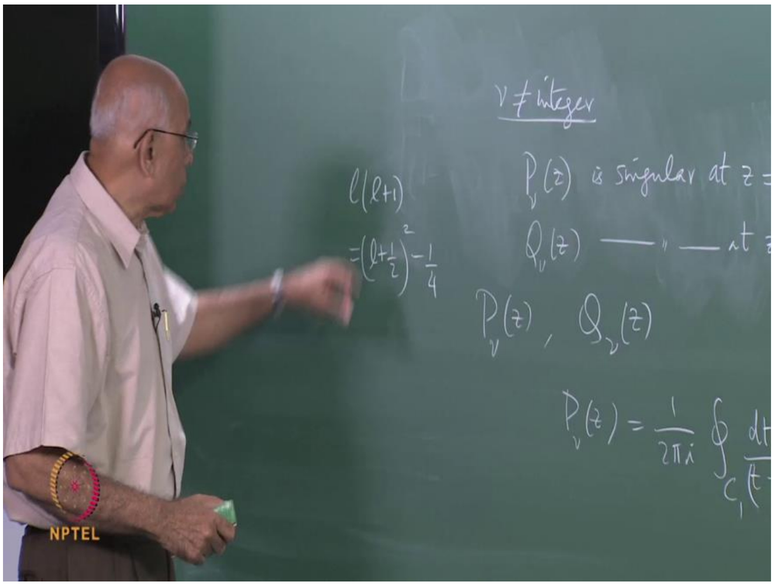

And you can write l times l plus 1 as equal to l plus a half whole squared minus 1 quarter, it is a square of this that appears. So, it is clear there is symmetry about the point minus half. So, that is why you have that symmetry you have P minus 1 equal to P 0 P minus 2 equal to P 1 and so on. In fact, it is true in general, because what appears in the differential equation is nu times nu plus 1. So, I would expect that symmetry to be valid, and it is in the true it is in the true that this is so for arbitrary complex nu, this P the Legendre function of the first kind has the symmetry property. But, we see again that P minus l minus 1 is also a polynomial, what about Q? So, let us look at what Q does Q is sitting there.

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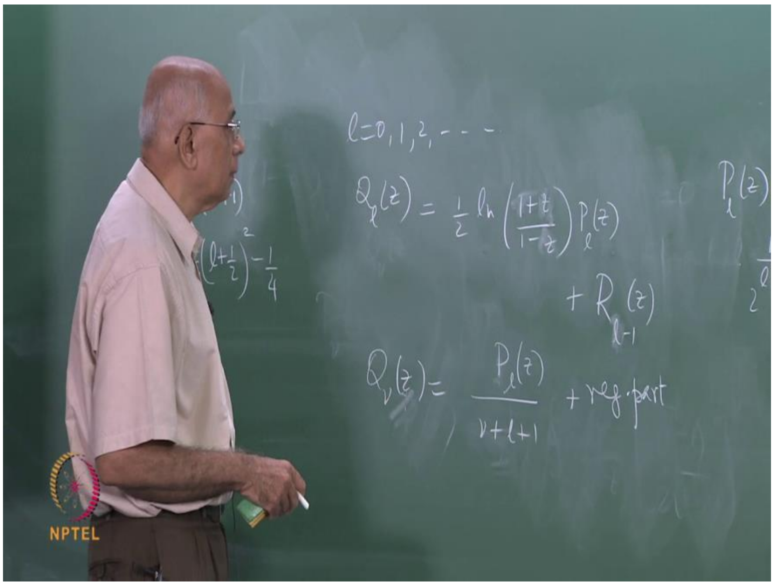

Now, we got to ask what does we have l equal to 0, 1, 2, etcetera, and we looking at Q l of z and if you look at it is integral representation we had here a nu. This contour now encircles that now that the singularity at the one positive sense and that minus 1 in the negative sense, and what does it do? Well, this thing becomes an integer, but you still have this sitting here at t equal to at nu equal to a positive integer this contour integral vanishes, because the singularity that it encircles has disappeared. It is t minus 1 to some power t plus 1 to some power which are integer power, so this integral vanishes.

