Selected Topics in Mathematical Physics
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Module - 5
Lecture - 11

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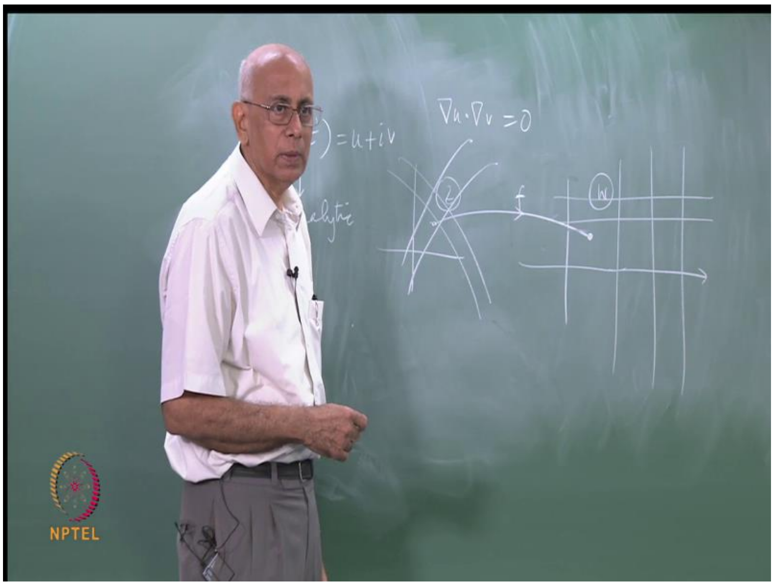

We can regard every analytic function as a map of the complex plane to the complex plane always regard by call w equal to f of z, and this is analytic, then what is happening is that every time you give me a value of z in the z plane you give me a value here you give me a value in the complex w plane. So, this map is f.

Now, the think about analytic functions is that this map has very specific properties in particular just as the the lines of constant imaginary z, and constant real z are right at cut each other at rectangular is the rectangular grid here in the z plane similarly the lines of constant u, and constant v, if I call this u could equal to u plus I v recall that, because of the coshiriman conditions one of the a criteria one of the consequences of that was that grad u dot grad v was equal to zero what that meant was that in the w plane if these are the lines for which the real part is constant namely u is constant these are the that v is constant in this side ok.

Plot it in the z plane they going to look like this, and something like this. So, these curve lines are getting map down to these rectangular grid here. Now these two will satisfy the property of confirmality namely that the angle between these is going to be a trite angles all the time, and this is the special case what is called conformal map in which says that angles are preserved between curves when you do this map in....

Now, you could off course say alright this is going to work for the entire plane well is the map invertible is the first question, suppose I go from z to z squad for every value of z there is a z squad, but for every value of z squad there is no unique z, because minus z would give you the same point.

So, the question is what are the one to one maps what is the most general one to one map that you can think of to take u from z to w it should be invertible, and it should be one to one it; obviously, cant involve anything higher than any higher power of z we cannot involve z square z cube etcetera it present nothing. So, the only thing it can do is be a linier map. So, z goes to a z plus b, and that is invertible completely.

But it is a sort of trivial map does not do anything interesting it is an extremely trivial map it means only linier functions nothing more than that. So, this question is a trivial answer if I say the z plane what is the most general one to one map see trivial can, and the reason is that the point z equal to infinity is map to the point w a equal to infinity a say w say z plus b if z is infinite w is also infinite. So, you kept that as a fix point now I say no no no I do not want to do that I have the remands sphere the extend that complex plane I have the remands sphere, and that is map plane analytic function to the remands sphere to another remands sphere, and I ask what is the most general one to one map from this sphere to that is sphere, and I do not say that the point at infinite a has to remain the same this set of points this sphere on this sphere should be mapped in a one to one fashion invertible fashion on to the set of points in the others function, and I ask what is the most general map in this case when the answer becomes very nontrivial, and turns out that the most general such map.

