Selected Topics in Mathematical Physics
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Module - 6
Lecture - 14
Multivalued functions: integral representations (Part I)

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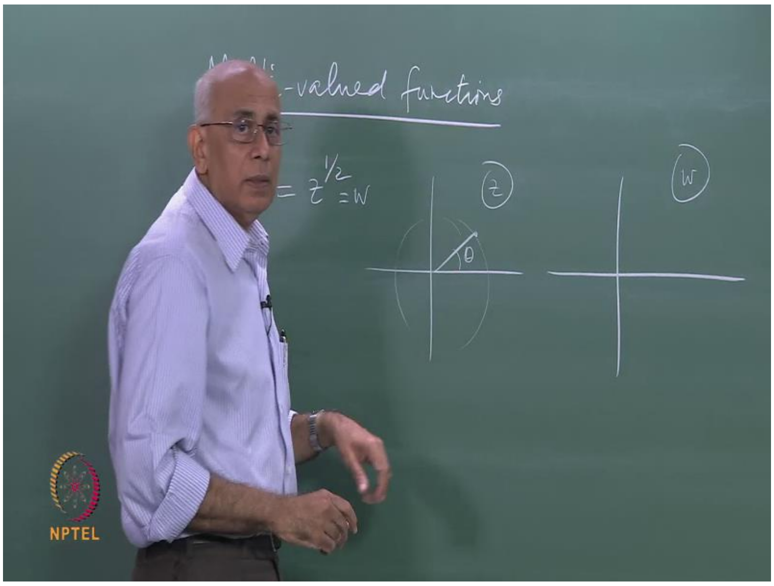

Multiple valued functions. So, multi valued functions. So, far we looked at problems where the functions we will concerned with I mean single value in other words for a given value of the complex variable z there is a unique value of the function f of z. So, we will looked at function we will looked at the general case where you have one to one map of the extended complex plain to the extended complex plain discovered that the most general form of base transformation is more base transformation, and so on. We have not looked at functions which can have more than one value for a give value of z, and this going to bring us to the idea of branch points in branch cuts, and so on, because most functions are infect multiple value.

So, let us look at simplest example of these let us look at the functions like a f of z equal to z to the power of half what I mean by square root of z now the first problem that it is you write a way is that as you go round in the z plain, and let us called this number equal to sum w this is the w plain since we take square root here always it implies that if z is sum r e to the I theta, then the argument of w is e to the theta theta over two it is half this which means that if you cover this entire plain the z plain zero to two pi in argument your only covering the upper half plain in w, because the argument of w will not exceed pi it will there for respecter to be upper half plain ok.

Be like to cover the entire plain in w, and to do that it is clear that you most increase the argument of z up to four pi. So, that be argument of z to the half can go up to two pi, but of course, if I increased by four pi than I no way of knowing where I am for instance if I am this point here at some angle theta I am it exactly the same point if I increase by two pi. So, the second time I go round two pi to four pi I am not sure where I am right if I just specify a point on the z plain in this fashion, then it is not clear whether the argument or theta plus two pi. So, it is evident imminently that in order to cover the full w plain zero to four zero two pi you need to be able to cover z zero to pi. Which means you need to copies of the z plain place it to one above the other if you like such that when I go around ones the horizon I go around fully ones in the z plain I hit regiment to pi are than zero than I go the second time I go form two pi two three pi to four pi which being's be back to zero cracker the I need to copy of the z plain in this case in order to be able to map on to the full w plain this to copy most convention where pitcher in them be will find that is ones you get little use you this it will quit easy to do this.

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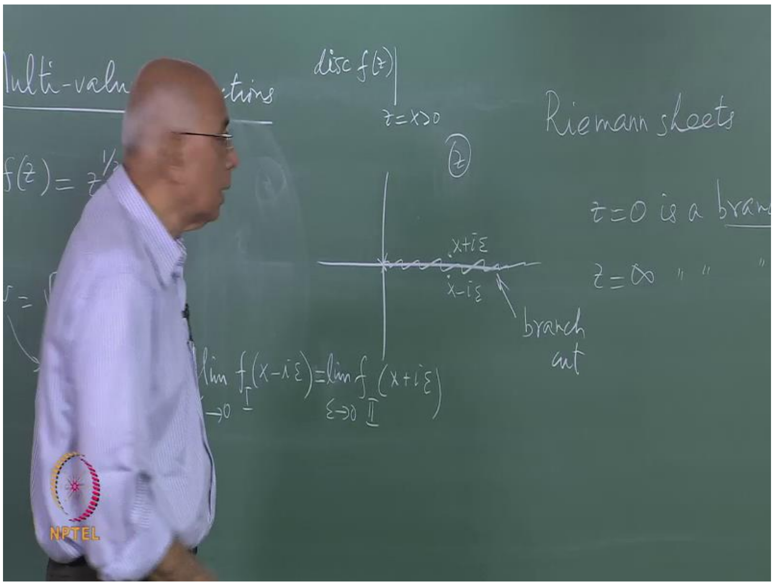

These to copy her called Riemann sheets, and in the case of the square root function we need to just of the sheets, because that well bring you back will cover the entire w pains. So, it is clear that if you had z to the power one for is some plus the function you would have to have three copes of the z plain in order to cover the w plain completely. So, would you three riemann sheets and. So, and but we also most arranged such that when I increase the argument of theta of z theta from zero two four pi, and increase be on four to pi, and back to the first a plain the top plain. So, the geometry of this topology of the sheet is not all together triviour, and be have to be little careful how we do this what is to be absurd is that on the fast plain. So, this speak you have w equal to square root of z, but after the argument of z increased by two pi, then the argument is two pi theta, and I take a half. So, z two the half will be.