On the other hand this vanishes to in the denominator. So, just as in the case of the gamma function, where for positive integer values of n we had in integral that vanishes and we had the denominator function sin pi nu or pi z are something which also vanished and we have to take a limit in exactly the same way, you have to take a limit here and the answer is some finite number. We will see what number it is, but let me give the answer here. It turns out this quantity here can be written as 1 half log x plus into 1 plus x over 1 minus x I am not sure about the sign of this log times P l among plus l of z plus a polynomial of degree l minus 1 in z.

So, typically what happens is that you have a logarithmic singularity at plus 1 and minus 1, and then you have a polynomial part, but I already argued from the general representation that Q l is expected to have a singularity when z it is plus or minus 1, because what happens is that this singularity at t equal to z comes and pinches with t the plus 1 or minus 1 pinches the contour. So, sure enough there are logarithmic singularities. We will see later why it is logarithmic rather than anything else.

In particular we know that p 0 of z is 1, and this disappears and you have Q l Q 0 is just a log that is another name for this log 1 plus z over 1 minus l. But the other Q l the higher Q l is could be some logarithmic functions multiplied by this polynomial plus some polynomial, what about negative values? Now, denominator slightly different situation in the case of negative values this guy here goes in the denominator, and the contour integral does not vanish. So, it is some finite number contour integral, but this vanishes here.

So, the result is Q nu of z as nu tends to a negative integer has a singularity in nu, and in terms out that Q nu z equal to 1 over nu plus l plus 1 is passion where l is 0, 1, 2, 3 etcetera, so the singularities at minus 1, minus 2, and so on multiplied by residue and the residue interestingly turns out be P l of z itself plus regular part. So, the Q (s) and P (s) are intricately intimately related to each other in this fashion. So, as a function of nu now, there is singularity of this function here.

So, the right way to look at all this function as is as function of 2 complex variables of the order as well as the argument, most of the time we are interested in the argument, but the order also displays very interesting properties here. Incidentally the fact that there is a log here that is sitting here, suggest that what is really happening is your integrating P l in some fashion with some kernel which is like t minus z or something like that.

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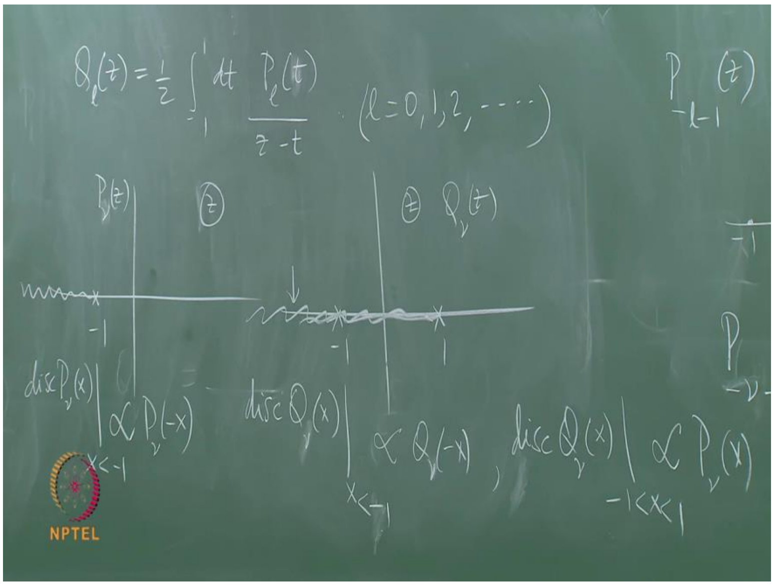

And sure enough there is a connection between the 2 which does the following sure enough Q l of z equal to 1 half minus 1 to 1 d t P l of z over t this is z minus t. So, it for non this is valid for l equal to 0, 1, 2 etcetera. So, you have an expression for Q l as an integral over P l multiplied by this cushy kernel 1 over t minus z from minus 1to 1, and this formula is valid only for non negative integer values of l, for negative integer this is perfectly fine, but this is singular.

