**Selected Topics in Mathematical Physics**
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**Module - 4**
**Lecture - 10**
**Analytic continuation, and the gamma function**

So today what I would like to do is to bring out a lot of connection. So, as between the gamma function, and Gaussian integrals, then I would like to talk about the mittag leffler representation for the gamma function which I mentioned briefly last time after that I would like to talk about the log derivative of the gamma function which to which to I mention last time derive few of thereof it is properties, and then give a very important couple of identities satisfied by the gamma function, which are extremely useful one of them is call the reflection formula, and the other is call the doubling formula very useful things, and after that I would like to introduce the beta function, and derive the connection between the gamma function, and the beta function, and we take it from there.

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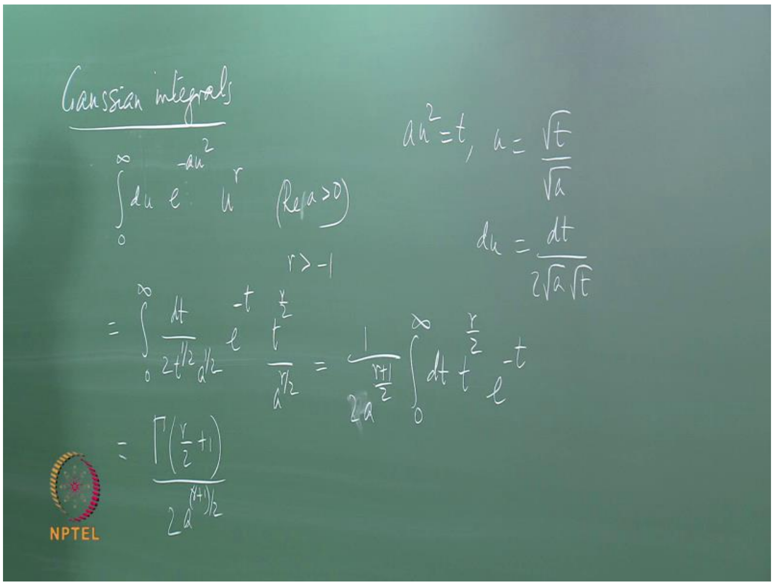


So, the first item is Gaussian integrals, and as all of you know general Gaussian integral the familiar one one they are most familiar with is this integral zero to infinity d u e to the minus a u square u to some power r say this integrals going to converge as long as real a is positive. So, that this provides a damping factor it infinity**,** and one should also ensure that this does not provide a divergence at the origin. So, you also require that r b greater than minus one. So, that it does not diverge at either, and that is the general Gaussian integral.

Now what is the way to do this well substitute put a u square equal to t or u equal to root of t over root of a, and then d u is d t over two root of a root of t. So, the integral becomes equal to zero to infinity d t over two t will the half is an a to the half here e to the minus t, and then a u which is t to the power r over two over a to the power r over two. So, that is one over two a to the power r plus one over two, and integral zero to infinity d t t to the r over two e to the minus t in that of course, is a gamma function, the original representation for a gamma function.

So, it immediately says that this integral equal to gamma function of r over two plus one over two a to the r plus one over two. So, try to way provides you with in expression for this general integral in terms of gamma functions a one can generalizes this further begin for a linear term here etcetera etcetera, and then complete squares and. So, on if it is minus infinity to infinity, then you can translate you shift the shift the integrals still remained by any constant amount, but if will zero to infinity you will end up getting error functions of various kinds. So, we want go to that right now, but I just wanted to point out that this integral is do or right a brilliant terms of gamma functions here in particular if you set r equal to zero this thing goes away, and then you have e to the minus a u square which is gamma of one divided by two, and then there is a to the power of half all clear they should be some they should be some, is it gamma for plus one by two into the r minus one plus two into the half t to the power here is it?

Yer sir.

Yes, that is right yeah, because otherwise in becomes independent of the wondering where this went yeah t to the power of r over two minus a half thank you.

And then what is a integral become gamma of r minus one over two.

R plus one over two r plus one over two yeah, thank you.

