Selected Topics in Mathematical Physics
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Module - 6
Lecture - 16
Multivalued Functions:
Integral Representations (Part-III)

Yesterday, we looked at some contour representations for certain functions. In particular we looked at the contour representation for the gamma functions which I claying was applicable for all values of z explicitly. And if you recall the way we depth is to start with representation the original integral representation which was valid in the region real set greater than 0. I show that it was equal to keeping set in this region I showed that it was equal to another integral the contour integral this time.

And then I have pointed out that the right hand side the contour integral, actually make sense for all values of z. So, it provides the neat analytic continuation to all values of set of the gamma functions. So, this was the trick you find a representation for a function in the region in which it is applicable you find another representations. That is an identity, but the second representation may be valid in a bigger region are at least in region which is partially over of fully beyond the original region. But they must be common region of over laps so that you can call 1 the analytic continuation of the other. Now, let us see what we can do about the beta function by the we can play the similar trick.

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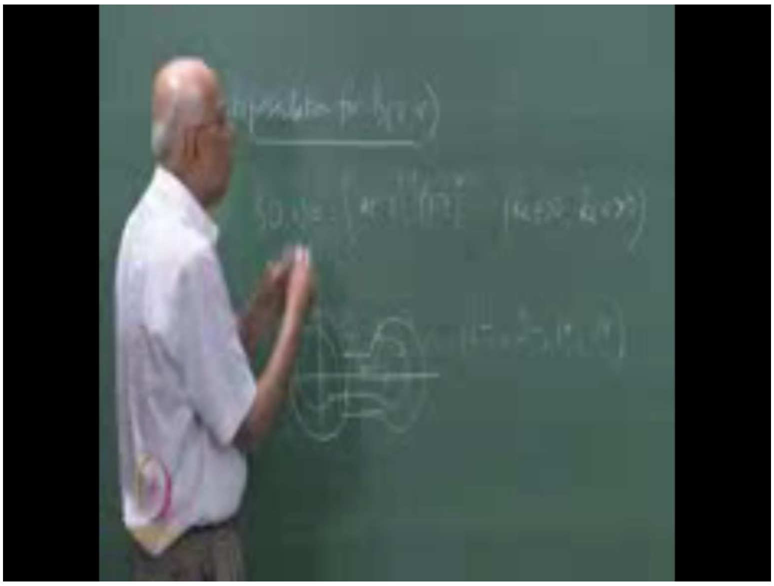

And this case so, let us write representation for beta functions of and recall that we started with a def defining representation. This was equal to integral 0 to 1 d t t to the power z minus 1 1 minus t to the power w minus 1 with the k v at that real set was greater than 0 real w was greater than 0. So, that was r original defining representation. I have need keep this condition so, I was not to 1 into trouble here and this So, that you do not to 1 into trouble that upper end of integration. And this case integrating by parts either of these factors did not work, because if we do that, then while you may improve the convergence in z by taking into the left to real z greater than minus 1. You will version the convergence as for as w is concern and make it real w greater than 1 and so on. So, we that that method does not work unless you specifically would like to have a representation and some step on the other. On the other hand we could exploit the fact that this function as a function of the t in the t plane has the branch point a t equal t 0 and another branch point at t equal to 1. And we know the phases of these functions on the principle sheet right.

So, in the t plane you have singularity of 0 and singularity at 1 and since in general z and w are arbitrary complex numbers in this region. It is clear that thus the winding pointed t equal to 0 and another winding point at t equal to 1. And that the cut what over cut you choose should on which our direction runs from 0 to infinity from here and 0 to infinity from here. And this continuity is given by examining in the integrant more carefully. In this case since it is a product of this forms this. So, does to speak discontinuity here does not cancel that would have cancelled both of them have been square root singularities and not many other case. But they do not cancel in general these cuts will run the all way to infinity. But now if I start by sink that the phase of this integrant 0 here just about the real axes this definitely a cut here. Then imagine what happens considered what happens if keeping this these conditions striking to this conditions I considered the following contour integral. The phase is 0 out here and move along like this and I go along this once in the negative sense in this passion.

Another go back in this direction and go around this 2 in the negative direction sense again the negative sense. I come back here and go in the positive sense I am exaggerate this figure and I go around like that in the positive sense and it is clear what our to do is to go around this 2 in the positive sense. So, I come back remember started like this I comeback go around here and then go around this in the positive sense come out and join up. And declaim that it joins now the function goes back to its original value, because when I went around this contour 1 and hears it symbol for it.

So, I went around 1 in the negative sense and then I went around 0 in the negative sense and then I went around 1 once second in the positive sense and 0 in the positive sense. So, let us the short hand notation for this contour then I have drawn and look and at what as happen when I went around this fashion this factor here picked up a phase factor e to the minus i pi w minus. Because we went around the sense and in picked up once second here down here and went around this in the positive sense in the negative sense it picked up factor e to the minus 2 pi i z.

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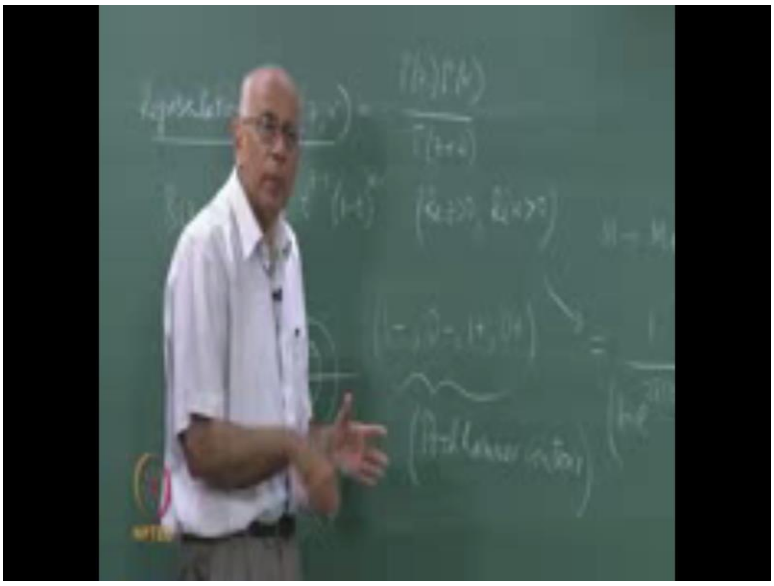