So, this thing will tell you that you can expect a logarithmic singularity, because when can you expect the singularity of this integral itself P l is a nice function is just a polynomial, but this integral they integrant has pole of t equal to z, and when z hits if the minus 1 or plus 1 you cannot move the contour away and you expect the singularity, and that is have you end up with this logarithmic singularity as we will see a little later. In fact in general, this is what happens to in the z plane to P nu of z for P nu of z there is a singularity at minus 1, and thus a cut running all the way to minus infinity.

For Q nu of z again in the z plane singularity at 1, singularity at minus 1 and there are cuts running all the way to minus infinity conventionally. The discontinuity of Q l across this cut of Q nu across this cut in general is proportional to P nu that is what we will see from this formula. This discontinuity here, so by what do you meant by discontinuity? I am asking what happens if I take if z approaches the real axis from about, and from below and you ask what is that the difference between the 2 functions that is the discontinuity.

So, discontinuity Q nu of x, x less than minus 1 is proportional to Q nu of minus x I am not going to prove this, but the jump here is proportional to Q nu at minus x out here, and the discontinuity between minus 1 and 1 is proportional to P nu. So, discontinuity of Q nu of x minus 1 less than x less than 1 is proportional to P nu of x itself. And similarly the discontinuity here discontinuity P nu of x, x less than minus 1 is proportional P nu of minus x. So, this shows you can actually write dispersion relation for these functions in terms of these discontinuities here.

So, the Q (s) and P (s) not only satisfy the differential equation, but they also satisfy the integral equations of this kind, because of these interesting analytic properties. Let us see where these comes from, I made statement about this logarithmic singularity and so on let us see where this comes from. So, let us ask for a very elementary treatment of how functions acquire singularities, functions defined by integrals acquires singularities. So, let us start with the simple examples.

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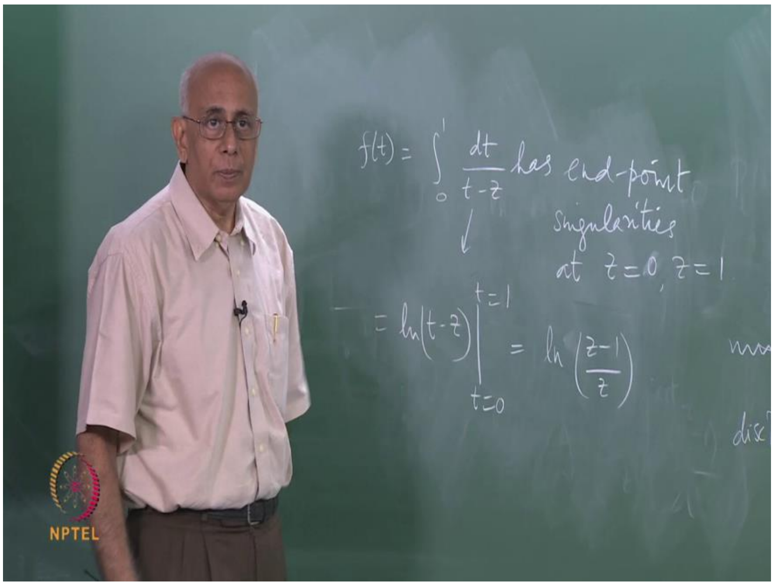

Let us look at an example like f of t equal to integral 0 to 1 d t over t minus z it is as the simplest possible thing. This is an analytic function of z where over this integral exist, but you expected to have some singularities somewhere. Now, where would you expect that? Well, in the t plane the contour is a line segment running from 0 to 1 that it is fix it expect that the 2 ends and it is like a rubber band, you can stretch it you can do all kinds of things, but you cannot take away the end points.