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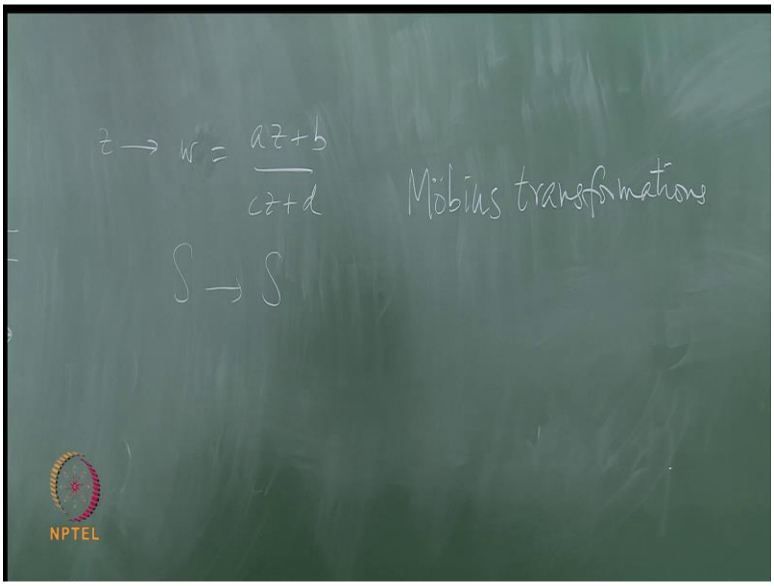

They make this very formal is a z goes to w equal to a z plus b over c z plus d were a b c d are obituary complex numbers, and in order to make sure that the map does not become regenerate you require that the determinant not be zero if the determinant is zero can this becomes a trivial map as real c.

So, this is the most general map of the remands sphere s to the remands sphere s which is analytic, and invertible, and this kind of map has a very large number of names one of which is linear fractional transformation sometimes we called fractional linear transformation, but we going to use for this the term mobius transformation.

So, in the next couple of lecture we will talk about the properties of mobius transformation once we finish that, then we will come back, and look at multi valuate functions. So, this means that we are still in the rime of single valued functions, but the fact is that now we got very nontrivial situation here, because this is not a linear map it should be immediately clear to you that if I multiply a b c d by is the same constant number, then in this ratio that cancels out, and the point to which z is mapped does not change at all.

Therefore its immediately clear that I can take these a b c d, and normalize them in some suitable fashion, and I will normalize it in such a way that ad minus b c equal to one I can always do this if a d minus b c is not equal to one I will divide each these coefficients by that number. So, that the determinate become small.

Now, this immediately should suggest to you that there is a kind of group property involved here, and we going to see that in grade detail you see you can write is a b c d as a metrics two by two metrics, and off course when you multiply two two by two metrics is you get another two by two metrics. So, all nonsingular metrics is well form a group. So, it is clear that the transformations also provided they also operate in the same way will also form a group, and we will see that the connection there is a deep connection between this group of two by two is, and the group of mobius transformation.

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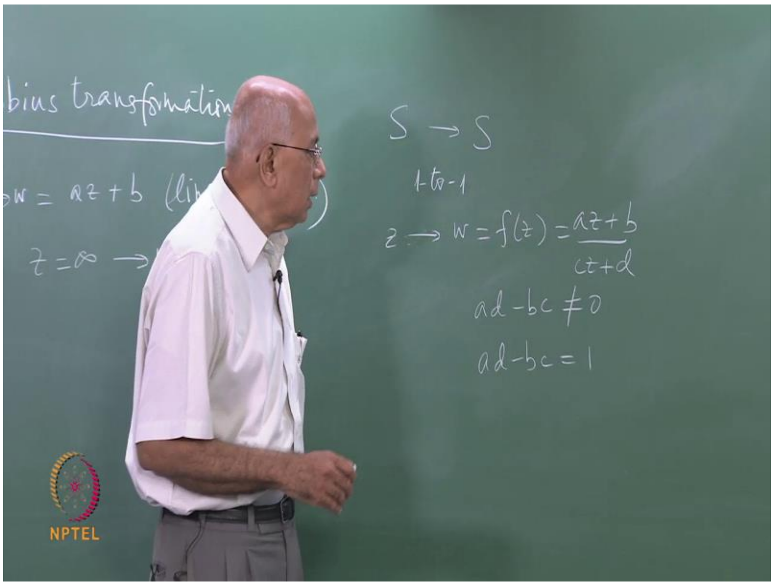

Lets now take up the study of the set of transformations call mobius transformations, and these transformations are defined as follows as I started mentioning the last time if you ask whets the most general one to one map of the complex plane to itself since it has to be one to one it has to be invertible there has to be there has to goes to unit w, and vice versa the only thing that you can do is to say a z goes to w equal to some a z plus b that is a linear transformation. So, sort of trivial linear transformation, but you see this is not very interesting, because we know all its properties immediately, and the other point is that is leaves the point at infinity unchanged. So, it is a z equal to infinity maps to w equal to w equal to infinity in this case.

