Selected Topics in Mathematical Physics
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Module - 5
Lecture - 12
Mobius Transformations (part II)

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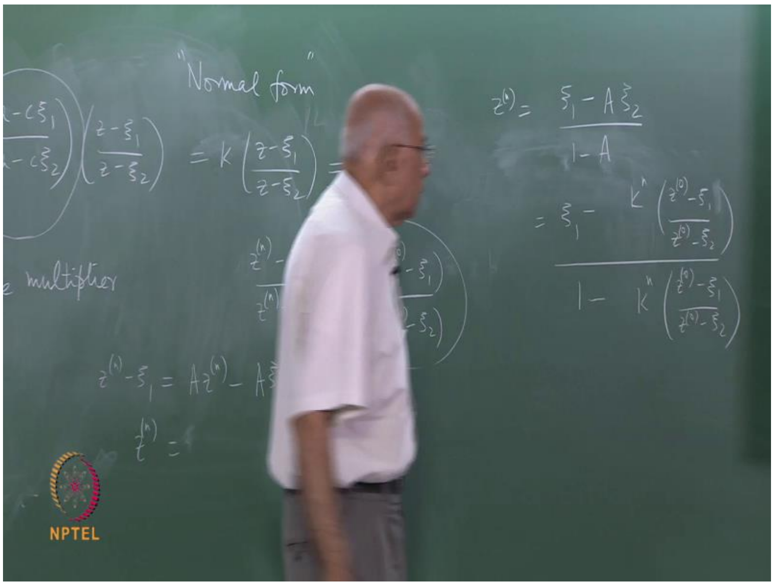

So our general statement is that when you got two distinct fix points then the transformation is of the form w minus this is equal to K times z minus psi 1 z minus psi 2. So, that is a much more natural way of writing this mobius transformation, because it say take this a issue and then all you do is multiply this ratio by a certain number. This immediately implies something much, much more interesting which is if you do this transformation over and over again.

So, let us suppose that you start with z naught and you make the transformation mobius transformation, it goes to z 1 and it goes to z 2 that is not use this notation. Let us call this z naught, z 1, z 2 dot, dot etcetera. Each time I make the same mobius transformation, I repeat it I iterate this transformation. Let us suppose that the original variable was the superscript 0, and after in iteration ((Refer Time: 01:22)) superscript n. Then what is the n going to become, well you can write this transformation repeatedly in each time it satisfy this. So, this immediately implies that z n minus psi 1 was the n minus psi 2 plus be K to the power n z naught minus psi 1.

So, the problem of multiplying this matrix a, b, c, d in times raising to the power n this kind of circumented by writing it in this form. And the end of the day after you do all that algebra it has to be this; this is called the normal form of the transformation this thing here is in normal form. And now you see why a transformation instead of specifying it by the co efficiency a, b, c, d with a d minus b c equal to 1, you should really specified by saying what psi 1what psi 2 and what is the multiplier, these three together take care of what the transformation is.

By the way you have some more information here lot more information, let us solve for this guy let us solve for z n, let me for a movement call this a number something of the other call it A. Then it is clear that is the n into 1 minus a minus a psi 1. So, let us write this all z n minus psi 1 equals to A z n minus A psi 2, let us says z n equal to psi, so let us write this all properly so it is z n equal to psi 1 minus A psi 2 divided by 1 minus A equal to psi 1 minus K to the n z naught minus psi 1 z naught minus psi 2 divided by 1 minus K to the power of n z naught minus psi 1 z naught minus psi 2.

So, it says after n iteration of this transformation, any arbitrary point z naught goes to z n which is given by this formula here. Now, you can see where it is going to go, as you make n very large as you make it larger and larger, depending on whether modules of K is bigger than 1 or smaller than 1 things are going to flow to different points. Suppose mod K is bigger than 1, and K remember it defending this form this is the definition of K where psi 1 psi 2 are given to you explicitly.