In the second sheet will be e to the I pi, then square root of z, because the argument of z is going to be what it was originally at such a point plus two pi, and if I take half that I get pi, and it will be I pi is minus one. So, it is evident that on the top sheet Riemann sheets, and rather picture very shortly the values we have to speak off z to the half is a z to the half, but on the bottom sheet it is minus z to the half now how do we ah picture this when it is minute you obvious that when z is zero both this values coincide, and whenever two different branches of a function coincide that point is called as branch point. So, z is equal to zero is peculiar kind of singularity it is not an algebraic it is not a pole it is not an accumulation point of poles it is not an essential singularity, but it is called a branch point.

So, z is equal to zero for this function is there any other branch point in this function well we know in this extended complex explain this in one point at infinity. So, infinity is another branch point, because whether it is plus infinity minus I infinity or I infinity it does not matter just at one point. So, the other branch point is that z equal to infinity, and the way two sheets are constructed is as follows you take the top sheet, and you take the bottom sheet together, and then at the point where this functions coincide in value joint the two glow this together. So, you glow them at zero, and you glow at them infinity once again.

And then once I cross zero to two pi, and cross the argument two pi, and increase the argument of z by two pi, and should descend down to the second sheet, and then I go around, and come back to four pi I should ascend back to the first sheet. So, the way to do this is to take two sheets of paper mark in origin at the center of this paper, and draw a line from zero to infinity in any direction what. So, ever most conveniently along the positive real axis all the way to infinity, and then glew zero, and infinity as other for the two sheets, and make sure as you cross two pi you will descend to the second sheet as you cross four pi you ascend back to the first sheet this means you make a slit, and take one end of the slit in the top sheet, and glew it to the bottom, and vice versa if I draw a picture

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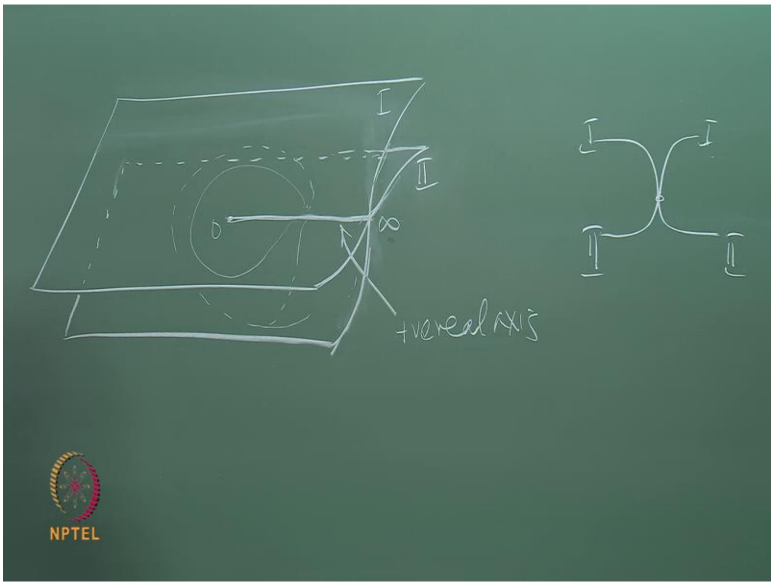

It is going to look like this here this is due to the bottom sheet these was bottom sheet looks like see. So, you get the picture these two sheets found zero to infinity you group crossed the sheet has gone to the bottom that sheet, and in these function if i. So, start this is the realize positive real axis this is zero, and emotionally infinity this is first sheet see I draw the selected better. So, you get a picture, you got a this portion this is the first sheet little bit more. So, conveniently illustrated this is the second sheet. So, I am just about the realization on the first sheet encircle all the way down come two paid we slip into the second sheet at the bottom travels on the second sheet on here at the I jump up from here I can climb on to the first sheet. So, it look at the right looking into the from last infinity go to like this for me these line these the line were.

The two branches the line between two branches points these line that why the cut it that is two increase epsilon why the sheet is common both sheets as you can see, and I one cross the cut, and jump on the another sheet climb down, and climb up the second sheet all right. So, epsilon the picture give you some kind of idea what this look like this split is runs from zero to infinity, but it could as well as go to infinity of any direction, but it more conveniently picture in this function, and these two are the remand sheet what you should in can understand is that all the z plane. So, this is sheet one this is one this is two, and this is two this is, and this is looking to the grand point in this is function cut in function.