So, it is as if on the remands sphere you have insisted that the point at infinity be unchanged, and then ask towards the most general one to one transformation between the remands sphere of z, and the remands sphere of w, but if I relax that condition, and simply say that the remands sphere has to be map to the remands sphere you know one to one fashion, and ask what is the most general possibility, then this is not the most general possibility, but a very interesting thing happens, and one can show that the most general map of the remands sphere s to the remands sphere s one to one the most general possibility is z equal to z goes to w is this a function of z which is of the form a z plus b over c z plus d there a b c near obituary complex numbers.

That turns out to be the most general map of the remands sphere to itself which is invertible one to one were a b c d are obituary complex numbers with just one provisor, and that provisor is that the determinate ad minus b c cannot be equal to zero, because it is easy to see that a from d there is d minus b c should turn out to be zero, then f of z is the constant for all z, and that is not a very interesting map all the points in the z plane map on to constant in that is they hm.

So, we will a d minus b c to be non zero in general and. In fact, you can go further you can always divide each these coefficients by the same number without changing the map, and you can divide by square root of ad minus b c whatever complex number that is, and once you do that, then the new new coefficients will satisfy ad minus b c equal to one. So, we will in general say ad minus b c equal to one always hm.

So, now the statement is that this most general map call the mobius transformation is a map from z to w which is defined by this ratio here such that these coefficients are obituary complex numbers satisfying this conditions a d minus b c equal to one, and that is what a mobius transformation is in the most general case now these transformations have a very large number of very very interesting properties a they will form a group a certain group the group itself will be to other groups and. So, on and. So, for, and will try to classify all possible transformations of this kind.

So, task is non trivial, but the first thing you have to note is what happens to the point at infinity, but it is clear that z equal to minus d over c is maps to w equal to infinite. So, this point z equal to minus d over c goes to w equal to infinite that is immediately clear um.

On the other hand this also implies, and is implied by the that z equal to d w minus b over minus c w plus a that is as the transformation. So, from z I go to w, but I invert this, and I get d w minus b over minus c w plus a I could write this down, because I am thinking of it always as a metrics a b c d with determinant one, and the inverse of that two by two metrics is just d minus b minus c notice that ad minus b c still equal to one in this case. So, the the transformation has determinate one the inverse transformation also is a mobius transformation has determinate one.

Now, what we need to show in further to show that these form a group is that every transformation as a inversed we need to show that the product of to transformations is again a mobius transformation of the same kind, and that is very reasonable to expect that that we will indeed be the case, because one make a further transformation here is clear that you are going to get something very similar, but just the metric is multiplied a b c d multiplied, but we will see how this comes up out.

By incidentally that also tells you that the point z equal to infinity is mapped on to w equal to a over c. So, in general the point at infinity in the original remands sphere is mapped to some other point, and vice versa some other point the point at infinity there is mapped to something else in ok.

Now, how do we go about analyzing all possible such transformations well the first important concept is that of a fix point, then we will be look a transformations one should always ask what points if any are left unchanged by this transformations, and then you work around those points just as in dynamical systems, and we will see that there is a very close connection between this, and about happens in distressed dynamical systems.

