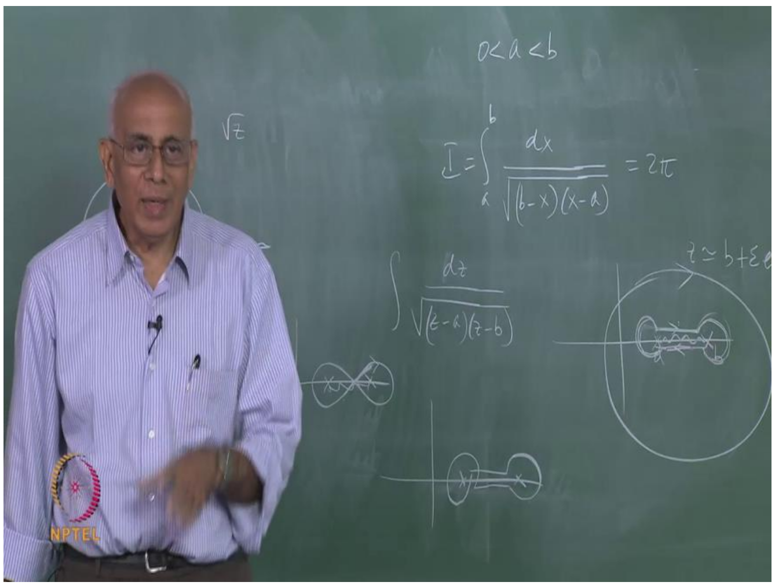
Selected Topics in Mathematical Physics
Prof. V. Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Module - 06
Lecture - 15
Multivalued functions: Integral Representations (Part II)
Outline
Contour integrals in the presence of branch ports
An integral involving a class of rational functions
Contour integral representations for the gamma function

(Refer Slide Time: 00:15)



Here is a simple trick which we use, for instance, to evaluate contour integrals. Very often you come across an integral like a to b, dx, so let us say, 0 less than a less than b, positive numbers. So, square root of b minus x, x minus a, basically this is simple definite integral. The answer is very well known. But, you can do this by contour integration using a little trick, and that is the following.

Consider the function, 1 over square root of z minus a, times, z minus b. We know how to draw graph for this function, z minus a to the minus half, z minus b to the minus half. So, you can draw a cut between a and b, that is it, and write the phases of this function down, etcetera. What is meant by this integral, is an integral over positive integrant. This is the modulus of this function here, between a and b. So, what should do is, I am going to leave this as an exercise. We consider this function, choose the cuts, the cut run from a to b; write down the phase here, arrange it so that the phases are known here and here.

And then, I already showed that across, if a square root branch point, the discontinuity is just twice the modulus. The reason is, on one shade it is modulus, and the other shade it is minus the modulus. So, exploit that, apart from factors of i and so on, which you must be careful about to convert the line integral from a to b, to a contour integral of this kind. So, a line integral, I have exaggerated here, a line integral from a to b, a line integral from b to a, the opposite direction coming down, by its different function there, because there will be a sign change, and then 2 small circles around this.

Consider this contour; this is the closed loop. I will come back to the original value because I have not crossed any cut. Now, what is this value equal to? Well, this part of it can be related to the original integral I, apart from some factor. This follow can be related also to minus I, going in the wrong direction. So, the whole thing will be proportional to, twice I, this part and this part. But, you added these little pieces, ok. We have to ask, what happens to these little pieces? But, now, what happens, near, z equal to b, for example, on this circle, z is equal to b plus epsilon a to the I theta, on this circle.

And, what you are doing is, integrating from phi to minus phi, and out here you integrating, when you come down here, from 2 phi to 0. But, look at, what happens to this function? You have d z on top, and there is a z minus b here; this for us harmless. In this factor, you could put z equal to b, in the neighborhood of b. This factor you have to put, z minus b, is, epsilon e to the I theta. This, b z, is going to give an, epsilon e to the I theta. There is a power of epsilon here, and there is a power of square root of epsilon here.

So, the net result is a power of square root of epsilon on top. And, what happens to that, when epsilon goes to 0? It vanishes. So, similarly, this portion also vanishes. And now, what is happened, is that your original integral, this real integral, this real number, is apart from some factor of 2 pi, or whatever it is, is given by this course contour. But, this function has no other singularities. So, you can rewrite the course contour like this, or the other way. You can glow it up, you can glow it up all the way to infinity.