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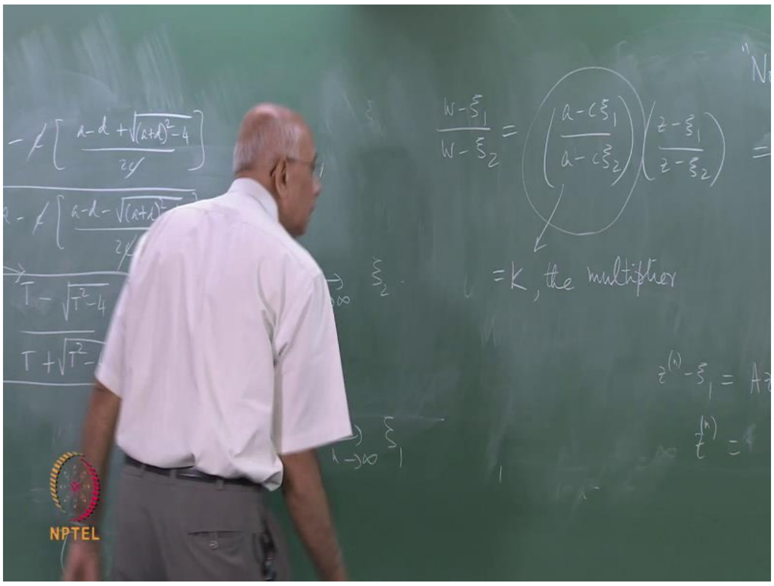

So, incidentally we should write a formula for this multiplied K, so let us do that K equal to a minus c psi 1, and what was psi 1 recall it was equal to a minus d plus square root of a plus d whole square minus 4 over 2 c divided by a minus c a minus d minus this square root a plus d square minus 4 over 2 c c (s) cancel out, and you 2 a minus a which is a tends be equal to a plus d. So you have a plus d plus the square root, which is the trace of the matrix, so let us write it as the T is a plus d. So, it is T plus square root of T square minus 4 over T, T minus over T plus square root of T square.

So, very compact expression for the multiply just determine by the trace of that matrix more than that, and just this ratio. And now once you have that in place, we can decide what is going to happen to this, so the mode K greater than 1 would imply that z n as n tends to infinity, remember this n is the iterate is the Mac of n iteration of an arbitrary point z naught. And where is it going to go, well this is going to dominate and this ratio cancels out. So, the whole thing tends as you can see there should be psi 2 I am I right yeah. So, psi 2 it says is tends to psi 2.

So, what happens is undergoes iteration, if not K is greater than 1 and mod K decided by this, then every point on the remains here with one exception is going to fall in to the points psi 2 the fix points psi 2, what is that one exception psi 1 itself psi 1 is already a fix point, so if you start with that you cannot move out of that, but everywhere else things are going to flowing to this points psi 2. So, I could calls psi 2 as a sort of attractor at attracts point and psi 1 repeller, because point in the neighborhood of psi 1are going to flow out of it in to this psi 2.

The next thing is of course to find out what sort of flow is this, and will look at the classification, but it already tells you like some interesting thing I going to go on in this, exactly like in the case of discrete dynamical system, where you have attract as an repeller stable fix points and stable fix points and so on. So, something very similar happens there, then n is like discrete time every time make rate it is like one further time step. One point if I forgot mention is that this formula here is true for negative and as well.

So, you could actually take this transmission backward the inverse transformation, so you could set n to be negative in negative number integer, and it will go backwards in time like. So, the inverse of a mobius transformation is again in inverse transformation, so it is clear that once you write it in this normal form the inverse map is with the K inverse sitting here and that you can ((Refer Time: 09:03)) So, in one short you can look at not only forward iteration, but also the inverse transmission going backwards, what happens if mod K is less than 1.

When the roles are reversed in this case z n tends as n tends to infinity plus infinity it tends to psi 1, because this term is going to get smaller and smaller, and will disappear in the limit and the answer goes to psi 1. So now, psi 1 becomes the attracter and psi 2 becomes the repeller in this case. Precisely what path is followed will depend on the point z, but there are generic cases will see that what happens really is a some kind of spiral, what happens of mod K is equal to 1.