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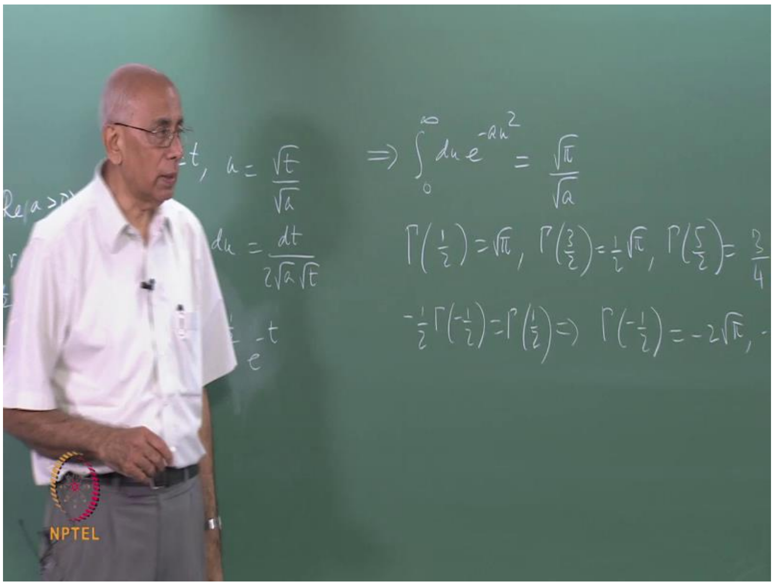


So, this immediately implies for instance that integral zero to infinity d u e to the minus a u square equal to gamma of a half which is square root of pi over square root of a now this integral. Of course, you can do by other tricks you can do this by squaring this this is the famous trick that pass on discovered way back in the early part of the night in century you square this in order find I squared go to polar coordinates, and then you come back, and you discover this integrally square root of proportional.

To square root of pi, and then scale factor one over routine now the immediate consequence of this is that you can start writing what the gamma function is for half odd integers. So, gamma of half equal to root pi gamma of three half is half gamma of half which is equal to half root pi gamma of five half’s equal to etcetera etcetera there is a three half. So, it is three quarters root pi and so on. So, all the half odd integers you can write down in a close form in terms of some factorials, and then finding as a square root of pi factor, and you can go backwards in the other direction also. So, for example, we have minus half gamma of minus half equal to gamma of half which of course, implies the gamma of minus half equal to minus two root pi, and so on, on the other side. So, you have the possibility of writing gamma functions of half odd integers in terms of some simple numbers etcetera not. So, easy to do it for other values of rational values of the argument, but certainly for half odd integers is the very useful formula this is this appears all time very familiar.

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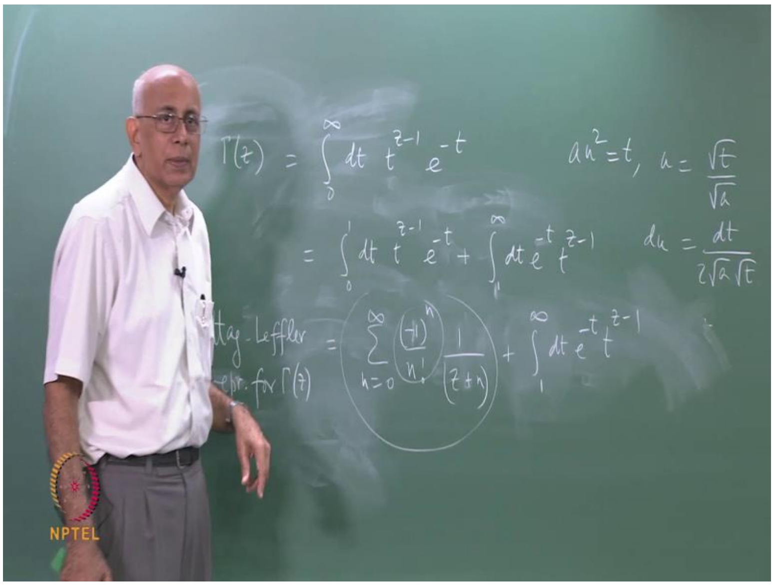

So, now let us go away to the mittag leffler representation for the gamma function now what is it we know about this function we know that gamma of z is a meromorphic function it has simple poles at all non positive integers, and an z equal to minus n the residue is minus one to the n over n factorial. So, what we need is the formula which writes down the gamma function in terms of a some overall the these poles with a corresponding residues plus a piece that is an entire function hm.

And now if you go back, and ask what is this what is this the singularity due to eventually it arises, because you had t to the power z minus one e to the minus t it arose from here from this factor here write. So, all we have to do is to write this as an integral from zero to one d t t to the z minus one e to the minus t plus an integral from one to infinity in this fashion, and do this integral explicitly we do this by integrating by parts for example, and then what you get the first term what you get here in this integral if you work this out.