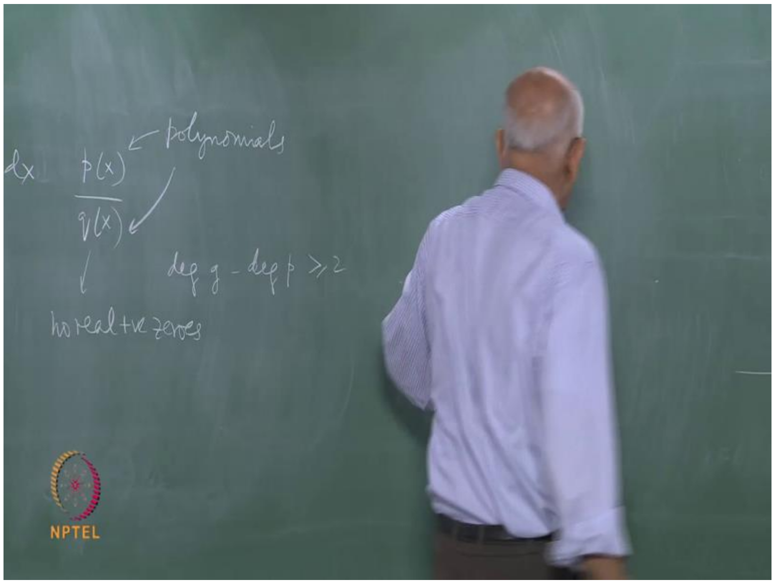
What happens when z goes to R e to the i theta? This goes to, R e to the I theta, I d theta. And, this guy here, a and b, it became irrelevant. And, you get exactly z down square is 2. So, the rs cancel out, and you have an integral over d theta, from 0 to theta 2 pi, which should be at 2 phi. So, the value of this integral, actually turns out to be equal to 2 phi. Now, we done this without any special trick, changes of variables or anything like that, simply by choosing the branch points appropriately, branch cuts appropriately, and it is exactly equal to 2 pi, and using Gaussi's theorem once again, we did not even evaluate a residue.

Notice that, I cannot write this contour by crossing this; I do not cross this contour. If I do, then the function has changed, and then I have to compensate for it, or do something else. There is another closed contour possible in this case, but that does not give you this integral, and that is this following. I get on this ones, I pick up a factor 2 phi in this case, minus 2 phi; but I could also come back and pick this up. So, on this way, I get plus 2 pi, and they will cancel each other, since they are both square roots, and that is the closed contour.

There is a cut here, but that is fine because the function has returned to its original value. And, if I go rewrite it in this form, what I am doing is the following; I start here, I draw a round like this, I come here like around like this, and join it, in the opposite direction. We will play with contours of this kind, quite a bit. But, after you gain a little practice with the phases of these functions, it is quite straight forward to do this.

You can already begin to guess that I have used a and b here, and I use 0 and 1, then there will be branch points at 0 and 1. And, if you recall the definition of the data function, there was singularity at t equal to 0, one at t equal to 1; and I am going to exploit this sort of feature in order to write a contour integral representation for the rate of function eventually. But, I hope, this little trick of doing these integrals, with a little practice, will do a little more, few more of this; this is an extremely useful way of doing things.

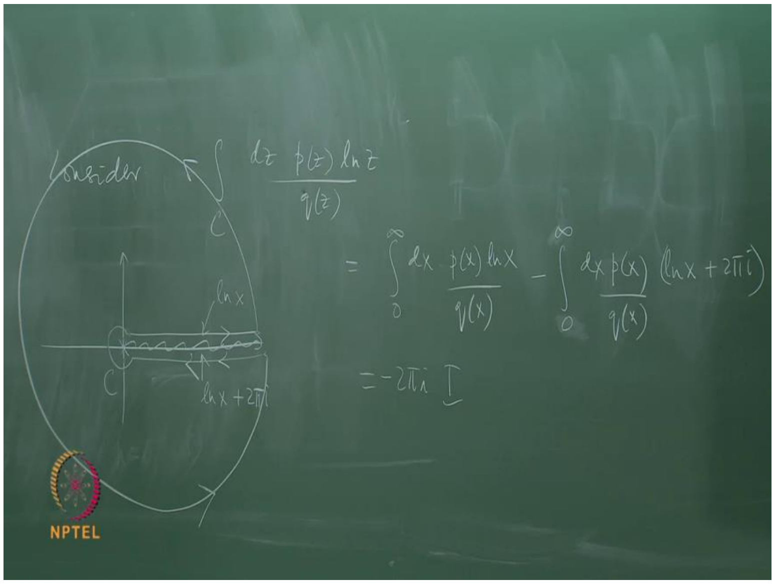
(Refer Slide Time: 07:34)