This is cut look likes, and showing the sheet separated, and this jump magnified and. So, you can get a picture what is happening, but actually by continuity when you here it is same as being here the continuity, and may here same as being here, and what types of being here it means at on the top sheet this is the z plane was a branch point here cut all the way to infinity this is called a branch cut, and when I am here on the top sheet I am sheet one at x plus I epsilon say, and I come down here x minus I epsilon on the first sheet, but I continuity that the same as being it here same as being here it an f x plus x I epsilon, and the second sheet.

As I show these things separated take meanings, and other words if I call this function f or z f one on the first sheet x minus are epsilon limit epsilon go to zero is same function on second sheet at x plus ix limit, and vice versa being here is same as being here, and being there is same as being here. So, when you have multiple valued function what you need to do first construct the remains surface these sheets together stuck in this fashion the branch cut is called remains surface, and then once you have a remains surface at every point of remains surface is function as a unique value if that not will case if did not have a remains surface, and I just do the z plane, then I any give point in the z plane clear whether I am talking about plus or minus z, but now it completely I am at every point on the unique value once you I got a unique value for this multiple value function called square root of z now it is clear that as you cross this cut, and the function goes down here from here to here there is jump there is a discontinuity in the function which is the same as saying the discontinuity this is x plus I epsilon this is x minus I epsilon there is a different between. So, it is a clear the discontinuity disc f of z of z equal to x from positive values.

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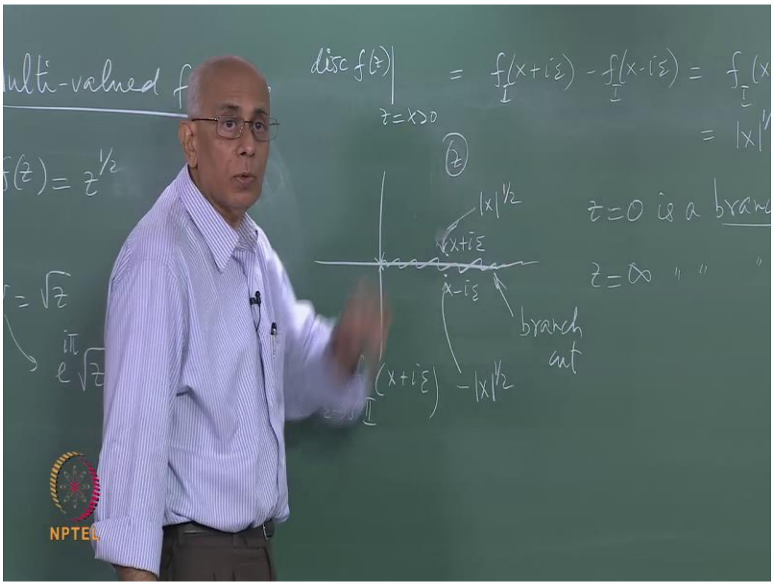

If on the cut positive relaxes this quantity this continuity defined of f f x plus I epsilon minus f of x minus I epsilon, and what this is b well that easy to answer on the first sheet when I am here what is the phase of z on the positive realizes just as epsilon goes to zero to the phase is zero feet as zero i. So, observed mean module I z next to the power off. So, out here the function here at this point as the value modules x to the power half needed becomes x x module this case, and what the value of function here when I go round z increases argument by to pi z to the half increases the argument by e to the by pi. Now we have minus modules of value of here with minus modules x square epsilon the same as the value on the second sheet directly below this point as I have explained that why called as discontinuity.

So, on the first sheet f one this is the f one this is the same as f one of x plus I epsilon minus f two, because being on the first sheet on this point these the same on being the sheet this is right of the up of the real-axes, and real-axes, and the below relaxes looking into, and what this equal to this equal to mod x to the power half minus minus mod x to the power half with twice mod x, and discontinuity vanishes at zero after that increases. So, this idea going to the next interest usefully one for us they do not always draw all the branch cuts, because lot of freedom in drawing in branch cuts, because reason is that I do not have to take the argument of z to run from zero to pi I could say it runs from minus pi to plus pi, then of course, the cut of like this in this direction or I could say runs from pi over to two plus pi five over to, and then to along imaginary access there is a most convenience choose along to relaxes especially the positive, because the always choose phase to the zero directly. So, more complicated function I wont to always construct the remains surface, but I do the simply indicate on the plane.

What faces on the talking about that immediately tell you which sheet time out, and what the discontinuity are. So, immediately create cannot have a one branch point the has to be a branch cut from one plus point to another point at least to branch point could be more when will see example of, but this is not all this is the simplest incident branch plus point now it is clear function like f for z equal to the z or one over in, and n equal two three execrate all these function have branch points z easy equal to zero, and z easy equal to infinity they called algebraic branch points, and how sheet this function have in general this would been end sheet e d r e mean service per in scene you had three, and equal to three than count part of this diagram would be form.