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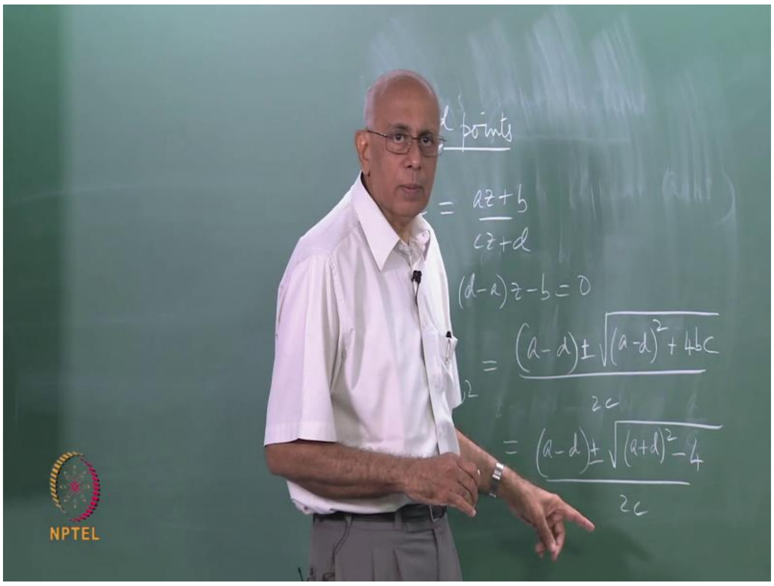

So, let us ask what are the fix points of this transformation these are those points which do not change, and other transformation. So, they must be solutions of this equation z equal to a z plus b over c z plus d or a c z square plus d minus a z minus b equal to zero.

So, let us call the roots zye one coma two is these roots are at a minus b plus or minus square root of a minus b whole square a plus four b c over twice c, but remember that you said ad minus b c equal to one. So, I could also write this as a minus d plus or minus square root of I make this an a plus d. So, that a plus d whole square minus four ad minus four ad plus four b c. So, that becomes minus four divided by two c these are the fix points of the transformation

So, it is clear immediately that at best you can have two fix points for any jungle transformation, and they given by zye one, and zye two now the reason I wrote it as a plus b is, because I am going to use this fact this a plus d is the trace of a metrics if I write this as a b c d. So, the something invariant going on here, and I let to write it in in a term in terms of this trace ok

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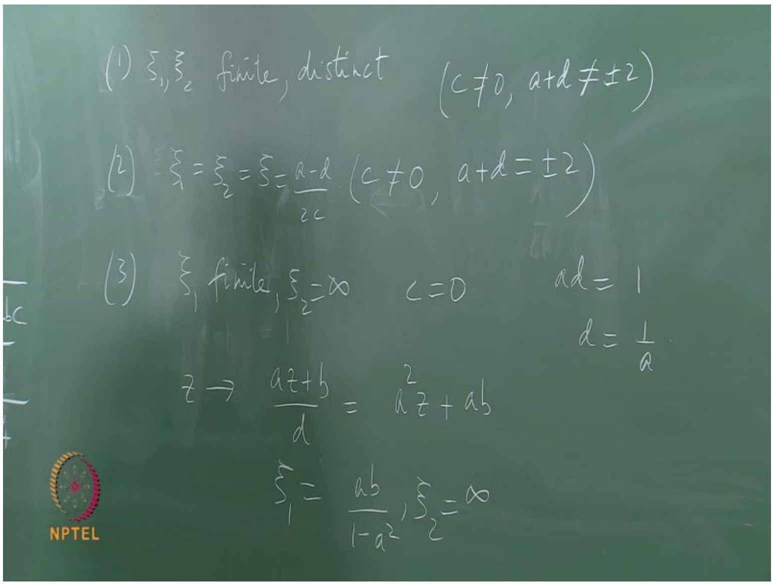

So, now the following possibilities occur the first possibility is that zye one zye two or finite, and distinct that would mean that c is not zero there are finite points, and a plus b is not equal to plus or minus two that is the most general case. So, c not equal to zero a plus d not equal to plus or minus two two distinct fix points some around the remands psi.

By the second case corresponds to zye one a finite zye two equal to well lets if the second case would be when that coincident. So, let say zye one equal to zye two equal to zye which happens c not equal to zero a plus d equal to plus or minus two, and this incidentally is equal to a minus d over two c.

So, when this thing vanishes the two roots coincident that the same point, and there is only one fix point here, and it is finite since c is not zero. Third will happen when zye one is finite. So, if c is zero zye one finite zye two equal to infinity. So, this would correspond to c equal to zero, and then you can see immediately that the fix point the two fix points, and remember that you must have now ad equal to one, because b c is zero since c is zero. So, this will imply that d equal to one over a immediately, and what happens to the transformation it says z goes to a z plus b over d which is equal to a over d z, but a d is equal to one. So, this can be written as a square zye plus a b, and this has two fix points off course. So, you have zye equal to if I bring this a square here its equal to a b over one minus a square zye one equal to this, and zye two by inspection is infinitely, because if I said z equal to infinity here that is infinite here to. So, z equal to infinity is a fix point, but there is another nontrivial fix point finite fix point which at a b over one minus a square. So, that is the third case, and then the fourth, and final case is that a completely trivial.