There is one more integral which also one can do. And, those are integrals of following kind. Very often you end up with an integral of the form d x, and then some p (x) over q (x), where p and q are polynomials such that for the strict towards the degree of q must be atleast 2 or more greater than the degree of p, for a reason it should become here. So, let us say, these are polynomials, p and q, such that the degree of q minus degree of p, is greater than equal to 2.

And, we also require for this strict toward that this integral being well defined; in other words, there should be no 0s of q (x) which are real; so, no real positive 0s, these are assumptions. When you have an integral of that kind, such that q has no real positive 0s, no singularities on the region of integration, line of integration. And, the degree of q is atleast 2 or more, greater than that of p, then this integral can be done in a very quick way by using contour integration with a little trick.

(Refer Slide Time: 09:09)



The trick is a following. So, consider the following integral, integral over a contour c which I am going to prescribe now: d z p (t) log z divided by q (z). And, the contour is the following. In the z plane, this function log z, that I have introduced, has branch points at z equal to 0, and z equal to infinity, so it is part which I chose from here to here. And, out here, the function is log x on top, and out here it is log x plus 2 phi i because the argument of z is increased by 2 phi i.

And, now consider this contour, a happen contour which comes along like this, and goes off to infinity, that is my contour C. So, it does not cross the branch cut that take me to a function which has different value on another sheet. So, among the principle sheet of this log, but the contour is the happen contour which goes just below the real axis on the principle sheet, goes round the origin, and goes off to infinity once again.

And, what is this integral equal to? This thing here is equal to an integral. Now, look at what happens of the origin. There is no singularity at the origin that this quantity in a limit tends to 0 because what happens here is that z is epsilon e to the I theta. And, you have a d z which has an epsilon in it. This fellow here will have a log epsilon, but epsilon log epsilon, the limit goes to 0, as the epsilon goes to 0. So, it does not contribute at all, this one, I added this and then I expanded it out.

And then, from here to there, is the integral 0 to infinity, d x p (x) log x over q (x), that is from 0 to infinity. I have taken the limit epsilon goes to here. And then, this integral has infinity to 0, the opposite direction plus, an integral from infinity to 0, d x p (x), that single value, it is a polynomial, so it does not change anything; log x plus 2 phi i, divided by q (x). So, let us call this integral that I want, let us call it I. I now considered this contour integral. What is this equal to? This is equal to, this infinity to 0, I convert 0 to infinity, and put a minus sign. And, miraculously, the log cancels out, the log integral cancels out on both the sides. And, I have a 2 phi i, that is the integral that I want, which is I.

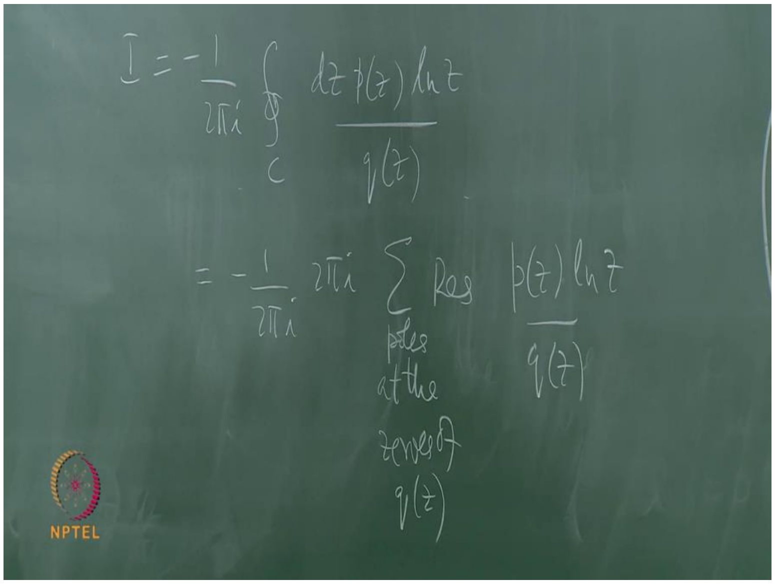
And now, the next step, off course, is obvious. What should be the next step? So, I reduce my, the integral that I want through this contour integral, but to do contour integrals you have to close the contour without crossing any cut. So, what should I do? I draw a huge circle and close the contour. Well, technically I should keep it to some R, let the limit r go to infinity and so on, but this is essentially what I do.