So, you started for starting point started with modules M then in this circuit when it when back in this from 1 to 0 it did so with M e to the minus 2 pi i w. And that function now, you went around the point 0 in the wrong side direction of sense it. So, picked up an extra phase e 2 minus 2 pi i w e to the minus 2 pi z and then I undate this 2 things. So, I got read of this so, it when to m e to the minus 2 pi i z and then it went back to M. I am sorry, because it got multiplied by e to the plus 2 pi i w and got read of this factor. And then we got multiplied by e to the 2 pi plus 2 pi i z got read of this factor you back to have. So, you guarantee that this wearied looking contour is that close contour every one of the segments have exaggerate this every one of the segments is running from 0 to 1. Just about just below the cut excreta I do not neat to draw cuts, because required the complicated structure. So, I am getting on to all kinds of sheets this function but, your guarantee that this contour see is at close contour.

And now, the point is what about the contribution from all these small circles? Take at typical contribution here round z this thing here for instance then this is called give you an epsilon. Because t is epsilon into the i theta this is called give you an epsilon to the power z minus 1 which is a epsilon to the z. And this does not do anything can set t equal to 0 and this and since real z is greater 0 epsilon to the power z times to 0 as epsilon times to 0. So, this contribution these contributions are guaranteed to be 0 in this region. But one should write it in this form then of course, you are away from the point 0 and 1 the contour never passes through be the 0 or 1 at all. And therefore, no matter what we do with z and w it is not go to change the convergence of integral. And each times you multiplied you gone to pick up the phase factor and I like to ask what is the integral over m itself equal to that is was, the beta was. So, it terms out that this beta also equal to 1 over exactly has be d1 earlier 1 minus e to the 2 pi i z 1 minus e to the 2 pi i w and integral lower c. It is a close contour e t t to the power z minus 1 1 minus t w minus 1 and that is valid for all z and w.

Now, all the singularities of this function I go on to show up the 0 of this denominated here. And when there is no singularity at all even though this might vanished this can be cancelled by 0 and the numerated. So, once again when you go to the left of this region you gone to encounter singularities poles, but you can extract what the residue at the pole is this expression here in principle. But of course, we also know that this quantity beta of this is equal to gamma of z gamma of w over the amount z plus w. So, you could extract the singularity structure of this beta function also from this representation that is valid for all z every z. So, where ever you are in z and w that formula is always true as an identity.

Therefore, you can extract all analytic properties of this function from there all from this integral here. This start of contour this kind of thing this called the poch hammer contour. And the notation is you have various branch point you simply telling me in what direction are you go along going around the branch point and in what order. So, let us a very compact you are waiting this contour here some moment you write at immediately know that the contour is running from 0 to 1 between 0 and 1 back and forth with this sign convention. Of course, this is not a unique we have writing in that are other contour which are equable in to this. What is interesting is that if you actually wrote this contour on the Raymonds fear that in it looks extremely symmetric. Because on the Raymonds fear remembers this also branch point to infinity there in this in this functions so on.

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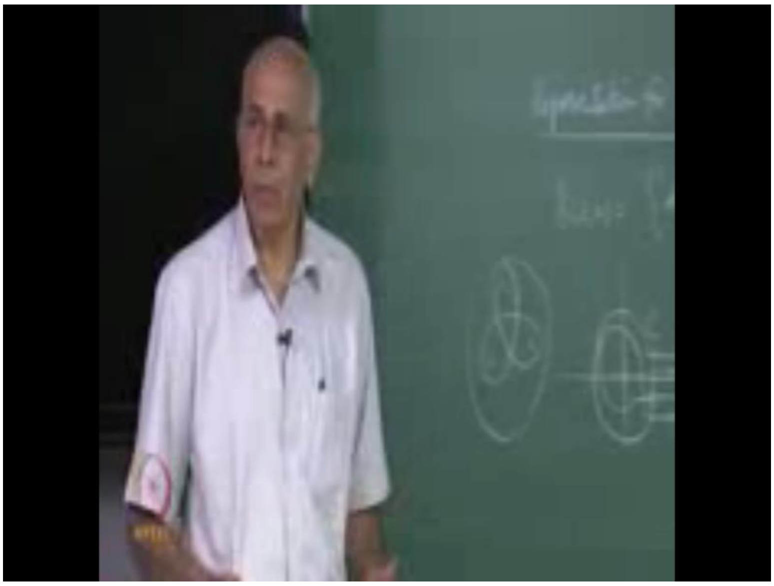

The raymonds fear what happens is that just for the set of representation we know that infinity setting there 0 sitting here and 1 setting on the equator out here. But for the moment let me write them and x 3 point here and then this contour looks like an eyes trefoil not looks something like that You can slipping around this start them around then So, on till its starts looking like this. And you guaranteed that is a closed contour once again. So, this measured of using contour integrals to provide representations for function is an extremely useful one and all kinds of analytic functions can be represented and this passion. Generally they provide you with representations valid in huge region if not the whole plane. Let us looks at another function with we looked at little bit earlier namely the z of function and ask what happens there can it find the representation? This of the remands z of functions, recall some properties which we deduced little earlier.