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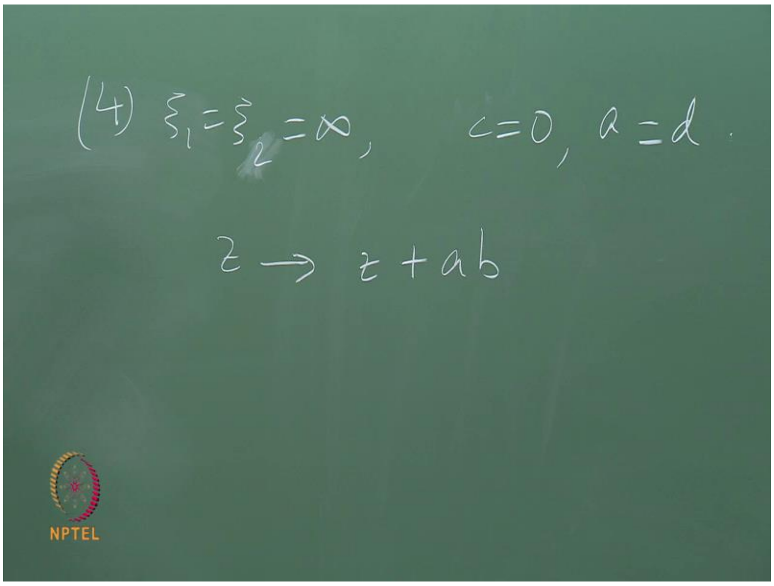

So, zye one equal to zye two equal to infinity in this case which will happen when c equal to zero, and a equal to. So, if you said a equal to d, and put c equal to zero, then you get just a translation. So, this implies z goes to a z plus b a over d that zye plus b over d that, and now remember that you must also have ad equal to one. So, it says a squared equal to one. So, this means z goes to z plus or minus b just a translation trivial translation these are trivial cases, but its important to have the in place. So, we have four possible kinds of transformations depending on the number of fix points, and were they.

In the case when c is not zero you have two possibilities either you have two finite points which are fix points which are distinct or the two point are finite, but coincide each other in both cases the transformation is nontrivial you still get a linear fractional transformation, but the movement you put c equal to zero this becomes a linear transformation immediately, and in that linear transformation there are two possibilities one of them has a non a finite fix point, and then another one at infinity.

And the last one is when the thing is completely degenerate you just have a shift, and nothing more than that. So, these are the four a to start with these are the four kinds of mobius transformations you could have but you must remember all the time that the most general case is this this is the most general case, and this is the special case slightly these two are is this are just linear cases that subsumed in the general transformations ok.

So, could, but the next question to ask is how do we classify these things I mean what is the flow what does it look like how do these points change and. So, on and. So, for for that it turns out to be very useful to introduce a concept corded cross ratio a four points.

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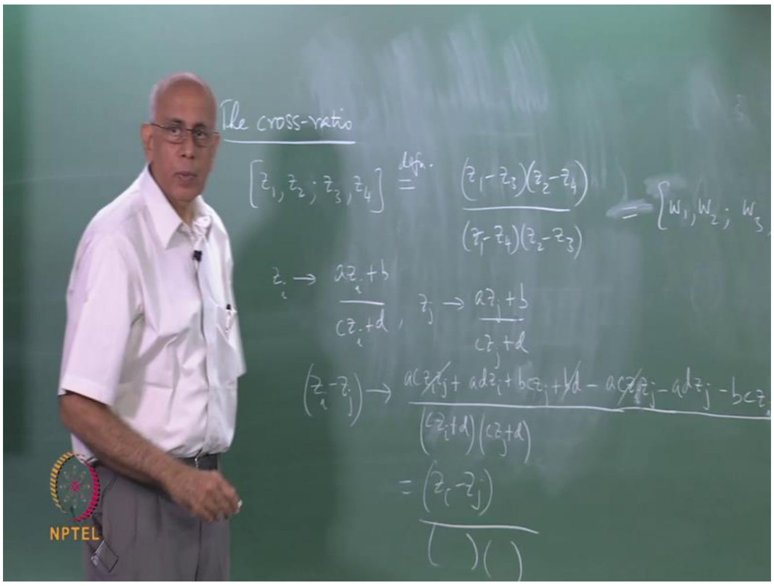