I must ensure that this large circle that I drew, the contribution from that is 0 in the limit, as r tends to infinity. But, that it is because you got a R from here, you got R to the degree of p, and then the log is irrelevant, and you got this r to a degree which is atleast 2 more than that. So, it takes care of this d z or also, because the degree of q is twice or, 2 or more greater than the degree of p; not 1 or more, 1 or more would be a problem because that be a r from the d z, but it is 2 or more. So, in the limit it goes to 0.

Student: There is a minus sign.

There is a minus sign, somewhere, yeah.

(Refer Slide Time: 14:30)



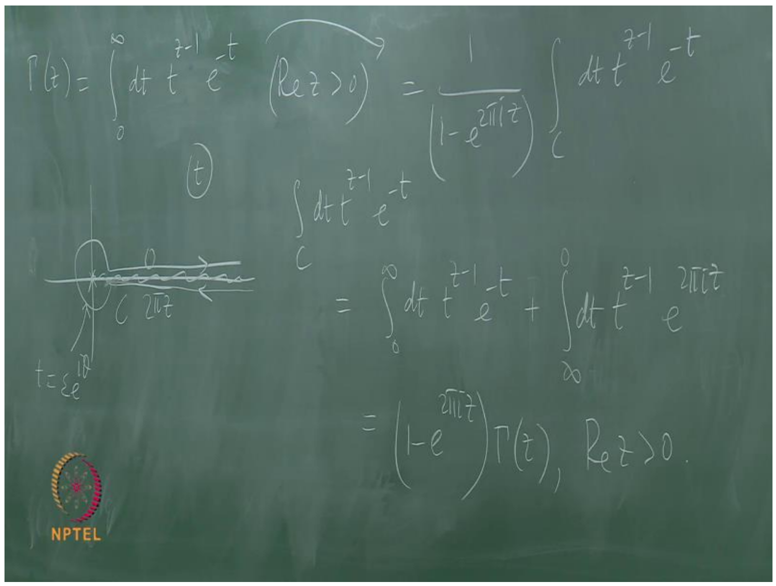
So, now we are in good shape because this immediately says, that this integral I, equal to minus 1 over 2 phi i, contour integral, and it is a closed contour now, d z p (z) log z over q (z). So, you introduce that log z gratuitously, but that helped us to do the integral, reduce it atleast to a closed contour. What next? Where are the singularities of this? Well, there are branch points here, but there are outside the contour. I can now shrink this integral. And, there would be singularities, there would be all the poles of the 0s of q (z), let be some poles, because it is a polynomial.

We assume, there are no poles on the positive real axis, no 0s on positive real axis, so these are sitting somewhere else, and they are inside the contour. And, they are being enclosed in the positive sense. So, this guy here, is equal to minus 1 over 2 phi i, times, 2 phi i, summation over the residue, p (z) log z by q (z), poles at the 0s of p (z). But, there is a log sitting there, so you can do a large number of integration.

For instance, you could do this integral. You know, you can do 0 to infinity, d x over x square plus 1, this is trivially done, tan inverse something. Suppose, if you want to do x cube plus 1, or x to the n plus 1, n greater than equal to 3, some positive integer, suppose you want to do this integral. This is called singularities of 0s, that all the routes of equation x to the n is minus 1. So, write this as, e to the 2 phi r i, the one, then move to e to the I phi because of the minus sign, and it got singularities at various places. For instance, regarding this 3, for example, then x cubed is minus 1, and the routes are at minus 1, something here, something there.

Now, all you have to do is to pick up the contribution from these poles, with the appropriate logs, and simplify, and that is the answer. There are other ways of doing this, but this will tell you, this is independent of how, what the power here is, straight forward to do this. So, certain integrals can be done using this. This is a simple example of how you exploit this log at the branch point, in order to evaluate definite integrals.