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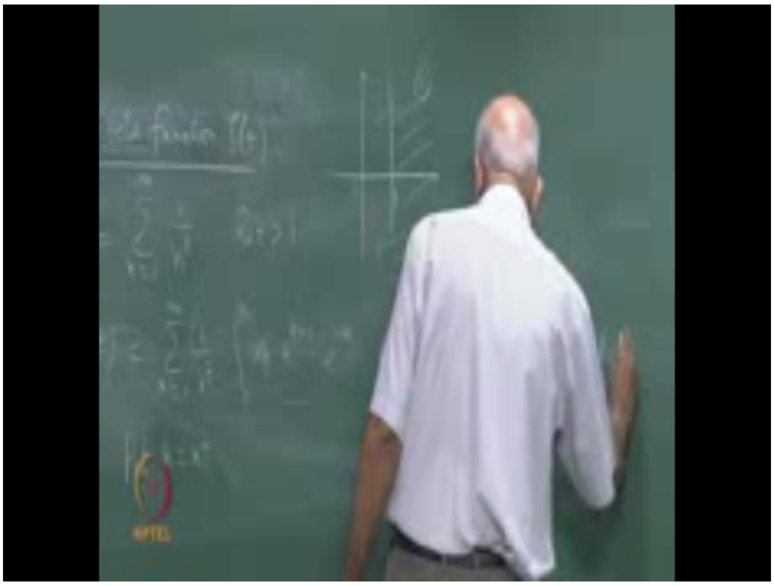

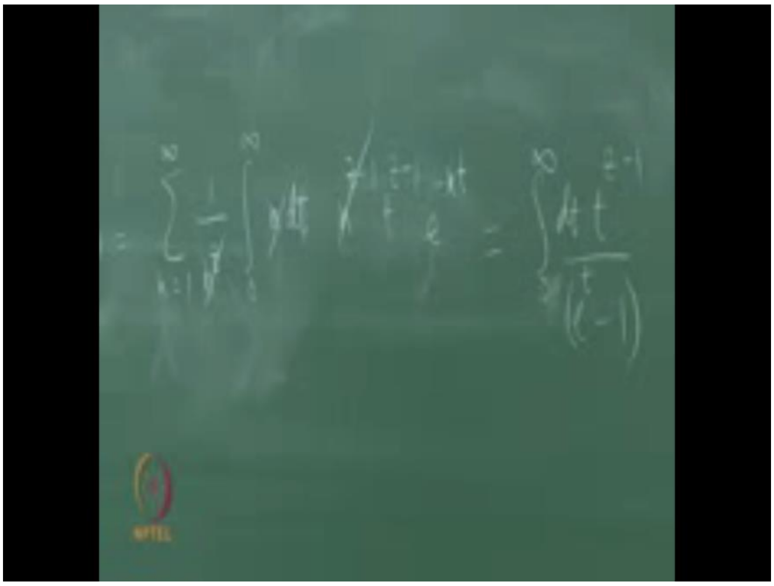

Recall that we defines z of z we summation n equal to 1 to infinity 1 over n to the power z that is defining representations. And this was valid as long as the real part of z was greater than 1. If its 1 it is logarithmically divergent as you can see. So, we have a representation that is valid in the complex z plane to the write of the point 1 and we are 0. Now, as we know from our experience with power series on the boundary of the region of convergence you expect 1 or more singularities of this function provided it can be analytically continue. In this case it will turn out that there is simple pole at z equal to 1 and its analytical everywhere else. And therefore, you can analytically continuity down here and now that request to find out what is this representation, what representation will show me expresitly that thus the pole at z equal to 1 with residue 1 as it is happens. And to the left of it function is well behaved once again. Well, the trick is to start not with z or z, but to start with gamma of z times z also and write representations for each of these. So, this is summation n equal to 1 to infinity 1 over n to the power z and integral from 0 to infinity d. Let us let me call its some other symbol d u u to the power of z minus 1 e to the power of minus u.

This was the defining representation for the gamma function provided real z was greater than 0. But this series convergence only real z is greater than 1 So, this whole formula is valid only if real z is greater than 1. And that region is the series convergence absolutely the integral converges absolutely and so on you can interchange the order of integration summation and so on. So, let set at, put u equal to n time's t so, this thing becomes equal to summation n equal to 1 to infinity 1 over n to the power z. And then integral 0 to infinity does not n d u and then and n to the power z minus 1 t to the power z minus 1 t to the minus n t. And the various factors of n, n 2 the power z that cancels of are d t this is n d t hence that and the factors of n cancel of completely. And then I can interchange we order of summation and integration. So, this becomes equal to and integration from 0 to infinity dt t to the power z minus 1. And this is the geometric series t to the minus n t starting from n equal to 1.

So, they answer is e to the minus t over 1 minus e to the minus t, because e to the minus t is less than 1. And multiple through by e to the minus t and it just becomes this divided by e to the t minus 1. So, that is an integral representation and we still we to keep real z greater than 0, because thus the problem here this type. In fact, we need to keep real z greater than 1 as it should be. Because we have done in anything this is true why is this necessary wants I have write that representation, because look at what happens to the denominator this is e to the minus 1. So, the 1 cancels out and the leading term is t in that denominator. This already a power minus 1 here so, there is the t squared in the denominator. So, this power had at least be better be at least 1 greater than 1 otherwise we have the singularity. So, it is still true that real z that greater than 1 that is still true that condition is still need it. But know we can place that trick that replayed earlier with regard to that integral to get read of the singularity at 0 and convert into account to it integral.

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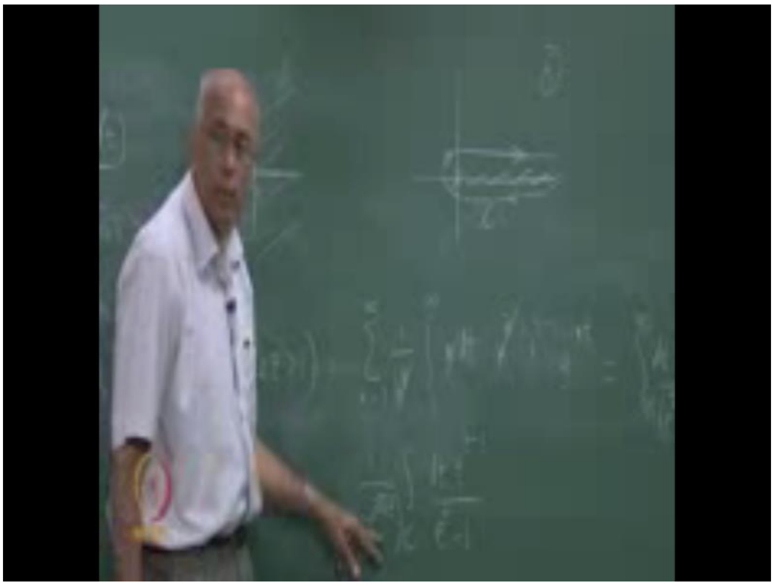