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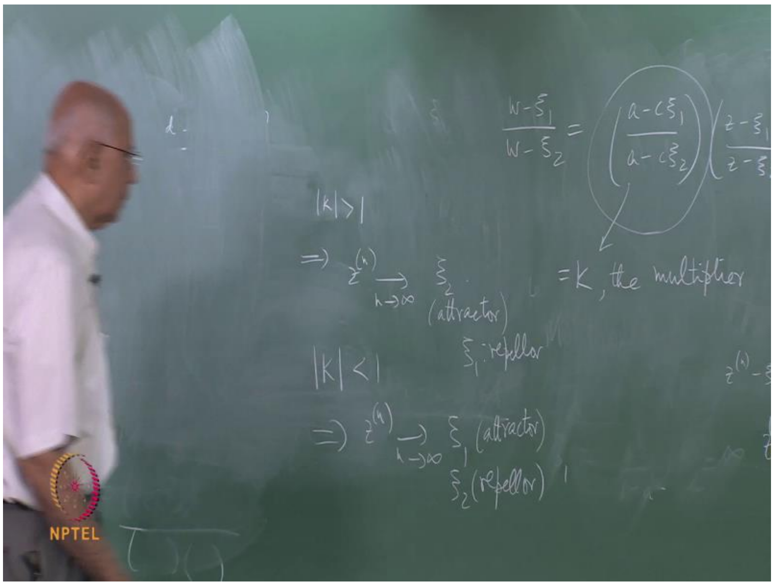

So, this one like is an attractor and psi 1 is a repeller, and here psi 1 is the attractor and psi 2 is the repeller, what happens if mod K is exactly equal to 1, then it is like an indifferent fix point it is like one of this fix point which is neither stable marginally stable marginally un stabilized as you call it. So, marginal fix point and the flow is neither ends nor out you are familiar with this kind of thing in other context, for instance if you look at the simple harmonic oscillators in dynamical system. And you look at the face trajectories their ellipse is going around the equilibrium point by that equilibrium point is a center.

So, thinks do not flow in to it or flow out of it, and the other hand I put a positive friction constant then everything will flow inverse spiral in to it, but the friction constant is go the wrong sign, then energy spam in to the system and it flow out. So, this is like that like a center if you like, where going to see what is going to happen if mod K is equal to 1. In particular could ask for happens of K is exactly equal to 1 look at that too. All this is true when you have 2 fix point psi 1 psi 2 distinct. Suppose there coincident what happens then?

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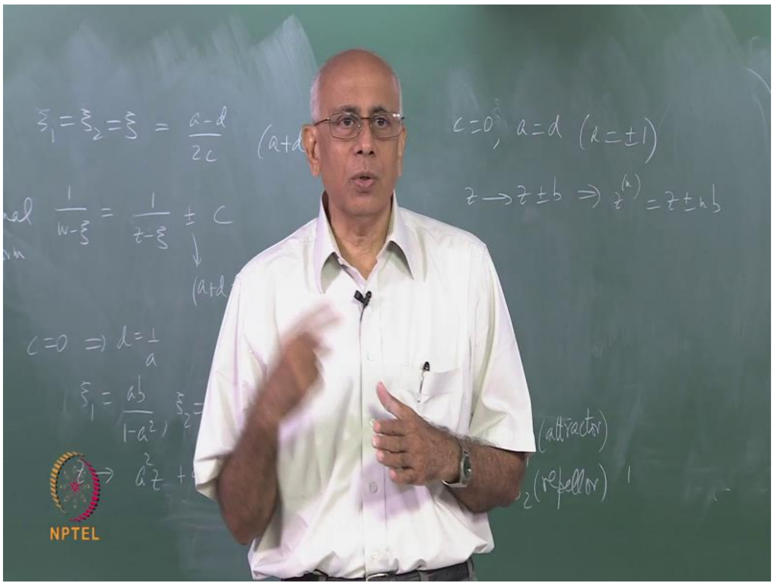

Then when you have psi 1 equal to psi 2 equal to psi this is equal to a minus d over 2 C, because now a plus d equal to plus or minus 2 in this case, then the question is what is the normal form look like in this case. It tends out the little bit of word you can show that the normal form looks like 1 over w minus psi equal to 1 over z minus psi plus or minus C. This will depend on whether a plus d equal to plus or minus 1 plus some constantly which I think is C, this is what happened when a plus d is exactly equal to a plus a plus d whole square is equal to 4.