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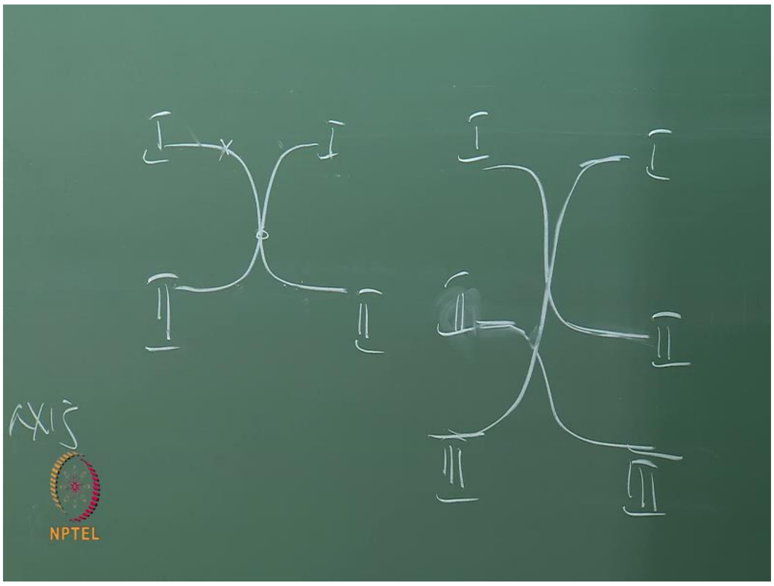

One when more on select three into two, and going to exited from two we shift tree, and form tree you back two one in this faction this is one this two is to when the same level, and this is sheet three this to are when the same level this to are the support to coning inside, and the hole think support to breach cut spouse to come out of the plain of the bold. So, this is the three sheet of the surface such that you go round once the first sheet the center the top of the second sheet of the realize go round come back, and send just above the realize of on the tread sheets go round ones you plain back to the first sheet.

So, aging what happened to this function p our q, but p our q intestate the last you cancel coming fetes of extra p, and q comprising say, and q is the integer grated than equal to two haw many sheet would be this faction have it would be just have q sheet, because you go round this you go round this, and q times to pi q hear q is will cancel this becomes the inter. So, you back plus one a ones you multiple by in i. So, this is a q sheet e d structure ones aging the fatter of you have a power in complete irrelevant after all z qua m this completely single value this no problem at all, but z to the power off has the problem you have the breach point am. So, things your all called algebra break branch points of some kind, but this more complicated possibility suppose you had f of z equal to z to the power of alpha, but alpha is the some complex number arbitrator y may be irrational number not a rational number.

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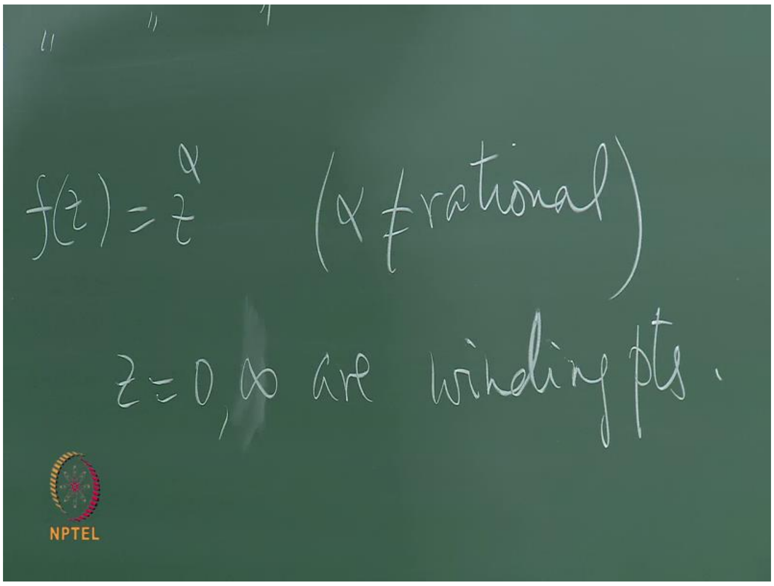

What happens? Now it is clear this is again multi appreciated, but you go round once, and z to the alpha this function phase changes from zero to two by alpha, and second time you go around becomes four time alpha we go around in the opposite sense it becomes mines two by alpha etcetera on the. So, called principle sheet of this function where the argument of z rounds from zero two to pi this function here on top the first sheet it has value which is given by phase zero, and then face two pi alpha just below real access.

The overall the second sheet, then the phase change increases from the two part alpha to four point alpha, but you could have gone round in the opposite sense, and it go to mines two point alpha alpha is not rational. So, matter of how many times you go round you never going to equal one what to, then b structure you should have infinity sheet e d structure in this case, and this thing is called a wind being point. So, this is z equal to zero for also branch points we called winding points they two are branch points, but you must remember that it is infinity sheet structure in this case what happens? Now if I have functions like log say. So, let us look at the log function.