So, once again as resort to contour integration by its saying I look at it t plane and this factor here t of the power z minus 1 has the branch point. And t equal to 0 and t equal to infinity and i draw cut between the 2 in this passion. The phase of the function is 0 here its t of the 2 pi i z here so, I consider once again this happen contour. Going below the cut form infinity to 0 going around 0 in the negative sense and then going off to infinity once again just about the cut. So, on this contour this portion of the contour this function is modulus multiplied by e to the 2 pi z and out here this modulus is they argument is 0. So, this can also be written as the equal to an integral forward this contour it is a not a close contour d t t to the z minus 1 over e to the t minus 1. And there is got to be a 1 over 1 minus e to the 2 pi i z, the 1, because this portion is just the originality integral and minus, because I am coming in from infinity with the phase factor e to the 2 pi i z. So, it is exactly as it was the so, this tells us that we have a representation for this z of function.

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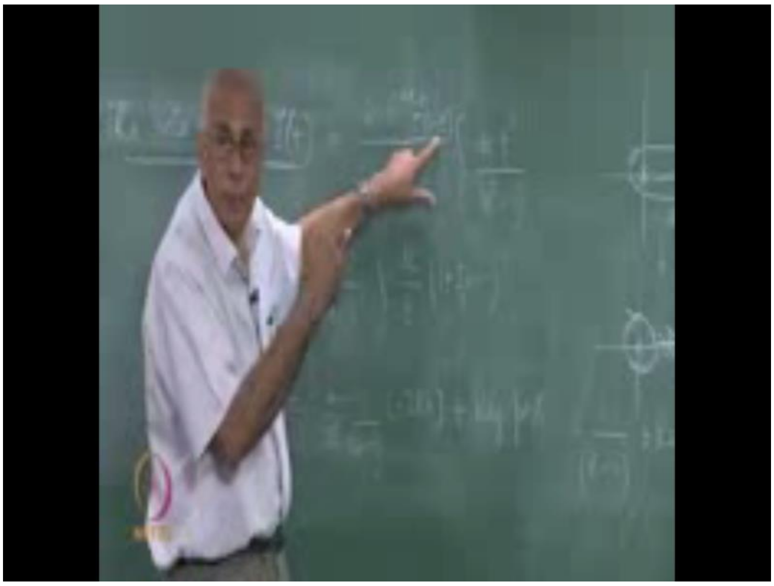

This is equal to 1 over gamma of z 1 minus e to the 2 pi i z contour integral d t t to the power z minus 1 over e to the t minus 1. And now, you see this contour no longer touches zero at all. So, in principle it is valid for all values of set this thing here is an entire function of z and all the singularities of this z of such as they are must allies from this factorial. A gamma function as singularity is e to the 2 pi i z could be equal to 1 for various values of z you could have singularities. Now, we can simplify by this little bit by using the following reflection formulae. Recall by use the beta function to derive this reflection formula which set gamma of z gamma of 1 minus z was equal to pi over the sin pi z. That was the reflection formulae for the gamma function if you recall and let me use that. So, we have I am go to pole out and e to the minus i pi z put it on top on that gives me e to the i pi z plus e to the i pi z minus z I multiply and divide by e to the minus i pi z So, this becomes minus 2 i sin pi z.

So, this is equal to minus 2 i so, let me write this i over 2 gamma of z sin pi z. And I will gives gamma z gamma of sin times sin pi z is pi over gamma 1 minus z. So, let us write this as 2 pi and gamma 1 sin and I am doing this. So, that I have 2 sources of singularities what is the gamma function they other sin pi z combine them So, that I can keep track of it this is 1 source of simplify. Now, where is the gamma functions singular? Whenever its argument is 0 minus 1 minus 2 etcetera which means that this thing here gives you a singularity at z equal to 1 as we suspected at z equal to 1. The some kind of pole this is an entire function and then possibly at z equal to 2 3 and so on. But we know that is not true at z equal to 2 3 and so on z of 2 is fine. It is pi squared over 6 z of 4 is pi 4 over 90 and so on. But we have to see what goes on for positive values of positive integer values subsets 2 3 4 and so on. On the other hand what happens near z equal to 1? Where this thing here at z times to 1 goes to 0 e t to the 0 which is 1 so, that is harm less. On the other this factor has the 1 over t and you know that integral d t over t is 2 pi i. It is a ((refer time 22:02)) round sense it is the minus 2 pi i.

So, this whole thing that was z of z goes as the terms to 1 to the following. It goes to i in all harmless factors you can puts z equal to 1 including this factor. But you got to keep this now, what as gamma of anything do gamma of w do 1 a w equal to 0 goes like 1 over w residue being 1. So, this thing goes like i e to the minus i pi over 2 pi and then this is 1 over 1 minus z. So, that is equal to minus 1 over z minus 1 in this passion. And then and integral d t z goes to 1 that is harmless over t 2 the minus 1 over this contour. But what is that equal to there is no branch point anymore; because of moment z becomes an integer the branch point disappears. And t the becomes the harmless polynomial are a pole at that point this case numerate just disappear. But you still have a singularity, because less quantity here e to the t minus 1 the 1 cancels. And then the next term is t plus t squared over 2 factorial plus dot which we could write as t into 1 plus t squared over 2 factorial t over 2 factorial plus dot this passion.