On the other hand in the z plane, if z is any value here, there is a pole of this integrant at t equal to z, so it is sitting here this pole. If I move this z around in this fashion, this pole moves around in the same fashion, but when this comes and hits 0 out there, then this pole attacks that point and you cannot move the contour. On the other hand if I took a path like this, and when down this no point problem at all, because this thing would come down like this and I can move the contour away ahead of it without changing the value of the integral. Because, the function is still analytic I do not let this singularity hit it, I just distort it as much as I like.

This thing is gone to fail only on 2 accounts one of them would be if the contour gets drag all through it infinity the integral may diverge. And the other is if this singularity hits the origin or the endpoint of integration I am in trouble and these are called end points singularities. When a singularities of the integrand attacks one of the endpoints of integration pad me

Student: ((Refer Time: 32:51)) close there on that

It is not the contour is not closed I am talking about an integral; I am talking about singularity of an integral which is a line integral from one point to another. If it is a closed contour there is no bound there is no endpoint to it, then what can happen is in pinching that I mentioned earlier and we will come to that. So, end points singularities occur when the integral is defined with 2 end points. This has end point singularities at z equal to plus or minus at z equal to 0, z equal to 1 on these 2 points

Well, we can corroborate this by just doing the integral, this by just do the integral be done with it. This particular case you can do the integral explicitly. So, what does this give you, is equal to log t minus z running from t equal to 0, t equal to z that is equal to log t minus z, z minus t, t equal to 1. So, that is equal to log 1 minus z over minus z. So, that is log z minus 1 over the z, and they we are I know that this is log z minus 1 minus log z, log z minus 1 is a singularity at z equal to 1 logarithmic branch point, and it has the logarithmic branch point at z equal to 0.

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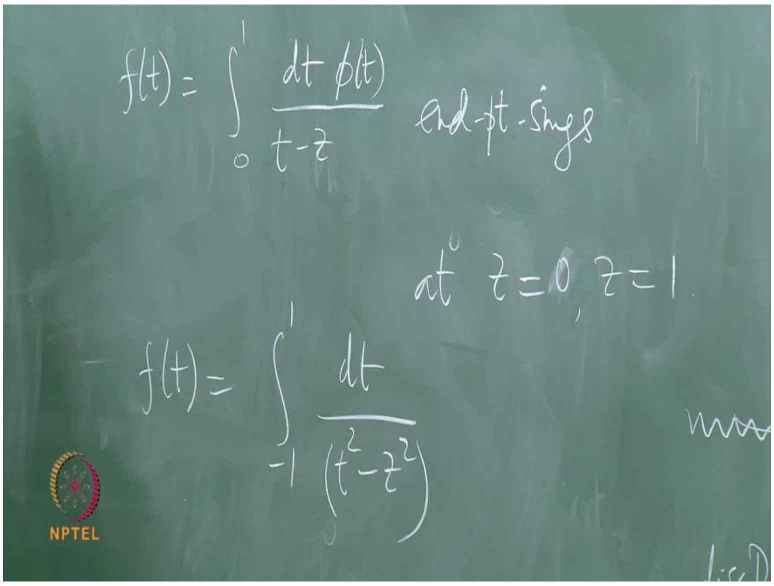

But I do not need to do the integral in order to come to this conclusion, because I could I had a more general integral like this multiplied by some phi of t some nice smooth function such that the integral exist, and you still have endpoint singularities at these 2 points. Of course, it is not equal to this anymore, but barring accidents generically it is gone to have end points singularities at these 2 points here. Now, what about this phenomenon of pinching?

So, let us look at the next example which is the following f of t equal to integral C minus 1 to 1 d t over t squared minus z square, please look at this squared this integral where would this have singularities. The first thing you have to do when you want to analyze how an integral becomes singular, this to ask where are the singularities of the integrand, and then look at how those singularities move as the variable z moves. Whether the singularities of this integrand at t equal to plus z and minus z and they both simple poles they both simple poles, and the contour runs like this.