So, integrate this, and differentiate the other part etcetera, and eventually you are going to get once I write this the next step is to say, let us expand this e to the minus t in powers of t. So, this we comes equal to summation n equal to zero to infinity one over n factorial, and then I do t to the z minus one e to the minus t which have expanded as t to the n over n factorial. So, there is a minus one to the power n here, and then integral from zero to one d t t to the power n plus z minus one which of course, gives me a one overs z plus n, and that is it plus supportion which is one to infinity d t e to the minus t t to the z minus one. So, we have what we want this is the mittag leffler representation.

See is representation for gamma of z here are the pole terms with the corresponding residues, and that portion is an entire function of z it has no singularities in finite part of this z plane now tell me I broke this zero to one, and one to infinity, but I could have really broken it up zero to any a, and a to infinity were is any positive number at all why did not I do that why did I choose one here what would happen if I choose an a some other a whatever remained would still be an entire function is you can see, but I why did not I choose an a.

Per me.

Series expansion.

Yeah, I want series expansion in terms of the poles the pole terms the presides pole terms with the correct.

Residues had I chosen some a that factor that be factors of a here sitting here those would not appear in the residue those are not the real residue. So, this is this choice is the only one that give you the correct residues at these points. So, you have to watched out, and that is what determines what this entire function parties ok. Now that we have this representation it is gone to look at some of the other properties of this gamma function, and one of them was I needed to remind myself yeah we will needed to define the log derivative.

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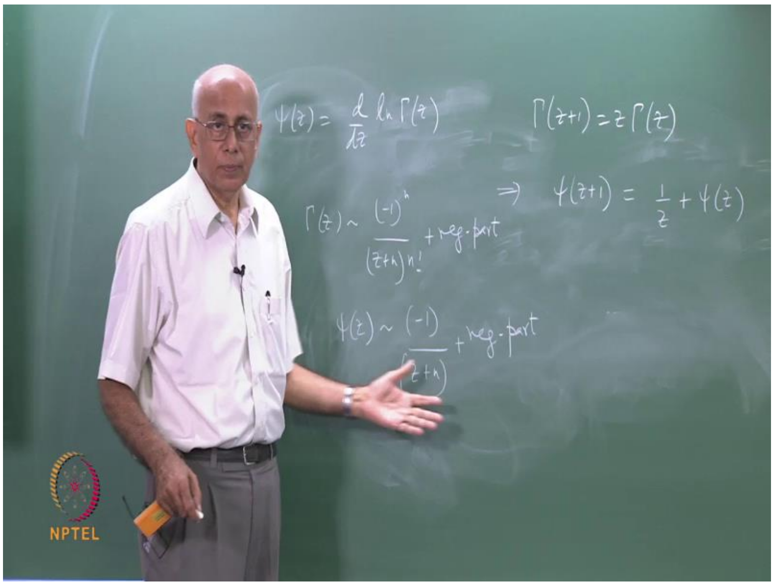


So, I defines psi of z equal to d over d z of log gamma of z that was a log derivative, and recall that you have a functional equation gamma of zee plus one z gamma of z which will of course, imply if I take the logs on both sides, and differentiate divide by the appropriate gamma function this is going to be psi of z plus one equal to on this side there is a one over z plus psi of z that is the functional equation satisfied by the log derivative of the gamma function. Now where does this get us well it'll help you to determine what this function does for specific values of the argument we know what gamma of one is gamma of two is gamma of half three half's etcetera is question is can I do the same thing for the psi function, and the answer is yes you can, because this equation this functional equation also helps you to analytically continue the psi function once you give it to me in some reason this will help me just like the gamma function did to continue this.

Analytically continue this function. Now I have already pointed out that gamma of z goes like minus one to the n over z plus n n factorial plus regular part near z equal to minus n near a pole near each of the poles this is the behaviour of the gamma function this is the singular part, and that is the regular part. Now if you differentiate this, and divide by the gamma of z once again I differentiate this I get a z plus n squares. So, psi of z as an of the derivative of gamma function as a double pole at z equal to minus n plus a regular part that part remains regular a divide one by the other it is clear that the derivative has a one over z plus in square the function has a one over z plus in square the function has a one over z plus n. So, this will immediately imply that psi of z was like minus one over z plus n plus regular part.

So, the log derivative has a constant residue at each of the poles it to has a simple pole add zero minus one minus two etcetera, but unlike the gamma function which has an independent residue this function has minus one residue just minus one as a residue at all these points now the question is what does psi of one do for incidents gamma of one be know is one, but psi of one equal to well the way to do this you need a little bit more information.