But the fix points at some finite part of the plain, then this is the normal form, what happens one of the fix points is that infinity this is the case from C was equal to 0, so C was equal to 0 implied d equal to 1 over a. And we have 2 fix points, we had psi 1 equal to a b over 1 minus a square and psi 2 was equal to infinity in this case. And it was a linear transformation this transformation is very simple, because z goes to a squared z plus a b in this case, and you see is to see what is going to happens if you iterate this transformation over and over again, it is to going to see what the answer is going to write this down in this case.

So, what is the normal form going to be here, the still are multiplier the stiller multiplier, so the normal form here is going to be of the form w minus psi w minus psi 1equal to K times z minus psi. And this case a over d the multiply reduces is to this in this case the multiply becomes 1 unity, and the last case was just a translation z goes to z plus or minus b, then both fix points are at infinity. Then of course, we have situation z n and. So, last case was you had the c equal to 0 a equal to d, so a equal to plus or minus 1 this implied that z goes to z plus or minus b implies z n equal to z plus or minus n b this shakes by n times x.

So, those are trivial cases this 2 cases that this are linear transformations nothing which happens, but this 2 are non trivial cases this one much important of all this. So, I like you to work out and I given that an exercise what happens as in intartically to points in this cases, what is what does K do in sort, so this is kind of classifies all move this transformation, what we need to do is to find out what the flow lines look like, and then what directions to be go etcetera.

Well, the cases when things will moves along latitude in the ((Refer Time: 15:38)) in the cases when the moving spiral towards the fix points etcetera. Typically that is what is where happen things are moving in little spiral and so on. And this spiral has in interesting name things out that this spirals are called ((Refer Time: 15:54)) this is the Greek word which essentially means a curve on the sphere, which makes a constant angle with longitude lines of longitude.

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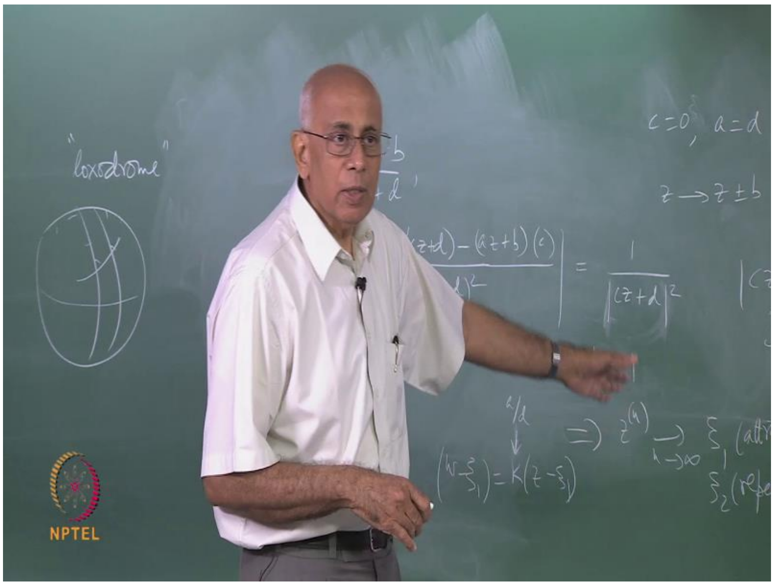

So, on this sphere we have these lines of longitude and I take a path which makes a constant angle with it, and of course it goes around in a little spiral as you can see if this angle is 90 degree is then you moving along a latitude the angle is 0 you moving along a longitude, but if you moving in between at some fixed angle k, this kind of spiral is called loxodrome. And this transformation are called loxodromic transformation this term comes from the Greek it is an navigation word, because they would try to navigate in the given direction by making a fixed angle with an art star, and that you tell as tell them where they are going.