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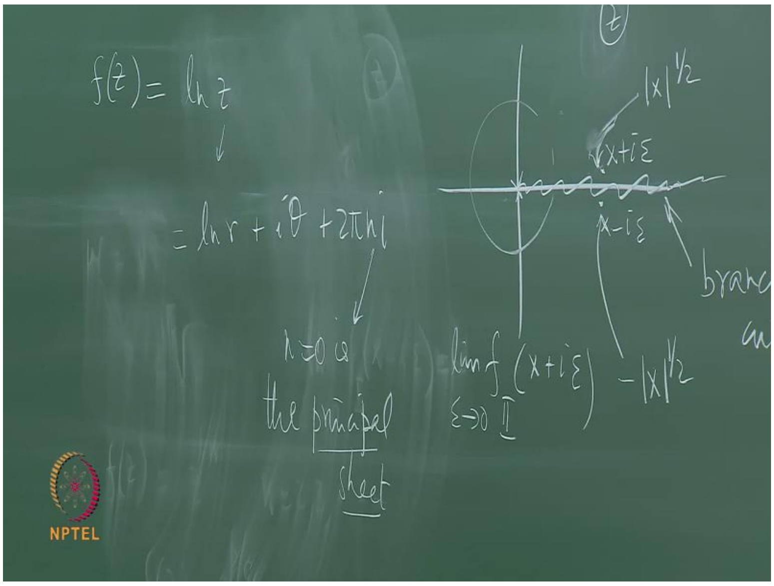

What would happen to this well it also has a branch point at z equal to zero the reason is that is for insurance this on the first sheet this is equal to log are plus, and this immediately evident that on the first sheet this I increases as you go from here by two pi it is add it your end of the two pi i, and you go round once again, then this log increases another two pi i. So, in this case it is not the function is changing sign of getting multiplied by phase factor, but you get in add it quantity two pi I in the first sheet four pi zero on the first sheet two pi on the first sheet zero on the first sheet two pi on next sheet four pi on the below that on. So, on, and similarly mines two pi, and mines four pi.

So, the sheet on which in general function logs it is therefore, equal to this plus two pi n I in general, and the sheet n is equal to zero is the principle sheet the sheet on which argument delta of z transform to zero two point this is the log r ethnic branch point z equal to zero infinity of logarithmic what about this function. What about the function f of z equal to log z mines one what sort of singularity does it have all you done should shift the origin by one right. So, it is not a branch point at z equal to one what sort of branch point is itlog r ethnics branch point.

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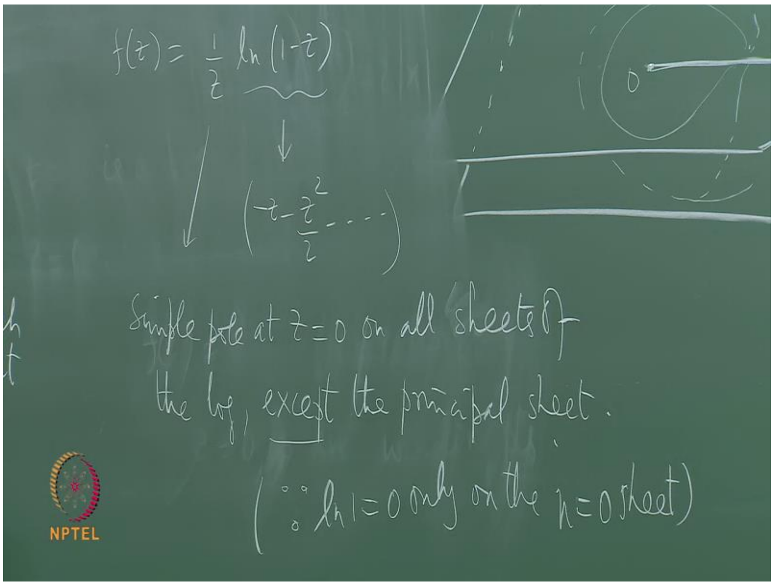

This branch point once they argument of z mines one changes by two pi now exactly the same behaviors before what kind of singularity or the singularity does this function hand by the way one mines z is e to the I pi time z mines once. So, that is not serious problem. This also a branch point z equal to one, and at infinity, and I can choose the cut run in along any ray from one to infinity, but what is sort of singularity is does it have it is definitely got logarithmic branch point z equal to one at z equal to infinity does it have any other singularities where is the singularity at z equal to zero looks like it has a singularity, but please note it this quantity as z tens to zero this quantity as z ten to zero this thing as a power series expansion. So, this is equal to mines z mines z square over to ex tetra ex tetra leading term is mines z is in that cancels this.

So, what sort of singularity does it have it has a remove able singularity right as said equal to zero, but you made an assumption in making that statement this thing implies that z equal to zero that log one mines z is zero when is log one mines z ah log mines one mines z is zero at z equal to zero implies that log one is zero, but when is log one as zero log one is zero in the principle sheet otherwise it is two pi n i. So, this is not true got it two pi n I add it here. So, what it is conclusion now it is got a sing lure got a simple pole at z equal to zero on every sheet except the principle sheet on the principle sheet the singularity has gone the vanishes is no pole at all. So, you have to be careful the it tells you have to be extremely careful

This function simple pole add z equal to zero, and all sheets of the logs except the principles that happens, because log one equal to zero only on the n equal to zero. So, you go to be causes they could be some singularity is some sheet, and could be other singularity is on other sheet does not always follow the singularity goes all the ways will construct more complicated examples, but this is stay why singularity only on the sheet other than any equal to zero on any equal to zero it becomes a remove able singularity.