So, let us write that down cross ratio. So, if you give me any four any four points on the z plane or in the remands sphere a z one z two z three z four I define the cross ratio of these four points by this this definition define z one minus z three z two minus z four divided by z one minus z four z two minus z three. So, this minus that time this minus this divided by this minus that, and this minus this that is defined as a cross ratio of these four points, and those of you what finally, with project a geometry will recognize at this cross ratio is fundamental concept in project a geometry, and places a very very important role in that entire subject.

Not this cross ratio have some very beautiful points beautiful futures which we go into analyze, and look at, but before that I would like you to verify that the product to mobius transformations is yet another mobius transformations. So, the actually form of group it is a trivial came to proof, and I am going to assume that I going to do this, and check out that is so.

We have already seen that there is an inverse for each transformation. So, now, the important thing about the cross ratio is the following under a mobius transformation this ratio does not change at all. So, it remains invariant that is not hard to see, because suppose z I goes to a z I plus b over c z I plus d, and z j goes to a z I plus b over c z plus d this will imply that this difference z I minus z j goes to I lets put the product here c z I plus d c z j plus d, and then its this times that minus this times this, and you can see that the z I z j term cancels, and you have ad times z z I minus z j, and b a. So, let us try to solve a c z I z j plus ad z I plus b d a plus b c sorry b c z j plus b d minus the same thing which is a c z j a z I z j minus ac minus ad z j minus b c z I minus b d, and this b d cancels and what are we left with this stuff also cancels here, and you have ad minus b c times z I minus z j, but ad minus b c is one. So, this immediately tells you the magic trick this is just z I minus z j divided by this denominator.

And now it is easy to see that if I take I j k l four points, then the cross ratio will be unchanged, because this denominators will cancel. So, this implies that this quantity goes to is equal to w is equal to w one w two w three w four.

If z I goes to w I were I is one two three four, and other transformation, then this cross ratio does not change at all its invariant. So, that is the basic property you will going to exploit, and everything, because this is a very crucial thing this property of mobius transformation which you make like for very easy now ok.

Now, the cross ratio itself a four points has a lot of interesting properties a lot of cemeteries first up all in choose four obituary points, and you are asking for a cross ratio, but you can define a large number of cross ratios you can define twenty four of them four, because you can write this one two three four in any order you like, and therefore, from a set of four points you can have twenty four cross ratios, but all of them are related to each other in there is only one independent cross ratio, because you can see the following cemetery which are easily established.

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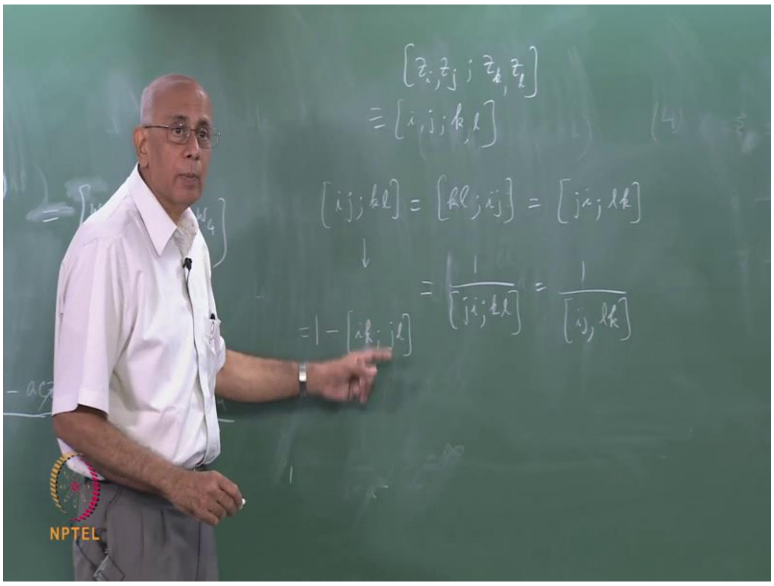

Let me just for shorthand let us call a let me write it as I j k l were this times for z I z j z k z l, then the following cemetery properties immediately hobbies a immediately established I j k l is equal to if you flit these to k l I j its equal does not change at all its also equal to a j I l k if y u flit these two, and flit these two does not change at all it is also equal to if you flit just to one part. So, this is equal to one over j I k l equal to one over I j l k.