So, and the curve that you trace on the sphere is a lots of garments since this iterates look like the points in also garments called loxodromic transformation. But otherwise you have things which look like the wants were you have mod K equal to 1 they called elliptic transformation and so on and forth. So, I classified them I write this down explicitly, but this gives you an idea how approaches this, so that crucial input is the invariance of the cross ratio that helps you get to the normal form.

Once you do that you have immediately consequence this consequence follow, the several other things which we will talk about such as the isometric circle and so on, but first I would like to appreciate one point which is the following. Since w is a z plus b over c z plus and d, it implies the d w if you differentiated is a what does it, a d z over some a over c z plus d minus next do this slowly equal to c z plus d whole square, and then you have a d minus b c 1 over c z whole square.

So, the Jacobean of this transformation the modulus of this and here is 1 over the smart square. So, this things here acts like a stretch factor or a contraction part the pending on how they get is this will tell you about d w is stretched or contracted in this function. So, where exist one curve which is major stretch nor contracted, and that is the curve let says c z plus d equal to constant equal to 1, what kind of objectives this? On that c plane what kind of objective is it.

Student: circle ((Refer Time: 19:37))

It is circle with centre at minus d over c and radius equal to 1 that circle will go to some other circle exactly maps to some other circle, but that circle is needed a stretch not contracted, what will happen to points inside that circle mod c z plus d is less than 1. So, 1 over that is greater than 1 which means points inside and map on to the terminal plain in it is stretch fashion, and points outside are contracted. Whereas, this itself does not change and this is called the isometric circle, what this mapped into...

Well, we know that circle have to be mapped into circles, so it is mapped into some other circle and the w plan, what is it mapped into I wanted to show that is mapped into the isometric circle of the inverse transformation. Remember there is in inverse transformation here which tells you z is equal to d w minus d over minus c w plus a, and that is got an isometric circle which is minus c w plus a mod equal to 1. So, I wanted to show that this circle is mapped into that circle. We will come back and look at the other property of this isometric chain. And now, let me also mention here the group theoretic connection, and tell you where this connection comes from because it is very, very interesting.

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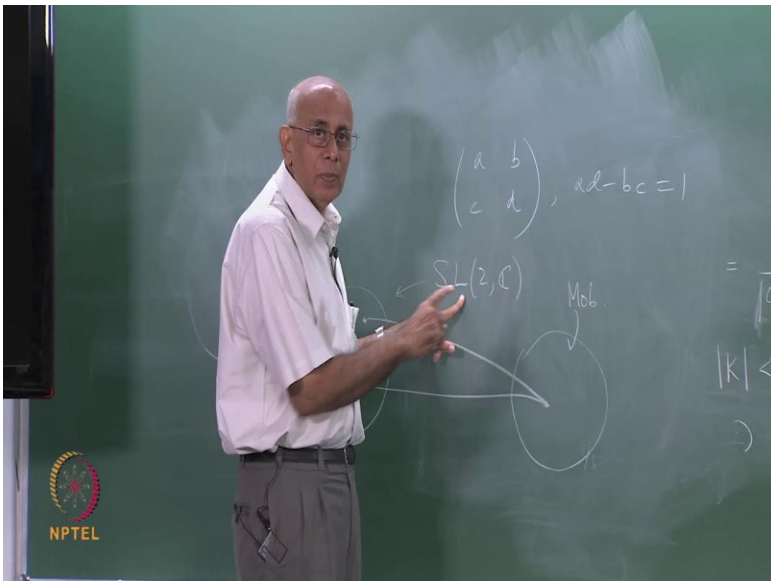

Look at the set of matrixes a, b, c, d and matrix multiplication 4 complex number, so these are 2 by 2 matrixes, which satisfy a d minus b c equal to 1. So, all matrixes a, b, c, d on the complex element which satisfy a d minus b c equal to 1they form a group, because all of them have inverses they are non singular matrixes, and the product of any to such matrixes again such a matrix, because determinate of a b is determinate determine b, whether or not aim become mute. So, they form a group and this group is called special linear group into on the complex numbers S L 2 C, now do you give that if you give me this 4 numbers a, b, c, d 4 complex numbers with determinate equal to 1have specified a mobius transformation for you avoid on that.