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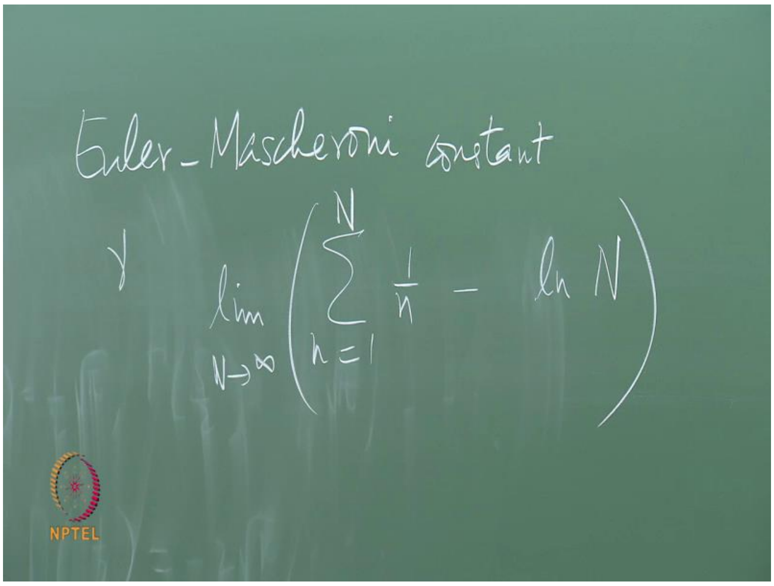


You need to know that gamma of z equal to near z equal to zero there is a leading term one over z the singular part, and then the regular part we need to know what the regular part is it is going to be a power series in z powers of z hm. Starting with z to the zero, and then etcetera, etcetera. So, this is going to be of the form plus summation n equal to zero to infinity c n z to the power of on this near z equal to zero in of z equal to zero the question is what are these constants equal to which requires a little harder work you need to start with the representation of the gamma function, and work through to find out what the answer here is, but it is not all together trivial turns out that the constant term c zero is known all the terms can be systematically found out, but the constant term is kind of universal constant is one of these fundamental numbers like e or pi or something like that it is an the same footing, and it is the following it is call the Euler constant, and it is defined as follows.

It is denoted by gamma, and it is little gamma, and it is defined as follows you know that if I took the harmonic series one plus half plus one third plus one fourth etcetera, and some the top. So, I start with a n equal to one to some capital n one over n this series diverges as capital n tends to infinity, because a harmonic series diverges it is a infinite right the question is how does it diverge, and roughly speaking I have something like one over n to some over, and that is like integrating over one over x up to some upper cut off now what is d x over x when I integrate I get log x. So, it is clear that this series is go to diverge very slowly like the log of capital n, but as n tends to infinity it definitely goes to infinity the next question is suppose I did this, and then I subtracted from it this n.

Here on this site a log n sorry this is the whole thing was like log n. So, I subtracted log n, and then took the limit of this as a n tends to infinity. So, after I subtract the diverge an part is the anything else left over is it finite or is it zero or it still infinity it could still be infinity, because there could be a correction which goes like log log n for instance, and that would also diverge well it turns out that in this case this has a finite limit limit is non zero it is finite it is a positive number, and it is given by this thing here gamma that is the definition of this euler constant, and it is numerical values found to any number of decimal places is approximately zero point five seven seven two dot dot dot etcetera. So, it is clear that this number has a well defined value.

But what is not known bots conjecture is that this is an irrational number. In fact, it is conjecture that it is a transcendental number not the root of any algebraic equation of finite degree with rational coefficients. So, this number conjectured to be irrational nobody has prove this proving the irrationality of given numbers proving the fact that they are proving the transcendental nature of the numbers individual numbers is an nontrivial task proving the fact that pi is transcendental took place only in the eighteen eighties or something like that similarly for e or me proved that e was transcendental, and gamma is another of those numbers which is believe to be irrational, but we do not know for sure it would be a big surprise if it to a rational on the other hand nobody has proved conclusively that it is irrational, but the chances are over well meaning that it is a it is a irrational number of some kind.

So, it turns out, that this is one over z minus gamma plus c one z plus etcetera this. So, gamma appears here in this place once we know that we can ask what gamma prime is you want to look like, and then of course, if you differentiate, and divide etcetera, etcetera it is not hard to show that is psi of z minus one over z, because remember I said that it is minus one over z plus n plus a regular part, and the regular part again turns out to be minus gamma plus etcetera. So, you put that together with this, and let z go to zero. So, this thing will imply immediately that psi of one.