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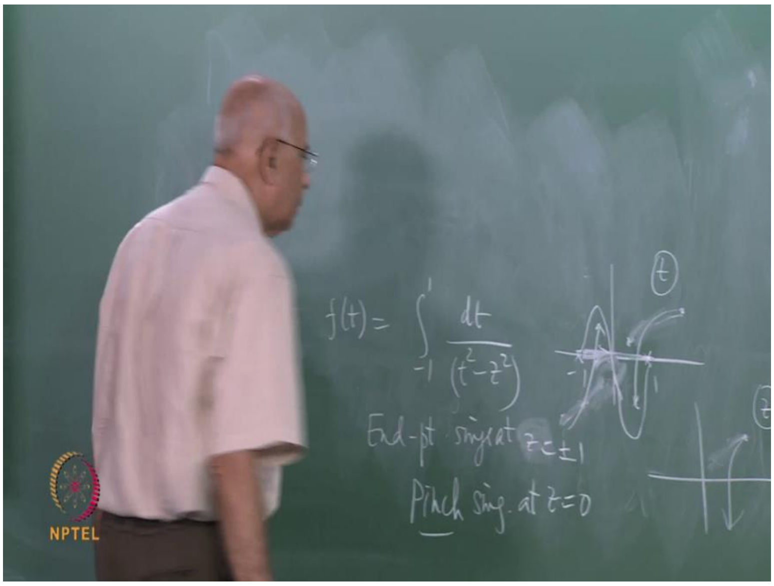

In the t plane you have 1 here minus 1 here this is your contour and in the z plane, so there is a singularity at z. So, z is some other is singularity at that point, and the other singularity is at t equal to minus z which is at this point. It is immediately clear that if z hits plus or minus 1, minus z hits minus 1 or plus 1 the other way, you have end points singularities. So, I expect that when z hits plus 1 or minus 1 you going to have end points singularities, but there is one more possibility as z toward moves towards the origin this point moves here, and the other point moves here to this side it remains on this side what will happen as z hits 0?

The contour gets pinched between the 2 and there is no escape, otherwise if z move down like this in this fashion is no difficult at all z hits this path, so I just distort this in this fashion. Well this point starts moving up, so I distort the contour here, and then you have this and you have that, and the integral still well defined. The only way to not be well defined is if z heads for 0, and then of course the pole at minus 0 and plus 0 coincides, but from opposite sides of the contour the pinch the contour between them, and then have a singularity of the integral, what would you call such as singularity.

But the last time it hit an end point and you called it an endpoint singularity, and now it is pinching the contour, so you going to call it a pinch singularity, this is the pinch singularity. So, it has endpoint at plus or minus 1 pinch at z equal to 0 this is all in z, what happens if you do the integral explicitly? Let us just do the integral and see what happens.

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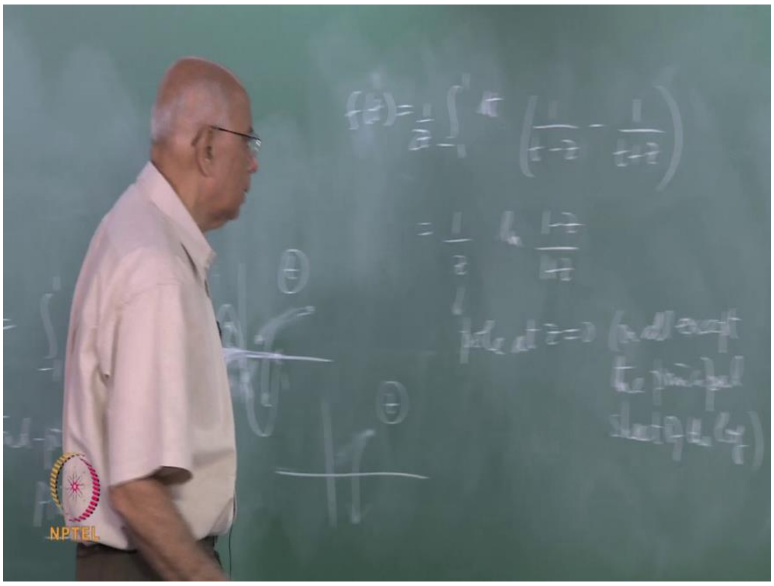