(Refer Slide Time: 18:07)



What we need to do next, is to ask some of the special functions we talked about earlier like, the gamma function and the beta function, have representations that go beyond, the representations we found by integrating by parts or special tricks of that kind. And, we call the first one we do is the gamma function. So, recall the gamma of z was defined as 0 to infinity, d t, t to the power z minus 1, e to the minus t; and carry out towards that the real z, the real power of z, is strictly positive. Then the singularity at the origin did not bother you.

Now, I like to continue this analytically, using a contour integral representation. And, the trick is the following. The trick is, in the t plane, this integral for general complex values of z, has an branch point, at t equal o, and at infinity. And, we choose the cut to run from 0 to infinity. What is the phase of this function, t to the z minus 1; by the way, this fellow is the entire function, no problem at all.

What is the phase of t to the z minus 1, just above the real axis on t? 0, because the phase of t is 0, right. So, the phase here is 0. And, what is the phase here? Not z, but t; t increases by pi, so what is the phase here? 2 phi z. Why do not we bother about the minus 1? Extra minus 2 phi, and that is one of course.

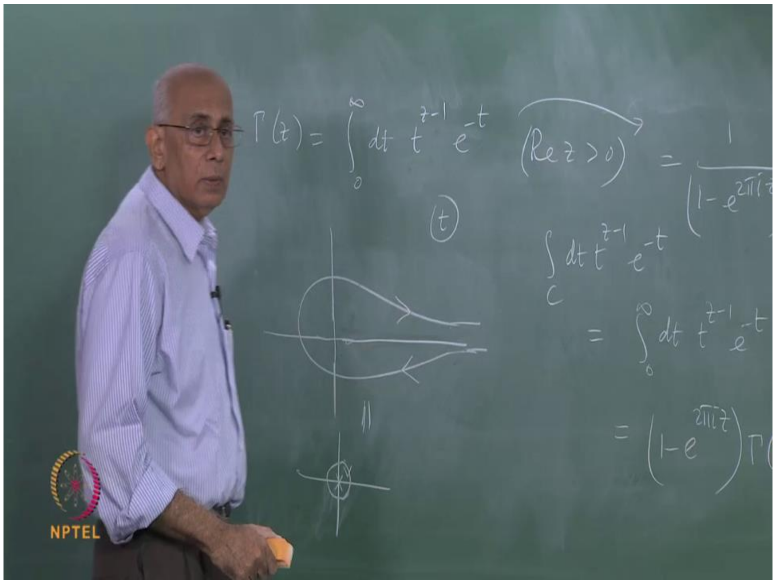
So, all that I am saying is that, t to the z minus 1, equal to mod t to the power z minus 1 above, and it is equal to mod t to the power z minus 1, e to the 2 phi i z minus 1 below. And, this one is gone because it is the minus 2 phi i is 1. So, the phase is 2 phi i z. Now, what is the contour I should consider? I want to convert this to a contour integrant. What is the contour ratio you consider? Exactly the same contour. So, I consider this one here, this line is contour, let call that c.

And, let us see what is the integral over c is? So, integral over c, d t, t to the z minus 1, e to the minus t, is equal to, an integral from epsilon to infinity, and I am going to write the limit epsilon goes to 0 straight way. So, this is the 0 to infinity d t t to the z minus 1, e to the minus t. By thie I mean a line integral from 0 to infinity, the original gamma function plus, an integral from infinity to 0. So, this is from infinity to 0, d t again t to the z minus 1, by t here I mean the positive number, times e to the 2 phi i z because that phase comes from this factor here, plus the contribution from this circle.

But, on this circle, t equal to epsilon e to the i theta. So, the contribution from that circle is of the form, an integral from 2 phi to 0, epsilon e to the i theta, i d theta, that is d z d t, t to the power z minus 1. So, you got, epsilon e to the i theta to the power z minus 1, e to the minus epsilon e i theta. There is an epsilon here, and there is an epsilon the minus one here. So, those 2 fellows cancel out, and you are left with an epsilon to the power z. But, what does that do, when epsilon goes to 0? It goes to 0, provided real part of the z is greater than 0, otherwise it is not true. But, that is why we started.