Is equal to one minus one over z etcetera in is a minus gamma yes. So, if you let said go to zero in this the leading term the pole cancels out, and lead zero with just the constant there, and it is minus gamma. So, this is another way of understanding what this gamma is it appears all over everywhere, and very interesting relation is that psi of what happens to be this minus the Euler constant here that'll off course immediately tell you psi of two's it is one minus gamma, and therefore, psi of three etcetera. So, all these numbers every one of these numbers in psi of we can write down psi of n plus minus gamma plus summation.

J equal to one to n one over three you can therefore, find just has we found that gamma function at the positive integers were all factorials of numbers natural numbers similarly the psi function at all the natural numbers is minus gamma plus something or the other. So, lots of identities of this kind can be established what I will still I want to do o yes, yes another representation for the gamma function where where little gamma appears once again.

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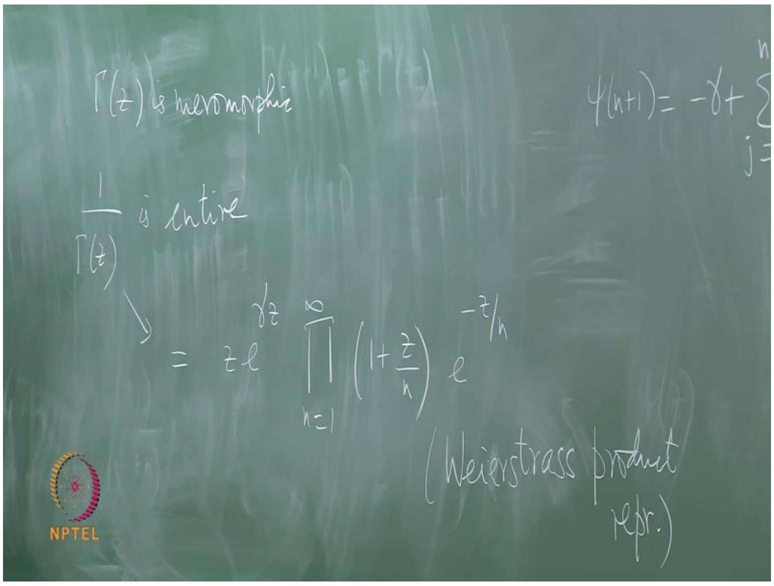


I know that the gamma of z is meromorphic turns out that one over gamma of z is an entire function what is the reason what is the what what condition should the gamma function satisfy in order that what condition should a meromorphic functions satisfy in order that its reciprocal be an entire function meromorphic means in the finite part of the plane it has only poles or whatever order it has only poles. Now I tell you it is reciprocal is an entire function what do you conclude I mean what condition should this function satisfy.

No zeros.

No zeros cannot have any zeros it has zeros, then off course the reciprocal will have divergence certain point right turns out one over gamma is entire, and entire functions have representations which depend on the zeros of this function just has you have a mittag leffler representation from meromorphic functions; these are all the pole terms, and then there is regular part which is an entire function similarly if you give me an entire function, then it is called zeros at various points, and you could ask suppose I tell you the points where it has zeros. And suppose zeros are at alpha beta gamma etcetera, and you say can I write it on the form z minus alpha z minus beta z minus gamma can I simply factor all the zeros together multiply them together, and ask is there a representation of this kind, and the answer is yes there is a representation call the weierstrass product representation, it is just precisely this, and in the case of the gamma function this turns out to be equal to well let me write it for yeah yes write this this is equal to z e to the power gamma z, where gamma is euler constant, and then a product N equal to one to infinity one plus z over n remember z equal to minus n is a pole of the gamma functions. So, the reciprocal will have a zero, and that point. So, these are the zeros, and then it is multiplied here by e to the minus z over n this representation is the. So, called weierstrass product representation for the gamma function, and it is typical that is such such a representation will typically have portions which have the zero, and then e to the power an entire function like this ok. So, it is a typical representative typical example of the weierstrass product representation for the gamma function again extremely useful.

Now you could take logs on this enable to all kinds of things get other representations etcetera. So, that is the other thing I wanted to mention now let us go on in the exercises are given a number of other identity is to be which are satisfied by the psi function, and you can work those out using the information given here with that let us go over to beta function let me mentioned here that we have…

So, far continued the gamma function analytically by this functional equation gamma of z plus one is z gamma of z, and I did that strip wise arbitrarily far to the left you can do this by doing it is strip wise on the other hand you can ask is that a master representation for the gamma function one single formula which is valid everywhere in the complex, and the answer is yes it'll involve a contour integral also it is an integral representation, but it also involves multiple valued functions. So, there are branch points branch cuts in… So, when we talk about multiple valued functions I will come back to the gamma functional derive for you this are the representation, but till than we use this functional equation as or main tool for continuation.