You have f of t equal to integral minus 1 to 1 d t, and this can been written as 1 over t minus z minus 1 over t plus z 1 over 2 z, t plus z, so write it in this form. So, this becomes equal to 1 over 2 z log of 1 minus z over 1 plus z times is the minus 1 this minus sign, so it is equal to minus 1 plus z over 1 plus z with minus i let us do this slowly, minus log 1 plus z 1 minus z which again write as plus log and invert this. So, I can write it as factor of 2 and that is it this is called that is it. And where are the singularities of this functions, you do end up with an end points singularity at z equal to plus 1 at logarithmic branch point, another one at z equal to minus 1 logarithmic branch point these are end point singularities, and what about this guy?

It is a pole it is a simple pole at z equal to 0, but where does this pole occur? On all sheets other than the principle sheet of the logarithm, because on the principle sheet of the log, log 1 is 0 this whole thing goes like z on top and that cancels this, so the residue gets cancelled out. We already saw an example of this phenomenon you can when you have this, when you have these poles multiplying logarithmic functions you got to you very careful could turn out that the pole vanishes in one of the sheets or exist on all sheets excepts if you them are something.

In this case, there is a pole at z equal to 0 on all except the principle, and that is a pinch singularity at this point. Now, the nature of the singularity depends on the nature of this whether it is a pinch are in end point singularity, and on whether they integrand itself as the branch point out of pole. In all these examples we looked at in integrals which I have got poles, but we could look easily consider an example where you have a branch point look at a thing like this. So, we want to keep this real.

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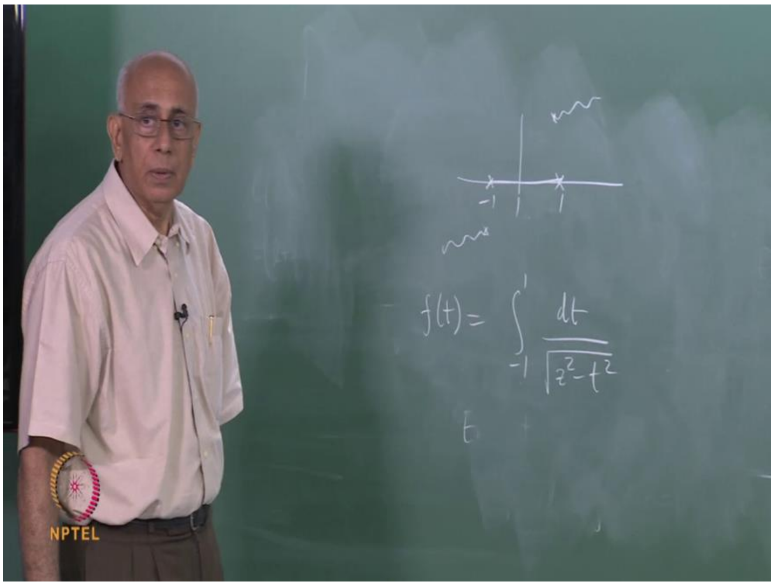

So, square root of z square minus t squared look at what happens now, where does the integrant have singularities at plus or minus of z, but what sort of singularities are those? They are themselves branch points they both square root branch points, because it just 1 over square root z minus t 1 over square z plus t. So, now you have a contour integral which runs from minus 1 to 1, but you have a branch cut and you have another branch cut with this. Once again when these fellows come and attack either the endpoint or they pinch the contour at z equal to 0 you going to expect singularities, but I leave you to verify what sort of singularities those are.

They not going to be the same as in the early case when you did not have this square root here, because now you really have branch points attack in these things here, and this could change things completely. So, once again this integral is doable it is very easy to do this you just have to make a substitution here and you can do this integral both are this. You expect that the singularities at the end points should be rather mild, and the reason is that what happens near t equal to z?