So, there are lots of such relations a its also equal to this is also equal to one minus cross ratio I j I k jl, and then you can use all this cemetery properties this on this, and derive all the relations a that that exist between any four points in cross ratio of any four points. So, it is a very very a its a quantity with a lot of cemetery properties, and the beautiful thing is its invariant, and a mobius transformations.

Now, the statement is a I should also mention one more thing suppose one of this points is infinity, then what happens to the cross ratio. So, let us suppose for instance that z one is a infinite does not matter any of these points could be infinity.

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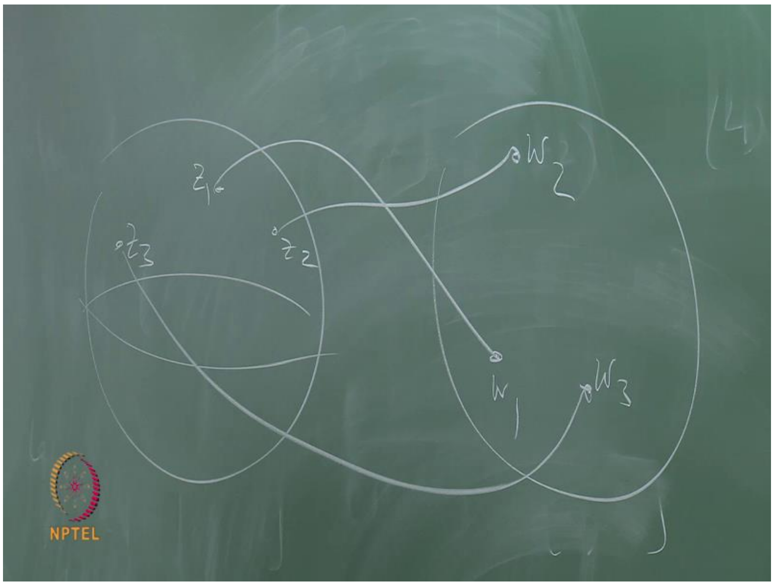

So, infinity z two z three z four is defined is a limit of this as, and one goes to infinity. So, it is just z two minus z four over z two minus z three all you have to do a limit z z one goes to infinity, and that is it. So, I treat infinity on par with everything else, because you have a point, and that is the definition if one of the points is that infinity ok.

Now, a this in place its now possible to look at what a mobius transformation looks like in terms of the fix points, because there something very special going on to these fix point under this flow a under this map a fix points alone remain unchanged. So, you would like to ask what happens to points in the neighborhood of this fix points what kind of to they go to…

Well a let me let me point out one thing first that if you give me if you give me any three points on the complex planes is a z one z two z three or something like that, and you get map to w one w two w thee, then there is unique transformation which takes you from the first set of points to the second set of points another words if on their remands sphere you specify for me a z one a z two, and a z three, and you tell me that these points should be mapped to w one w two, and some w three, and you specify these, then my assertion is there exist the unique mobius transformation which will do this map in one to the other, and the way, and the way to prove this is to construct this transformation this explicitly construct it, and you construct it by considering the the following cross ratio z z one z two z three, and I know that this is equal to w w one w two w three I know this has to be truth, and a this transformation.

And now just write this out, and that is it, and solve for z a for w in terms of z, and you know that its a mobius transformation, and its unique. So, there is a unique transformation is takes any three pre prescribes at a points here any specified at a points to any other three specified set of points is on the one, and only one mobius transformation which takes you from one to the other.

Now, you know that once you specify three points on the complex plane you also specified a circle. So, this is a way of showing that the mobius transformation map circle is on to circles. So, mobius transformation on the remands sphere now let us go back, and look at what are transformation does a in term of the fix point. So, lets consider as following I will do this for the general case, and then we will look at what happens in the special case. So, look at zye one zye two distinct, and finite, and remember that zye one two was equal to a minus d plus or minus square root of a plus d whole square minus four divided by twice c were I use the fact of ad minus b c is one.