So, can I say that for every 2 by 2 matrix determinant 1 there is exists a mobius transformation can I say that, what happened if I change the sign all 4 numbers a, b, c, d they make it changes, because minus itself what happens to the transformation it remains the same. So, this means that there is no 1 to 1 correspondence between these 2 groups, but there is a 2 to 1 correspondence between 2 groups. So, in the space of these matrixes this is S L 2 C here are all the matrixes in S L 2 C every point here is make the S 2 by 2 matrix etcetera.

And then here is the set of mobius transformations, this is mobius transformations on the complex numbers in general, because we later going to look at mobius transformation, we just real coefficients, but the corresponding between this and this is not 1 to 1 is two of this guy go to one of this. So, this 2 to 1 homomorphism between these two sets between these two groups. This thing here up to a sign is isomorphic figure. And the way you say it is up to the sign is by introducing, what is called the quotient group or modulo a group which consists of just two elements.

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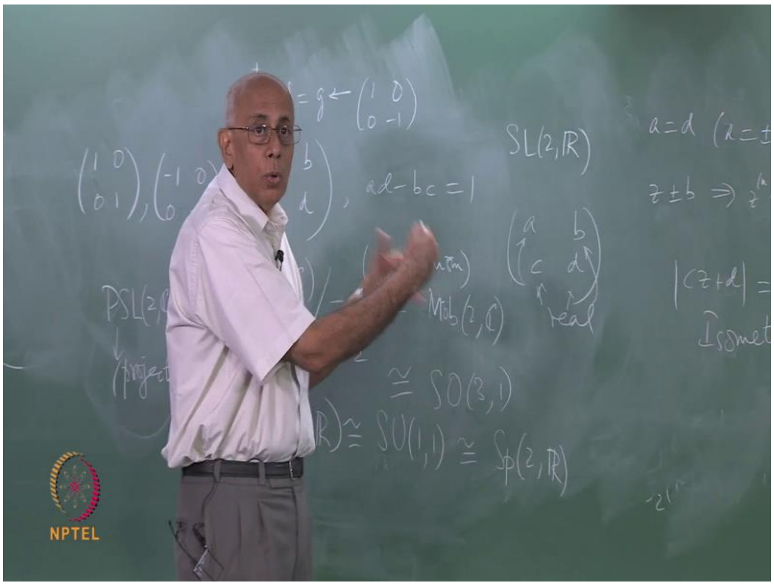

And that group those elements will be 1 0 0 1 and minus 1 0 0 minus 1 these to form a group among themselves, as you can does not identity element square of it is 1the square of this is that, this time that is equals to this that times this is equal to this. Let say gets a cyclic group of order 2, it is isomorphic to binary set of a integer and binary issue. So, addition modular 2 put all the even numbers in one basket all the odd number in one basket, call that one element and all the even numbers one element even plus even is even, even plus odd is equal to odd plus even equal to odd, and odd plus odd is even.

That group is a slightly group of order 2, and it is written as z 2. So, up to that sign this is isomorphic and this symbol values isomorphism, that is isomorphic to the mobius 2 C. This group of transformations this is same as this group a, and this is got another name this is also called the projective group and it is written as P S L 2 C, this P stands for projected both the both an element and it is minus are projected on to the same object. So, there is a group property and of course if you know the group property of this, when we know a lot about the group properties of this set of mobius transformations.

The very intriguing connections it turns out that for every group, and I will mention little bit later I will talk about parameter space of these groups. A group like this it is parameters space is simply connected, and other words if you look at the space of it is parameters that is some continues space. And every point on it you can go every other point in an ((Refer Time: 26:45)) wise fashion, and you can every close loop on it can be shrunk to point without living the space and that is sort of thing is called the simply connected loop. So, this group is call the universal covering group of this group here, it is simply connected Lie group.