So in this region, this has gone, and the contour integral is equal to 1 minus e to the 2 phi i z, gamma of z, in the region real z greater than 0. So, it is a delicatized element. We keep real z greater than 0, and then what I am showing you, is that this happen contour is equal to 1 minus e to the 2 phi i z, times gamma z. So, this implies that in this region, this is also gamma of z is also equal to 1 over 1 minus e to the 2 phi z, integral over contour, d t, e to the z minus 1, e to the minus t, in that region, point by point. This is where the poles of the gamma function arise from because you can write this as sign, whatever it is, perhaps it is minus 2 i pi z, and minus 2 pi z i in the denominator; and that is why the poles would come from.

(Refer Slide Time: 24:35)



But, you see, once you have done that then you can distort this contour. So, here is the branch point, and I can write the contour like this, going off or on the real axis. Can I now close this? Can I do that? No, because this function is bad news, this fellow is bad news, it will not converge. And, if you open it out, you can open it upto a certain point, but this is not going to help much; real z must go to plus infinity, otherwise that integral will not converge. So, I leave it like that; I leave it in this form.

But, the point is, important point is, does the contour touch z is equal to 0, p equal to 0 anymore? No. And, the problem with negative z, to the left of real z, is equal to 0, arose because of what happened to the integrand at t equal to 0. But, now the contour does not touch t equal to 0. So, you put any value of z you like, and the integrals will make sense. So, this is valid for all z, and it is an analytic continuation.

So, let me go with the argument again. You started with this, and showed that, in that region, this line integral could be written as this contour integral; they are equal, point by point, for every values of z. But, the contour integral make sense, it converges for all z. So, it is a representation of the gamma function. And you can now read out what is the singularity structure of the gamma function is, from this.

We know, from other considerations that the gamma functions has simple poles, and z is equal to 0, minus 1, minus 2, etcetera, but minus sign, residue is minus 1 to the power over n factorial.

And, it has no singularities for positive integer values of z, like the factorial. That should show up here. What happens if I put z equal to a negative integer, including 0? What happens to this branch cut? Suppose, t z is equal to minus 3, then you got d t over t to the power 4, e to the minus t. What sort of similarity does that have? It is got a pole, at t equal to 0. Is that a branch cut? Is any multiple value function left? No.

So, the branch cut disappears, the discontinuity disappears, and you have a pole at this point, and you have a contour of that kind. You can also see, there is no discontinuity, you can join this 2. And, this becomes equal to, at this pole, that is 2 phi minus 2 phi i, times, residue of that point. This will give you a simple pole at z is equal to minus n, because the sign in the denominator. We must be careful to write the residue for phi factors and so on. And, this fellow of here will also give you factors, when you evaluate the residue at the origin.

And, that is not hard to do. What you need is a coefficient of 1 over t. So, suppose this is t to the minus 4, you look at the coefficient of t to the 3 here, and that gives you 1 over t. And, that is going to be 1 over 3 factorial or whatever. So, it will also give you, straight away it will give you the residue, minus 1 to the power n over n factorial, minus come when expanding it to minus t factorial, and then the n factorial comes from the expansion of minus t. So, this will recover for you the fact, that the gamma function has simple poles at negative integers, non positive integers z.

But, we also know that when z is a positive integer, this function is regular its n minus 1 factorial, that should show upto, that is tricky because z becomes a positive integer like, 6 for example, you have d t, t to the power e to the minus 5, and there is no cut here at all, and no singularity here at all. So, in the limit that integral vanishes. And, it vanishes and kills the 0 of this function at z equal to a positive integer, such that the ratio of the 2 tends to n minus 1 factorial.

So, the fact that, the gamma function is just the factorial at positive integers will also come out, but it is a little hard to establish. You might as well establish from it here directly by integrating by parts. But, this will, no doubt, gives you the right answer. But, it happens because the cut disappears, but the pole also disappears, and the contour integral vanishes in the limit. And, it vanishes, so does the denominator in the ratio is finite. But, for negative integers, this contour integral does not vanish, it just picks up the residue of the pole at 0, and this gives you a simple pole, and it gives you the right residue at this point.

But, for other values of z, this is a representation. For arbitrarily large values of real z on the positive or the negative side, does not matter. It is still a representation. So, if you like this thing here is a master representation, a sort of master representation from which you can write down the value of gamma of z, for all z. But, typically, it is an integral representation. And, the important thing to remember is, in this instance, you cannot open this contour, you cannot do that. This is the best you can do. It is useful in many ways. So, we will try to derive a similar representation for the greater function, if you take that up next time.