So, you could write this whole thing as t here on due and then you have 1 plus t over 2 plus dot to the power minus 1. So, this thing tells you clearly that this integrant has a simple pole at t equal to 0 the t plane. If and moreover this contour goes us in this case to just a simple pole this portion of the integral cancels out. I can start this goes away and you left with just this that is minus 2 pi i times the residue. And remember d t over t around unit circle is plus or minus 2 pi depending on the u towers it in the clockwise my anticlockwise or clockwise that is. So, that gives you another minus sin and 2 pi i. And that of course, tells you that is equal to thus n a is just as in the whole we got have these minus sign cancel. And then you have an i over 2 pi 1 over is z minus 1 and then that was the minus 2 pi i plus. The regular parts due to all other terms that are to do all the other terms are come even from this as well as the gamma function. All of them have regular parts at z equal to 1, but well exacting on the other singular part here.

And we know do that is inside no this is a simple pole as equal to 1 i put z equal to 1 everywhere else. And keep just the 1 over z minus 1 term yet to find the restive to here.

And this is equal to 1 over z minus 1 plus regular part. So, we have now, come to this conclusion that z of function has a simple pole at z equal to 1 with residue 1, because this co efficient is 1. So, we know that this representation what we are original summation representation is actually valid even to the left. I mean can be continued analytically to the left of real z equal to 1. But add z equal to 1 thus the simple pole and no other the singularity at all what about positive integer values of z like 2 3 4 and so on? The 2 same to the giving as So, problem, because the moment you have 2 or 3 or 4 out here in this representation this gives your pole at all these points So, if a put z equal to n.

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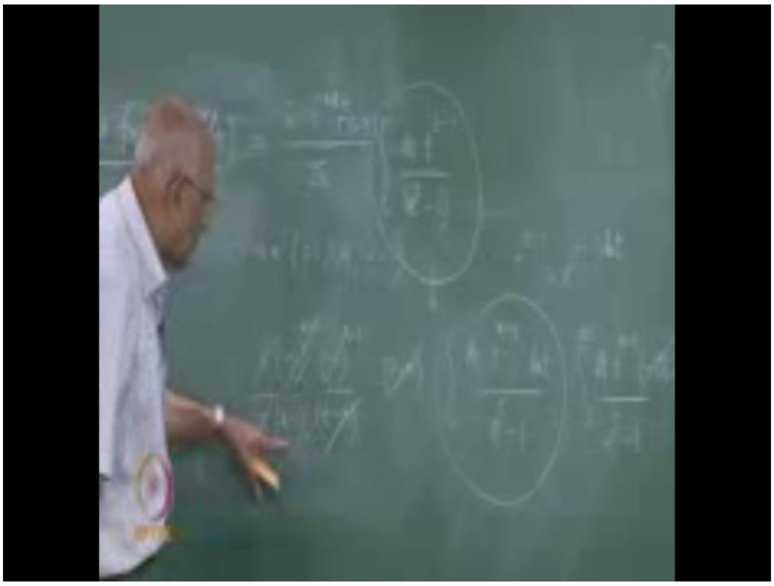

So, as z goes to n equal to 2 3 4 etcetera these integers this function here is going to have and negative integer argument. z equal to 1 the argument of 0 and gave you a pole otherwise it put z equal to n you go and get gamma n minus of 1 that is got a pole. So, the pole terms is going to have i times want to our exact what happens? So, you have e to the i pi times z so, e to the i pi and ten minus 1 is g1 to be set as. So, let us minus point to the n minus 1 and then this gamma function well, but that is not as singularity. So, this look like minus 1 to the n over z minus n. And then less than n minus 1 factorial a minus point to power n minus 1. Because that is what negative integer is minus of n minus 1 in this 2 pi also setting here and then look at what happens to the integral? This integral here will vanish, because in this case the moment I put t equal to 2 3 z equal to 2 3 etcetera you have a powers starting here at least at t. May be t squared t cubed and so on and this kind here gives you t in the denominator.

So, there is no singularity at all and whole thing vanishes. So, the moment I have put t equal to n over our that term vanishes the integral vanishes identically. So, I have to go to the next term and then next term is going to be proportional 2 z minus n. Because that is what this whole integral is going to be. So, expanding this integral as integral c d t t 2 the power n minus 1 over t 2 the t minus 1. And that I have are give it to as 0 plus the next term. So, the next term was z minus n times whatever is that derivative of this function? And what is the derivative going to be? Let us write t to the power or z minus 1 as e to the power z minus 1 log t. I differentiated now with respect to z and it gives you again the derivative of this is gone to give me and integral contour integral d t t to the power of n minus 1. If I differentiate i this i pole at lock t in the denominator.

So, thus the lock t over e to the t minus 1 times z minus n which cancels of 2. So, I have an i over 2 pi this follow gives me 1 n minus 1 factorial and then I have do this integral. But yesterday I showed you how you could an integral to 0 to infinity by putting in log and then going over to the hairpin contour. Now, we want to do that in reverse here. So, it is clear that this poch hammer contour up of the cut this log of log modulus t. But below the cut its log modulus t plus 2 pi i and its come with a minus, because we going from infinity 2 0. So, I could write this as an integral from 0 to infinity d t t to the n minus 1 times minus 2 pi i over e to the t minus 1. And then the 2 pi and the i cancels with minus 2 pi i and you left with 1 over n minus 1 factorial 0 to infinity d t t to the n minus 1 over the t minus 1.