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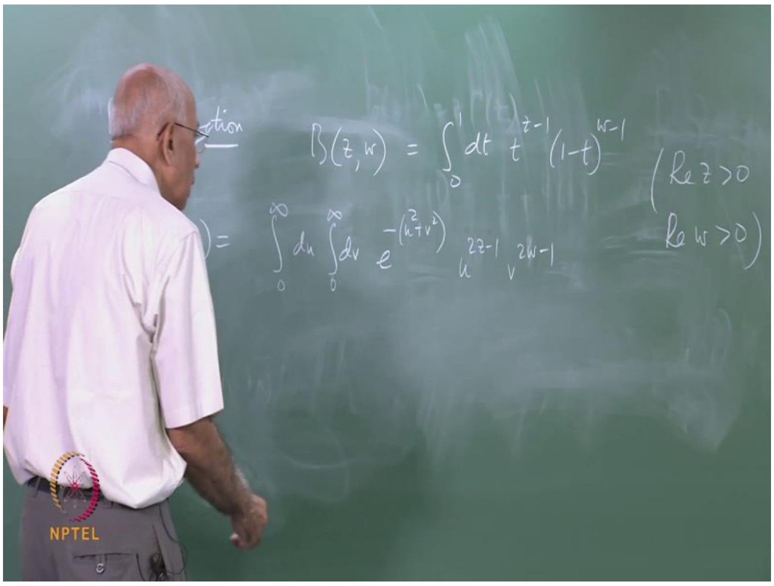


Now, let us gone to the beta function again due to Euler. In fact, the integral this call the euler integral of the first kind, and the gamma function is the euler integral of the second kind, and recall what I said I said that you could define it for complex z, and w to be equal to zero to one d t t to the power z minus one one minus t to the power w minus one, and you needed to have real z greater than zero real w greater than zero ok.

We need to derive a connection between beta function, and the gamma function, and the way you do it is to consider the following integrals. So, consider the following double integral let us call that I z, and w equal to integral from zero to infinity d u d v e to the minus u square plus v square u to the power two z minus one v to the power two w minus one you consider this integral it is like the Gaussian integral I wrote down earlier you go to ask where does this psi converge where would this integral converge to start with infinity does not pose a problem, because of this factor, but the zero would pose a problem. So, where is the conversions reason for this.

You want this exponent to be greater than minus one which is this region real z greater than zero real w greater than zero the right half plane in each of these variables. So, I start with this, and I can do two things one of them is to write it immediately.

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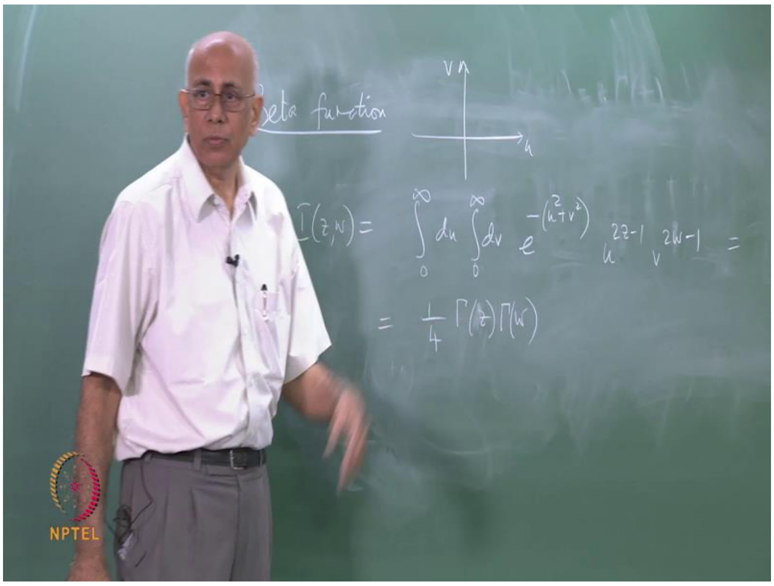

It is just a factor form two integrals multiplying each other, and I can read of those integrals. So, what I have to do is to put u squared equal to some variable. So, let us do that see what it gives us. So, you put zero to infinity d u e to the minus u squared u to the two z minus one. So, let put u squared equal to t or something like that u square root of t. So, this is equal to integral zero to infinity d t over two t to the power of half e to the minus t u to the power t minus a half.

So, this cancels at that t minus one sorry, but I am a saying u to the t to the power t to the power zee minus one, and then a half there is, but off course this is equal to one half gamma of z. So, on the one hand this is equal to a half from the u, and a half from the v integral. So, this is one fourth gamma of u gamma gamma of z gamma of w quarter of each of these all I did was small to gamma functions together, but you could also do this integral in pole a coordinates plane pole a coordinates. So, you said u equal to r cross theta.