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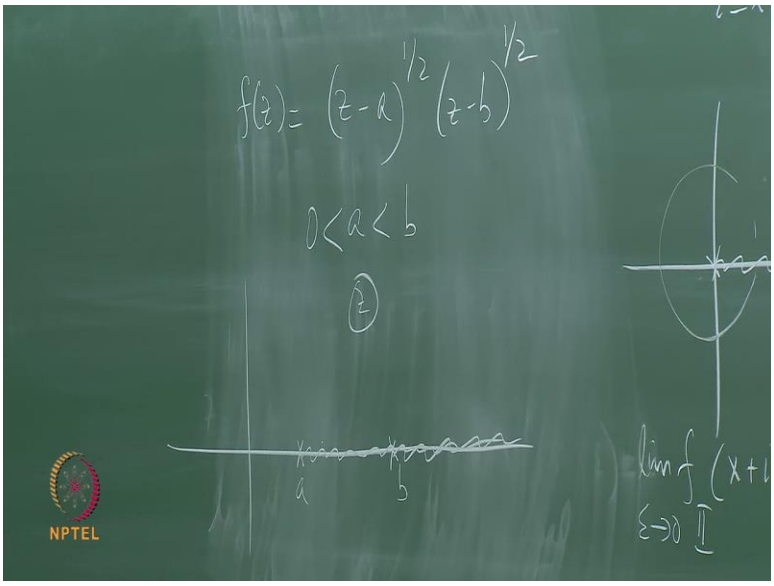

So, do not blindly put log one equal to zero, but ask which sheet are you want what sort of singularity is this function had F of z equal to z mines a to the power of half z mines b to the power of half let us say a, and b are just for simplicity let say a, and b are positive conscious zero less than a less than b not a server a singularity less I have it is clearly got branch points it is clearly got some square wood branch points, and where ever the the square would vanish they argument vanish is not branch points those points.

So, you agree that it has a branch point on this now I m not even triangle to try construct the remand sheet, because you need two sheet it is a surface of this you need two sheet it is a surface of this, and they have to be you know put together complicated way, but let us sign if you can see simplify this picture just found draw single z plain, and I m going say alright I start here I have zero phase at this point defiantly at the point a this is the singularity at the point b is the singularity, and if I what to do what I did earlier is square root of z, and draw branch cut running from a to infinity, and another from running from b to infinity let us do that. So, here is the branch cut running from a to infinity, and second one on top of it running from b to infinity now let us try the system of this function that is all that matters.

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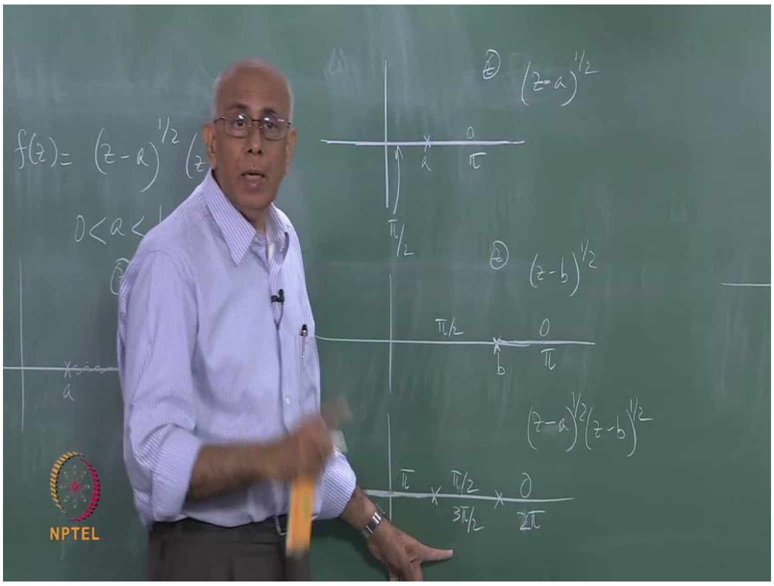

So, here is a, and now going look at z mines a to the power of half in the z plane, then the phase of z mines a to the power half just about the real access on this sheet is zero at this point faced down by this function either therefore, mean the modules multiplied by e to the I times your phase just write the phase down. So, out here value of the function if z is if a is six for instance I look at the point seven, then it says seven mines six to the power half one to the power of Positive square root module what is the phase at this point z increases z mines a changes phase by pi. So, z mines a to the half changes faced by pi over two. So, the phase at all along here is pi over two in other words what I mean by z mines a for any point z to the left of a on the real access modulus of z mines a to the power half multiplied by e to the I pi over two which gives me an I that is what happens when you have a minus inside this square root you get on i. So, the phases pi over two when I come here what is the phase it is pi, because z mines a has increasing by two pi. So, the phases pi now all you got ask is.