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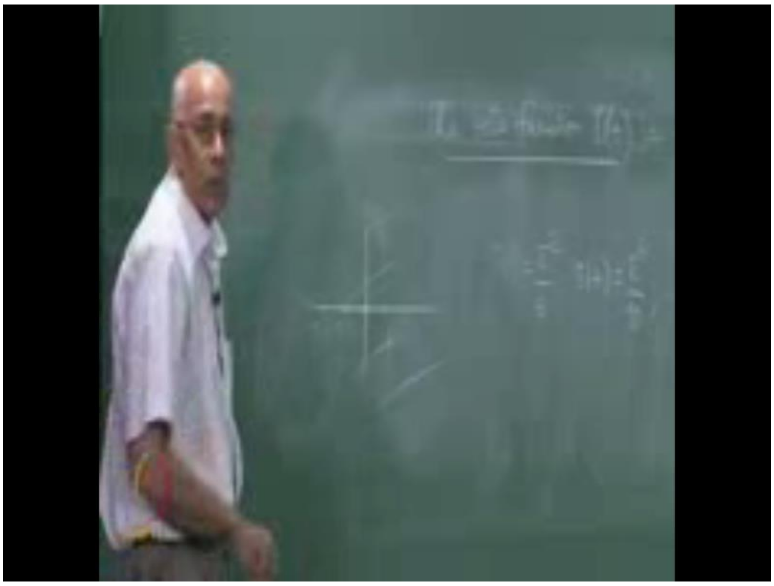

So, we are set that it or z of z; z or n equal to 1 over n minus 1 factorial integral to infinity d t t to the power n minus 1 over e to the t minus 1. And equal 2 3 dot that is a well behaved integral there is no problem at the origin due to this factor. And I can infinity this gives you convergence e to the minus t. And in fact, if you go back and look at the original representation for its z of function it will put this in immediately. Because recall that we started by saying gamma z of z z or z was the certain integral. Put z equal to n in the formula and use that fact that gamma of z n minus 1 factorial bring it to the right hand side that it given the factors and there is rest is presides this. So, this just checks out that what we have done is consistent this is impact the starting a representation. So, we know that is z of n is well behaved and z or z in fact is well behaved for the else greater than 1.

And that this integral representation does not give you in a pole and in a poles at any put 2 3 etcetera, what happens for negative values of integer values of n including 0? That is interesting let see what happens to this function when we have n equal to 0 minus 1 minus 2 and so on. I already pointed out that is z of 2 was pi squared over 6, z of 4 was pi 4 over ninety and so on. But I have also set as that time that there is no simple formula z of 3 z of 5 and so on. But this always this formula the 1 that we wrote down now 1 over n minus 1 factorial this integral with an end. That is solve we can always do that numerically of problem on to it what about negative integer values of z? Well, I have set it that does the singularity that z equal to 1 and then nothing else its actually well defined the beware else. So, in particular I like to know what happens at 0 what happens at minus 1 minus 2 and so on. For this all we have to do its substitute the value of z inside their and look at what is going to the factor.

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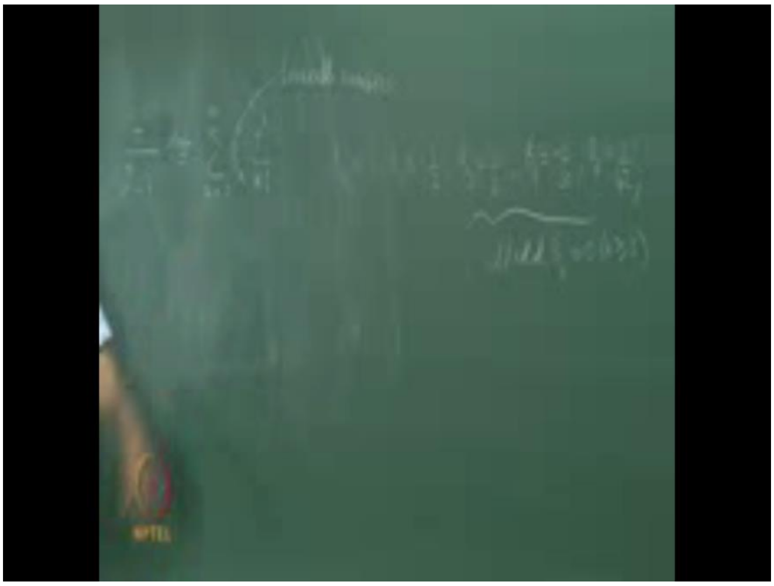

We need the little preliminary for that the preliminary that we need as following. The function t over e to the power t minus 1 considers this function here. This can be expanded in a power series in t, because near t equal to 0 this thing goes like t in that denominator in that cancels again this. So, gives you 1 is a leading term and then the next term of order t is t squared etcetera. So, if you expand this in power series this by definition is written as n equal to 0 to infinity t to the n. And then you put an n factorial here, because this is an e to the t here and will explain what about the displays. And it is multiplied by something called b and these are the. So, for a Bernoulli numbers and so simple matter to write the power series out here and then keep a try to track down the coefficient t to the power and gets a little t d as sometime.

But this nothing in principle to stop you from doing this and b 0 equal to 1 that is obvious. Because if you set the going to 0 t as the denominator and numerator both cancel and give you 1 b 1 equal to minus half that is also we obvious, because the next term here is going to be t squared a 2 factorial take out the factor of t you get on 1 over 2 with that t. And then when you take it abstract, because you are arising power minus 1 is going to give you minus half. Then the next term b 2 terms out to be 16; b 4 times out to be minus 1 over 30; b 6 terms out to be 1 over 42 and so on all odd b n equal to 0 and greater than equal to 3. So, that is the peculiarity of this function. It does not have any r powers except the leading term proportional to t. So, this thing goes likes 1 minus half t plus 160 squared and then everything else is powers even powers of t then you make an expansion. So, it looks like all the b are decreasing as you go to large of values of n.

All the collinear numbers appeared to be decreasing under this 1 over n factorial here. So, does not it look as the whole series is going to haven the radius of convergence that is infinite? Because is the 1 over n factorial and then it is multiplied by coefficient. If you did not have the b n than this is just e to the power t which has you know has in infinite radius of convergence. And now, you are multiplying by it which is sort of decreasing So, what would expect is radius of convergence Infinity this is what would expect, but what happens on the left hand side is this an entire function? I agree the pole at t equal to 0 that is cancel out that is removable singularity, but other any other singularities? Well, is that is exist where would they b at what points it denominator vanishes? I went as the denominator vanish by mean.

Student: At equal to 0.

At t equal to 0, but thing complex when is t equal to the 1 by t is 2 pi and i at t equal to 0 you have a removable singularities. Because of the numerator but, certainly this functions as poles at t equal to 2 pi n i where n is any non 0 integer.