V equal to r sin theta, and see what happens. So, let us do that this is equal to by the way what is the region of integration in the u v plane zero to infinite ya the first coordinate. So, when I go to plane pole a coordinates little r will run from zero to infinity, but theta will run from.

Zero to pi over two.

Zero to pi over two. So, this becomes equal to integral zero to infinity d r there is an r here, and then e to the minus r square which is this multiplied by zero to pi over two d theta, and then there is an r to the power two z plus two w minus two.

And then u is cross theta two z minus one sin theta two w minus one. So, let us bring the r's together this r to the power. So, there is an e to the minus r square, and then an r to the power let us leave the r here, and write this as twice z plus w minus one this factor is out you have this, and there was an r r d r times this integral, but r d r had I if I had a two r d r, and I put a half here.

That's d of r square, and I change variables immediately to r square. So, let us call r square equal to something are the other equal to s for example,, then this is d s e to the minus s s to the power z plus w minus one, but this is a gamma function e to the minus s a that is gamma of z plus w. So, we find have on this side half gamma of z plus. So, we have an identity with says half z plus w times that integral is equal to this integral product of gamma functions here on this side now it is not had to see that this sky here if I put cross square theta. So, a let us put cross squared.

Theta equal to some new variable let us call it z zai or something like that, then says two cross theta sin theta with the minus sin equal to d zai, and this integral will become what zero to pi over two d theta. So, this becomes equal to half gamma of z plus w when theta is pi over two zai zero I mean theta is zero zai one, but there is a minus sin there. So, let us make this zero to one, and get read of that minus sin there is an extra two.

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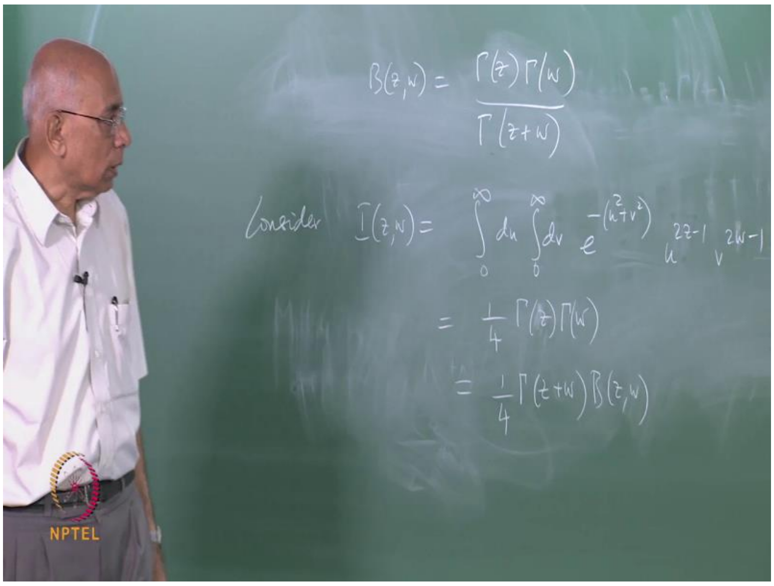


So, that gives me another one fourth on this side, and then there is a zai. So, I make this a two here a two here multiply by cross theta sin theta that gives me exactly d zai. So, that takes care of this, and then it is zai to the power z minus one one minus zai to the power w minus one its exactly that, but what is that equal to that is the beta function that is beta of z, and w by definition. So, it is says this is equal to one fourth gamma of z plus w beta function of z, and w. So, that finally, gives us a relation a basic relation with says beta of z, and w is equal to gamma of z gamma of w divided by gamma of z plus w is an identity.

So, this tells you all about the analytic structure of this function you can use this to analytically continue it once you know the continuation of gamma you can continue it to all arbitrary values of z, and w using this. So, the entire singularity structure of this function is known to you completely from this identity very fundamental relation between the two Euler, and teclaris. Now one of the consequences is the following I am not going to derive it, but I am going to mention it here what are the consequences of this formula.