There any line across faces jumps discontinuously yes on the positive real axis a to infinity that is why you have a cut from a to infinity. So, there is the cut hear, but I am not going to bother even one draw this cut this the… Now look at z mines b this the keep track now look at z mines b this is the z plain z mines b to the power of, and b is setting from hear this is b what is faces z mines b to the power off one second exactly there is before the phase is zero to the right of b every were to the left of b the phase is pi over two, and below the phase is pi, but what I have is the product of be show the function. So, the phase is died up in the product. Right now we are ready to write down what was the faces of z mines a to the power of z mines b to the power half suspected singularity at a is the another one is b, and less write the faces down to the right of b the faces of this skins zero the faces this is zero. So, the faces is just zero of the realized between a, and b the faces of between a, and b the phase below o let you cut the left the phase hear is till zero, but the phase hear is pi over two. So, between a, and b on I have to do is to add of zero, and five over two. So, is pi over to the, let us come to the lift of a is pi over two hear pi over two hear. So, the phase is pi now let us go below out here between a, and b phase of this factor is five now let us go below out here between a, and b the phase of this fact is a, and the phase of the pi. So, its still. So, its three pi over two, and to the right, but to the below the real laks the phase here is pi two. So, the faces two pi this town interims of what the actual factions is, then at says function looks like this.

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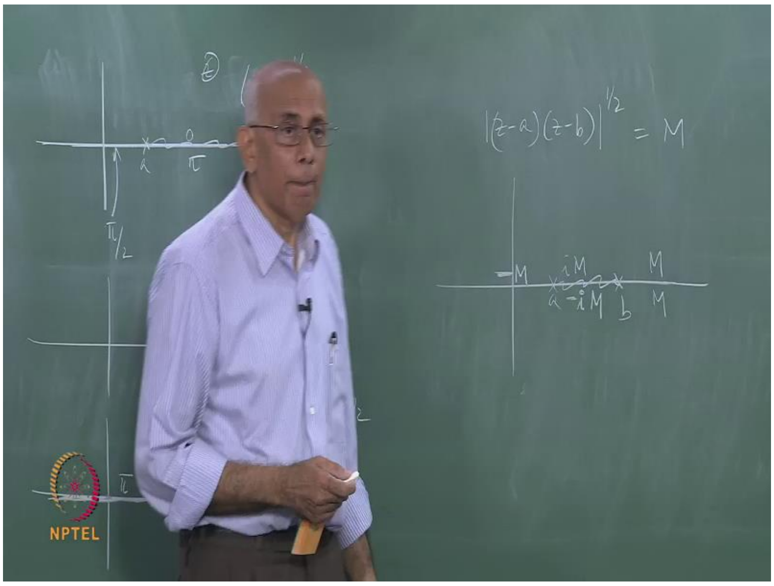

Let us puts modules z minus a z minus b to the power half that puts is equal to sum m is the modules when it is clear from the picture that here is a here is b, and the faction is modules at this point to the right of this the faction is eight of the eye pi of the modules, but the eight of the pi over to this eye. So, its eye times of the modules here to the left is a phase pi to the pi is minus one. So, out here its minus the modules here when below three pi over two, but what three pi over two minus I minus i. So, we have minus, and out here its m time e two the two pi i, but what he does the two pi I m its self to phase of two pi I is change the topic. So, m its self what is happened now where is the cut the started with two cuts, and cut going all the way from here to infinity another cut going from on the way from infinity super pose the two. So, expect the cut from the a to infinity in this continue b to infinity another discontinuity b to infinity with another discontinuity, but what happen is the discontinuity between the a, and infinity what cancelled out this means that u can now raw the cut from just a, and up to b. So, it is a final cut that it, and that why the discontinuity, and there is a no discontinuity which means they no branch point infinity for this function, but that now obvious by looking at the function itself if you took this function, and you let z go to the infinity compare to a, and b u can draw, and b u get z to the half z to the half which is z, and that is analytic at infinity. So, there is really now in a similarity, and infinity at all, and the cut can chosen to lie between a, and b in this case ok.

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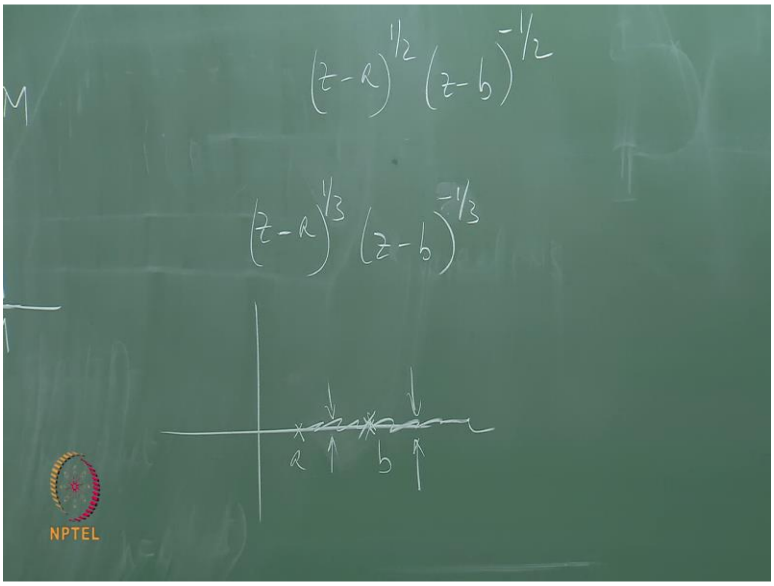