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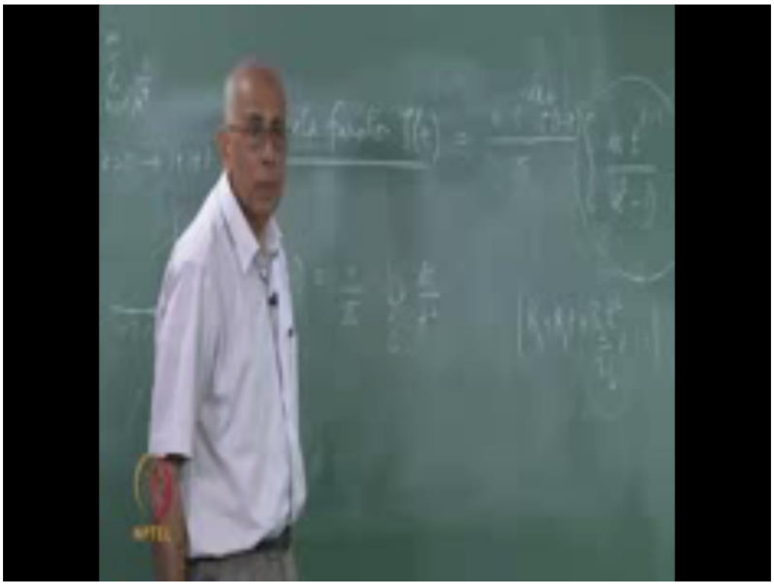

So, this left hand side as the function of t definitely has poles at all the 2 pi i. So, it has the pole 2 pi i 4 pi i and 3 pi i and so on minus 2 pi i 2 pi i and so on. And you making a power series expansion about this point. So, what will be the radius of convergence of this 2 pi i?

Student: 2 pi i.

It is 2 pie. So, this series guaranteed to converge only inside a circle of radius 2 pi what is that imply about that b n by the way? What must be happening as an increases? They should start increasing at along in rate. So, looks like a few first few of the first small but, eventually that they have to kill this they affect of this So, the indeed increase wide rapidly it is so on. So, they show you that even if you define at thing I can say lot about this. Because we have the representation for the function explicitly note here now, look at what happens here? And go to find what z of 0 say it z of 0 equal to thus an i over to pi that I can get over and then e to the 0 gamma of 1 is 1.

And then integral over this contour d t over t e to the t minus 1, but let me write this as t squared and then its t over e to the t minus 1. And it looks very much like this a pole this definitely looks like this a pole of some kind. Because you can see that this function here when i set this equal to 0 thus already 1 over t here and this can also is that t. So, this goes like a t 1 over t squared and then there are in terms of order 1 over t or t to the 0 and so on. So, this thing gives initial of that let me write this as b 0 plus b 1 t plus b 2 t squared over 2 factorial plus and so on and what is the contour now? Because there is no branch points it is just the circle round a small circle round 0 in the negative sense. So, the contour becomes this around the origin. But we know that d t over t squared the contour integral 0 give only thing that is non 0 is d t over t d t times any positive power of t 0 power of t is 0 negative power or 0 except 1 over t.

So, this answer is equal to i over 2 pi minus 2 pi i times b 1 which is equal to minus half. So, z or 0 is minus half exact value well, defined numerical value is minus half. Now, if you put z equal to 0 in that defining representation. So, if you had n equal to 1 to infinity 1 over n to the power z and i put z equal to 0 here I end up with the c v goes 1 plus 1 plus 1 plus 1 all the way to infinity. So, the series makes no sense it is not a good representation anymore it is not valid in this region. But the analytic continuation exists and gives you a perfectly well, defined value which is minus half what happens we put z i equal to minus 1? But exactly the same thing is g1 to happen if we put a minus point here thus already t squared and the denominator I put an extra t here so, there is t cubed.

So, let us put z of minus 1 is this whole thing and what is that equal to we can read of what is going on happen? The terms that going to contribute is this alone that is the only termed that is gone to contribute right and what is that give you? It gives you this terms b 2 and then this the minus sign, because the doing this contour clockwise sense. And do you end up with 1 12 as the answer should be a minus 1 12 I am not sure where I am missing the minus sign z of 1 is minus half z of minus 1 is 1 12 it is correct. It is 1 12 and this was the famous result that ramanujam wrote in as second letter to hard in 1913 when you wrote to him saying that he had the theory of divergence series. Ramanujam did not know about analytic continuation it did not use complex analysis. He had this own as the hard we let in put it then yours mean of deducing these results. But they were correct results and this latter he wrote saying that his according to this theory of divergence series.

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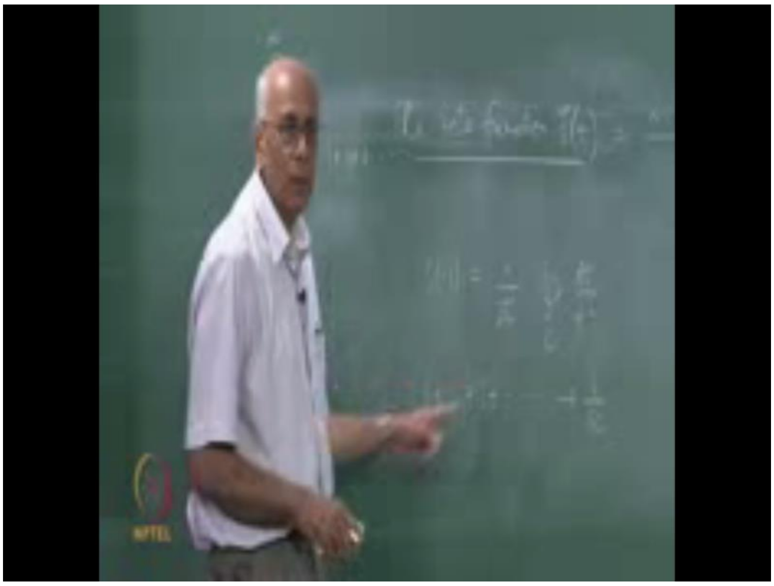

This series 1 plus half then equal to minus 1. So, it is 1 plus 2 plus 3 plus on the way to infinity it is a gives is 1 12. So, what it deduce its value of z of minus 1? By zone means by z1 methods and then we write that saying the 5 out to twelve you that the sum of the series 1 plus 2 plus 3 all the way to infinity is 1 twelfth. You will advance point out to be the way to the lunatic assigned at, but that is what my theory gives and so on. Today, we know that what it done was to find that correct analytic continuation on the z of function that this factor what happens if is that f n minus 2 or minus 4 and so on.