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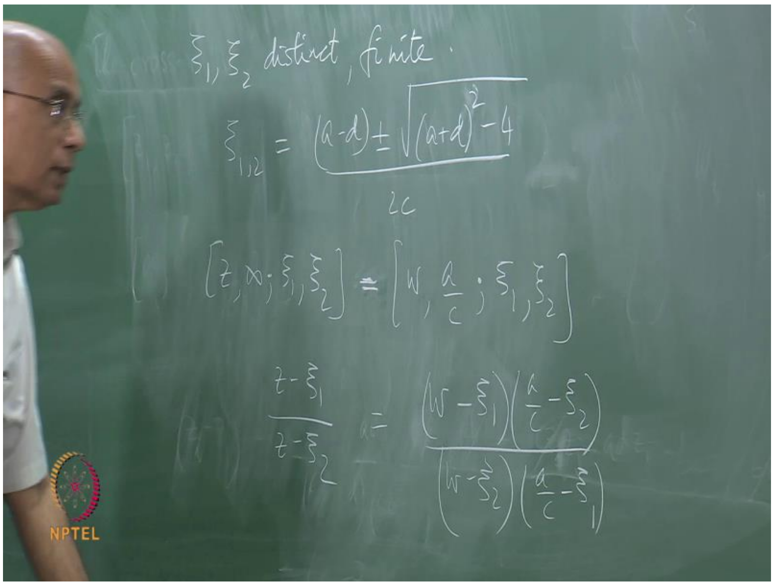

Now, consider this consider a z infinity zye one zye two, and I make a mobius transformation, and what is this go to well z goes to w what is infinity go to a over c. So, this goes to a over c, and zye one, and zye two go to themselves their fix points right. So, this, and the cross ratios in variant, and a this transformation. So, this must be equal to that were z is an obituary point, and its map is w. So, this must be true which immediately tells me that z minus zye one divided by z minus zye two. The infinity minus zye two is part cancels out this is equal to w minus zye one times a over c minus zye two divided by w minus zye two a over c minus this must be true.

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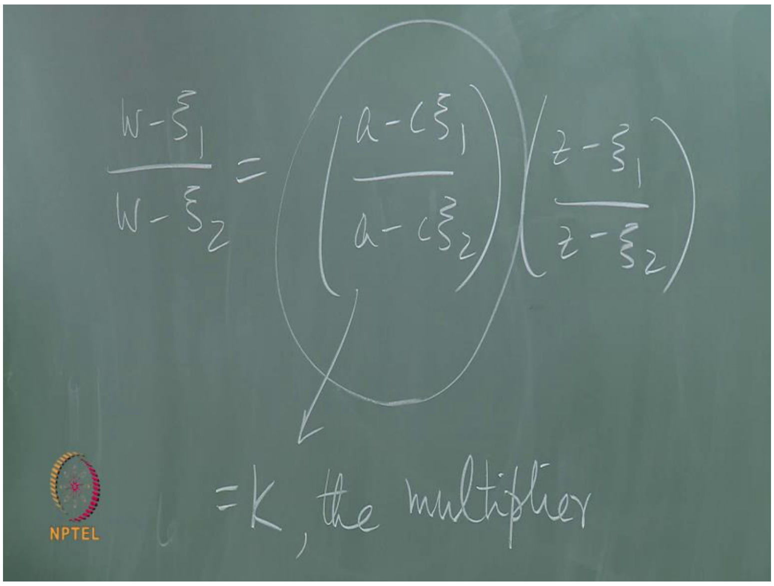

So, it has therefore, says that w minus zye one over w minus zye two is equal to a a minus zye one over a minus c zye two multiplied by the same ratio. So, its rewritten this project this mobius transformation in a very suggestive form it says under this transformation this ratio the distance between z on zye one, and z on zye two the ratio of these two numbers is equal to the new ratio is equal to the old ratio multiplied by certain quantity or certain complex number in general which is the function of the coefficients a b c d of the transformation. So, this number is a symbol for it its equal to k, and its call the multiplier .