You have essentially 1 over t minus z square root in the denominator there is a square root and integrated becomes the square root on top. So, we expected to be a very mild singularity nothing the divergence, but there is a singularity or some kind. In this case that will turn out to be squared root singularity on top z equal to plus minus 1 will have square roots singularities, and there be a logarithmic branch point at z equal to 0. So, I urge you to check this out, and now finely there you should stop him well I finish this and them we could.

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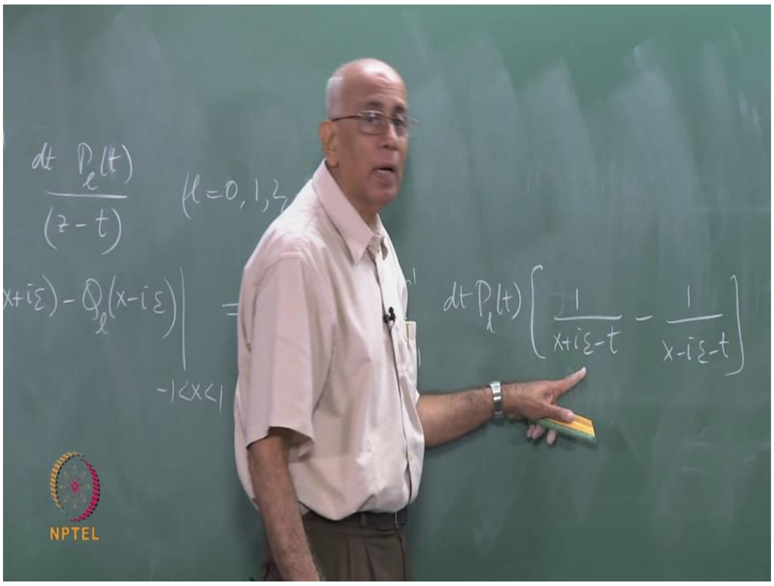

We could look at what happens to this representation Q l of z I mentioned was half minus 1 to 1 d t P l of t over z minus t that was the representation we wrote l equal to 0, 1, 2, what can be say about this singularities of this function as a function of z. Now, P l of the t is a polynomial. So, it does no singularity in the finite part of the t plane, but the integrant has a pole a simple pole at t equal to z, there is no question of pinching here there is only one singularity. So, the only possibility is an endpoint singularity when this t hits it the minus z e the minus 1 or plus 1 nothing more than that, and you expect log, because the P l is regular at that point, and d t over z minus t is essentially a log function.

So that now explains why you got this value, why at all you had this value, why we had this relationship it is at that Q l was equal to half P l times this log 1 plus x over 1 minus x plus the polynomial, so this tells you why. In fact, you can go further, you can ask as a function of z we know there is a branch point at 1, it is a branch point at minus 1 for this for Q l and we could ask, what is the discontinuity between these 2? So, if I come in z and stay here or here, what is the jump equal to we can compute this, because we are really asking what is Q l of x plus i epsilon minus Q l of x minus epsilon where minus 1 less than x less than plus 1, and what is this equal to that is the discontinuity.

This is equal to limit epsilon goes to 0 half integral minus 1 to 1 d t P l of t times 1 over z is x, so x plus i epsilon minus t minus 1 over x minus i epsilon minus t. We really asking for the difference of this kind, but we are familiar with this sort of thing, we very familiar with it from what we studied about the dispersion relations. The pole at t equal to x has been displays the little bit, because the pole is now at the x plus i epsilon, and this pole is at x minus i epsilon, and one case you pick up a plus i pi, in the other case you pick up a other minus i pi.