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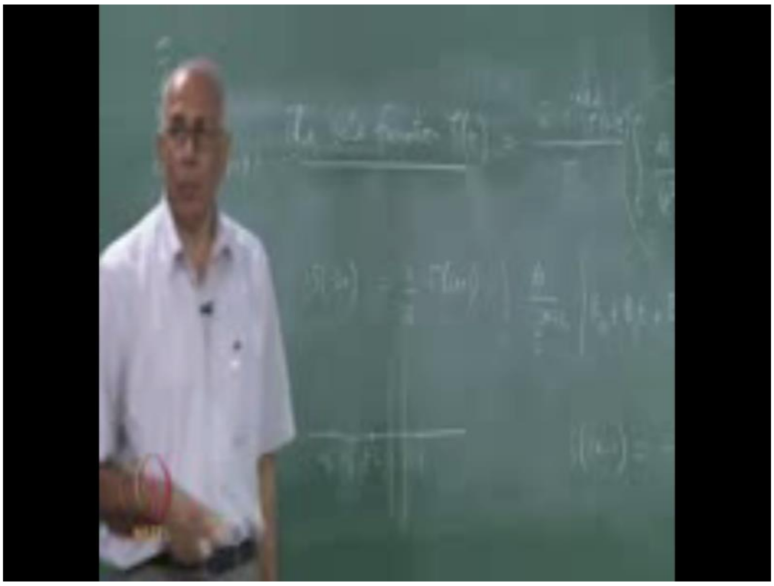

So, let us looks at z of function z of minus 2 n and again all we have to do this to substitute the formula here. This is equal to i over to pi e to the and this is minus 2 1. So, that is you gives you 1 and this gives you gamma of 2 n plus 1 which is 2 n factorial over 2 pi out here and then the contour integral d t over and this is minus 2 n. So, you have 2 n and then added 2 here and then i put in B naught plus B 1 t plus B 2 t squared over 2 factorial plus B 4 t 4 over 4 factorial plus dot although and there are no r powers. And if n is equal to 1 n 2 3 and etcetera when n is 1 then we looking z of minus 2 out there then this is the t to the 4 and you needed t cube that to give the non zero contribution. So, what is the value of z of 2 n? 0 vanishes.

So, this is equal to so, this z of function has 0 at all the even negative integers. These are called this trivial 0 of this z of function. So, what we found trivial 0 and in the easy to find so that the trivial. And of course, you know the famous hypothesis of remands say is that all the non trivial 0s of this function. So, when as 0 trivial 0 at minus 2 minus 4 minus 6 and so on that follows directly from this expression here, but in addition this an infinite numbers 0s on the line this is 1 so on. The line with real part half Raymond hypothesis was that this is infinite number of 0s on this line. And that all the non trivial 0s lie on this line that is the famous hypothesis. Now, what we are able to shown numerically is that the first 30 in 40 in billion 0s an lie on this by the by z of function. It is real analytic function its real for real values of argument. So, that implies that it you have 0 here you would have 0 here 2 by reflection.

And then all the 0s lie on this should not trivial 0s and this is certain distribution of this 0s which is of great interest of many problems many unexpected problems. But today, it what has been shown as for us I know is that all non trivial 0 lie on this almost surely in the regular probabilistic sense. Namely with probability 1 then lie on it, but this not the same the showing that all 0s which are nontrivial lie on it. It is also known that in infinite numbers 0 lying on this what is not in prove and completely is that the non no other 02s other than those here and all this line that not be improved. But it is continuously there number of 0s continuously increasing and there is in certain interesting level spirit interesting spacing or distribution of this 0s which is connected to the distribution of primes; which is connected to the energy icon values of quantum mechanical system.

So, classical contour parts at chaotic. It is connected to the icon values spectrum or distribution of icon values of certain classes of random matrices energy levels of complex nuclei and so on. So, it is the very large number of un expected connections with the 0s of the Raymond function and this hypothesis is mathematic the most the most important problem and solve problem in mathematics. Because if it is establish then huge number of other theorems followed immediate sucrucial thing we can to try, but it is worth understanding this a problem completely. So, that is for z of minus 2 n it is also possible to find out what z of and all negative integer is minus 1 minus 3 and so on. I leave that you as an exercise minus 1 we already take, because all we have to do is to put the 1 here and then make this 2 1 plus 1 its a 2 n and then find out what we appropriate to coefficient is.

So, I believe z of 2 n minus 1 is i thing I not sure about this minus b 2 1 over it 2 1 are something like that. Because this going to here 2 1 plus 1 factorial here and this 1 be 2 1 factorial here are 2 1 minus on here 2 1 there and is got it 2 1. But in any case because are easy to write all the comes upon the numbers so much for this z of function. There are many other properties of this whole of analytic number theory is based on the very intricate properties of this z of function. For which I have written down here as some as in at representation, but is also very interesting representation to gaoiler as in infinite product which involves on the prime numbers. So, this very close connection between the z of function and that is distribution primes and that is how as you know the primes of the building block.

So, if all arithmetic and this is our, the connection of analytic number theory intimate connection with this z of function analysis. Now, I like to turn next to some other function like the Legendre function, but we will do that next time. Hence we will run out of time and I will talk about to an integral representation for the legendry function. And then perhaps the few other special function as well, since you have already familiar with the real value real representations. As the representation of these functions as solution have differential equations power series and so on. It will make sense to be able to write them down in times of contour representations. We will do that next time.