What happens the other thing z minus a two the power off z minus b two the power minus off. So, now, I have a ration of to square roots by the by one over the z also have the point, and z is equal to zero. So, one important lesson is u have a branch point he function could either vanish or it been infinity as you already know from the case of z to the half now what sort of z branch point z it have most certain is called the branch point the z equal to a another one is z equal to b, and u have to do exactly the same think has before except that one that calculate its instead of plus half, and put minus half here, then the phase would still three zero out, but they should minus pi over two this would be minus pi on the sight, and then I to exactly the same think has before if its minus pi, and u add that pi zero it is clear the cut vanishes once again from b upwards to infinity once again u can chose the cut to run from a to b apart from a signs and. So, on plus I minus I into change that exactly the themes before what happens to the gain you can see that this think here, and z turns to infinity it is quite regular tens to one. So, there is no singularity at infinity for the this function what about this function what happens?

Now why would say there are branch points yeah at a is definitely branch point algebra a branch point, and three sheet structure and. So, on and. So, also at b, but u except the discontinuity to cancel from b to infinity to this case no no, because you can see that this think here goes to the z to the one third z to the one third. So, the whole thing goes z to the two thirds at infinity.

And that definitely has the branch point at infinity. So, in this case u cannot have a cut line just between a, and b it has to go upto infinity, and different discontinuities with different discontinuities. So, in this this function here would have a structure the branch structure here is a here is b. So, there is a branch cut here going all the way infinity there is one discontinuity a jump a cross, and there is a another discontinuity when u jump a cross how of this one third, and minus one third what do you think is going happen in the cut from b to infinity would be cancel out, because u can see that whatever phase.

This acquires is can by the faces it sin the opposite direction. So, from b to infinity will cancelled out did u can see that has s tends to infinity this followed tends to one. So, there is no problem at all that u can do, but it is only for the square root branch cut if u have two square root branch cuts the products of these two guys u can have a finite cut the ration u can always have a finite cut, but not for the product to t he product you really need half power half. So, some of these tricks one cane place simply, because these things has finite cut for instance, and now let us come to a very important point mind that is the following I have been talking about using coase's therm in order to evaluating integrals using the residual theorem and. So, on and. So, forth when you have a branch cut u cannot have a close contul u have encircle the branch cut once, because u go back u can another Sheet.

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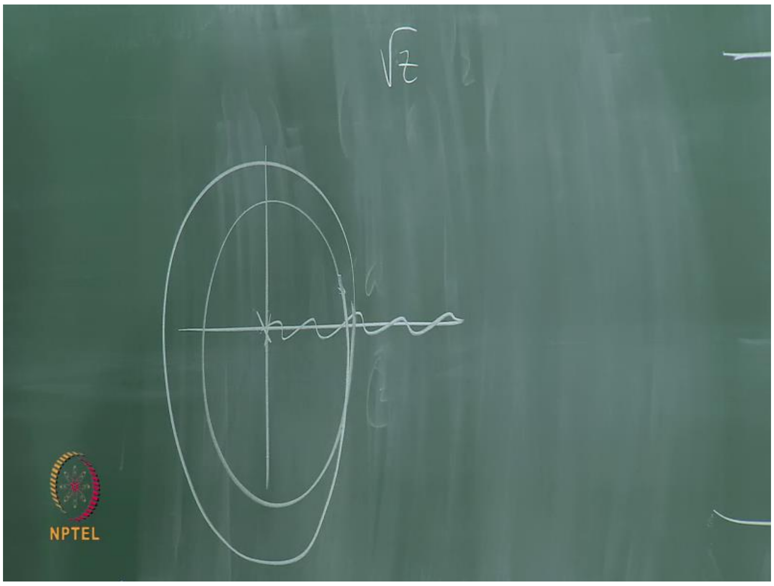

So, u no longer r close, if u have a square root, and point from zero to infinity. So, square root of z say, and is a cut from here to there, and I start at the this point on principal on the top sheet I would like to the encircle the branch point at zero how do I draw a close cut close if I do this, and I come back, and the on to the next sheet. So, I need to go round to that once again, and come back that is a close one. So, there is a no harm u can still in the presence of the branch cuts still try to find close the tolls, and apply (( )) thermo and so on, but you need to ensure that the function returns to its origin value one the comes back to the starting point otherwise not it is a close that crucial we will do that over, and over again will try to find close in such a way that will return to the original value. And that would be in to one sheet, and now very often, and I am not going to draw a multiple sheets, and just going to draw the faces. So, we have to keep track of this point.