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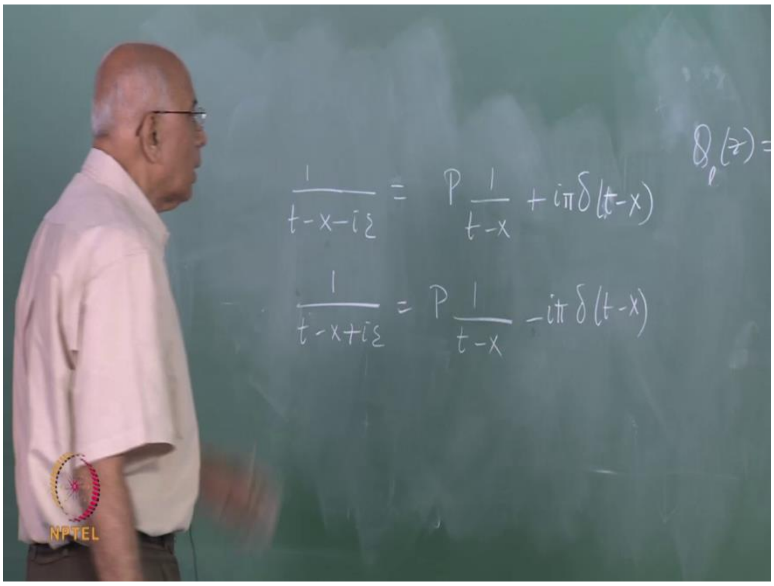

So, we have all to do is to go back and use this formula 1 over t minus x minus i epsilon. In the sense of multiplying by a function of t and the integrating across the range which includes x, this is equal to the principle value 1 over t minus x, and the pole is below the contour which is equivalent to integrating on top. So, this is plus i pi delta of t minus x, on the other hand 1 over t minus x plus i epsilon equal to the principle value t minus x minus i pi delta of t minus x in this fashion.

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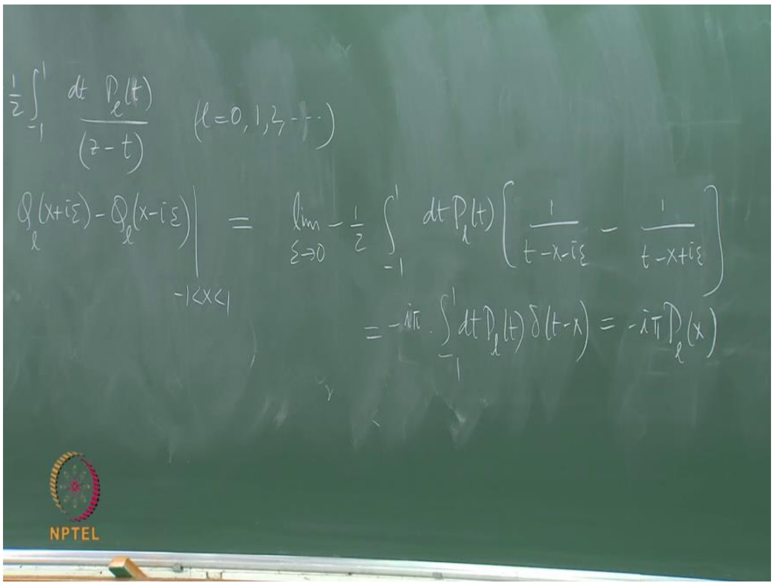

And then this thing is trivial, because the principle value parts cancelled out completely, and you left with this is equal to 1 half integral minus 1 to 1 d t P l of t and then there is an i pi, and there is a factor it should be careful. I should write this as 1 over t minus x minus i epsilon, and write this as 1 over t minus x plus i epsilon and pull out of minus sign, so that minus sign it is. And I get a delta of t minus x here, another delta of t minus x here when I subtract, because one of them appears with a minus sign, and there was 2 pi i. So, i pi the 2 cancels integral minus 1 to 1 d t P l of t delta of t minus x which is equal to minus i pi P l of x.

So, the jump is just proportional to P l of x itself, and that is why I mentioned earlier that the discontinuity of Q nu of z across between the cut between minus 1 and plus 1 is proportional to P nu of z. And we views the special case of on non negative integer l to establish that explicitly the constant of proportionality is minus i pi explicitly. So, there are logarithmic singularities, you could also do this by writing the expression for Q l in terms of P l and the log function and so on, but this tells you even without doing the integral that this is what it is explicitly. So, this is how evaluate discontinuities using this essentially using this form like here.