And this kind of connection this form of morphism in variably leads to very interesting consequences for this group here this one is easier to analyze, because space is simply connected, but the space of this is not simply connected becomes little more complicated. So, it is easy to look at the covering group find out what it is property are in use this back portion property in order to deduce properties group. In this particular case this group S L 2 C also happens to be the covering group for another very interesting physical group namely, as you know in 3 plus 1 dimensional space time.

We can make Laurence transformations, we make Laurence transformations of two kinds there could be in homogenous and homogenous once, the in homogenous once means you would also shift the origin of space and time, the homogenous once means you live the origin of space time unchanged. And either micro rotation or boost to other frames moving with constant velocity. And then among them you can ask other rotation proper rotations or improper rotation namely, is left handed coordinate going system go to remain left handed or is it going to change right handed coordinate system.

So, if you do not include parity and things like that are reflections, then you have a group of proper Laurence transformations homogenous Laurence transformations. Among them you could ask or you could changing the sign of time or not, if you do not it is called orthogonal. So, the future remains the future to both frames, in past remains the past without getting in to change. So, the group of homogenous proper orthogonal Laurence transformations that is denoted by S O 3 comma 1, special orthogonal group in 3 plus 1 dimensions, this special orthogonal means that group is not end times end transpose equal to identity.

But there is a metric sitting here, which says that the time the time component and the space component will have different science in the metric, I will come back to this and talk about metric. It turns out that this group and these groups are isomorphic to each other. So, there is very deep connection between something totally different, you have set of mobius transformation Raymond's Fay that turns out to be exactly the same group as the group of Laurence transformation in 3 plus 1 dimensional space time. There all this intriguing connections, one has and it is exploited this kind of thing exploited both of them have this same universal covering group.

Similarly, I just mention this later will come back to this, if you restrict yourself to real coefficients a, b, c, d, so I put a, b, c, d are real all parameter are real. They two form a group among themselves multiplies two such matrices in the unit modular, they two remain in the same group. This group is especially in a group S L 2 R on the ribs, once again it turns out that S L 2 R quotient to at z 2 is a set of mobius transformations with real numbers are alone real a, b, c, d they two form a group among themselves, subgroup of the more general mobius group.

And again you have this same thing S L 2 R quotient at z 2 is in fact, mobius to are, they could ask what is S L 2 R itself, what an earth is that isomorphic to if anything. It turns out to be isomorphic to this special unitary group, so let me this write this write prove this later S L 2 R is isomorphic to the special unitary group1 comma 1 pseudo unitary group S U 1 comma 1. So, you write on all 2 by 2 matrices, let us see with complex centric which are not unitary, but which satisfy U dagger g U equal to g and this g is the matrix 1 0 0 minus 1, they call pseudo unitary.

It is a pseudo unitary group, and this group is isomorphic to this. It also happens to be isomorphic to the group of canonical transformations in Hamiltonian mechanics of systems of one degree of freedom, and that is called the simplistic group, so these all are so isomorphic 2 1of R. So, at the mathematical level there are all these very, very interesting connections one and the other.

Student: ((Refer Time: 31:54))

Pardon me.

Student: ((Refer Time: 31:56))

No, no, no, no this is the isomorphism, but the mob have so if I quotient this z to then I get mobius 2 R. So, that is the connection will write down what is the most general metric will see explicitly where this comes about write down mobius transformation. And look at what are these, what does this group what kind of matrices ((Refer Time: 32:21)). So, in the process will answer questions like, what is the most general 2 by 2 unitary matrix etcetera, what is the parameter space? And what is the most general pseudo unitary matrix of this kind. So, there are this by the way has got many other intriguing connections, it is also what is call the little group for this special group of special Laurence transformation, and 2 plus 1 dimensions once again.

So, there are all kinds of at the group periodic level where lots intriguing connections between apparently very, very different things together. Incidentally, this connection here is used very much in optics, this connection here that is called Lie optics field of study, which again points to very, very intriguing mathematical connections. You will touch up on some of these things, but what I am do next is to classify is look at the flow mobius transformation, what do this flow lines look like. And then come back in point out, where these group isomorphisms come from.