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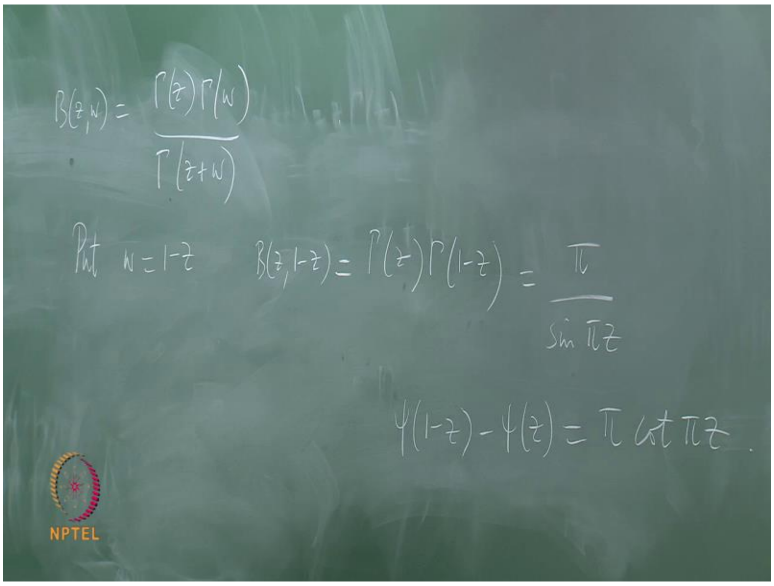


And I am going to give this is a next as an exercise is if you put put w equal to one over z, then you end up with relation it says beta of z one minus z equal to gamma of z gamma of one minus z divided by gamma of one which is one. So, you have a relation, but says the product of these two gamma functions here gamma of z gamma of one minus z. So, it is sought of symmetric product about the point half about the line half instantly that is just a beta function here, and you can write a representation for this function here go back to the original integral representation, and play around with that representation, and you can derive I given in the in the hand out I am I am told you how this should be done what substitution you should make it turns out that you can show that this quantity is equal to pi over sin pi z now to surprising, because this has singularities

Poles at z equal to zero minus one minus two etcetera which is where this function also singular, and it also has singularities this one has singularities at z equal to one two three etcetera, and that is comes from here. So, as you know the consequent has got pole simple poles at all the integers, and the once at the negative integers, and zero provided by this, and the once at all the positive integers are provided by this all clear now again you can take logs on both sides, and differentiate. So, you can get again a relation which will tell you psi of z minus psi of one minus z is equal to pi, and then if I differentiate this I take log, and differentiate it when I take the log I get a minus sin. So, I have to get rid of that minus sign. So, it will becomes psi of z one minus z minus psi of z on this psi is equal to log pi that does not get differentiated, and then I differentiate this log sin. So, there is a one over with the minus sin. So, it is minus if the minus I got rid of. So, you have a sin in the denominator you get a pi cross on top. So, this is equal to pi cotangent pi z another identity if you like the reflection formula for the psi function clear. So, there are lot of these very beautiful relationships remember the last time, we saw one of these relationships with said the similar to this we had a relationship for cosecant squared to one over z minus, and the whole square some overall integer values of n. So, this function will also lead this identity leads to similar identities lots of identities of this kind this this formula by the way this formula here is called the reflection formula for gamma of z.

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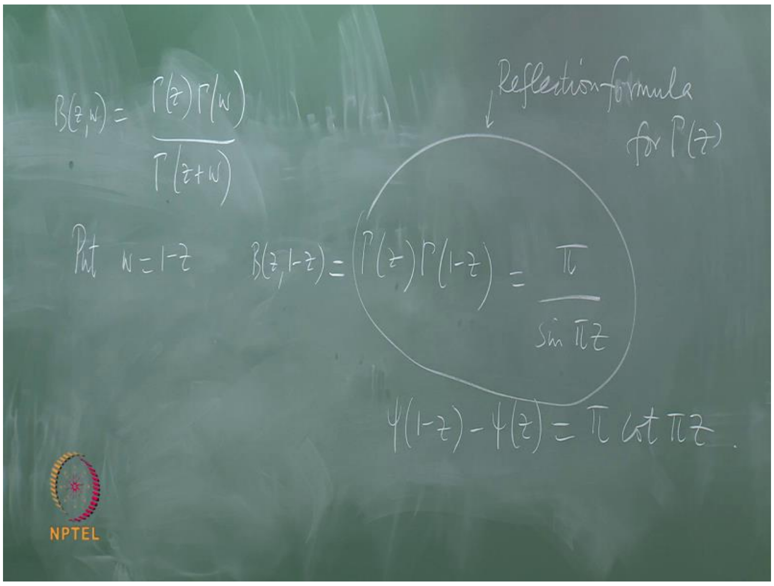


Now, the next thing I wanted to talk about was the doubling formula, and that is something which is once again, if you play with the relation between the beta function, and the gamma function you put z equal to w then…

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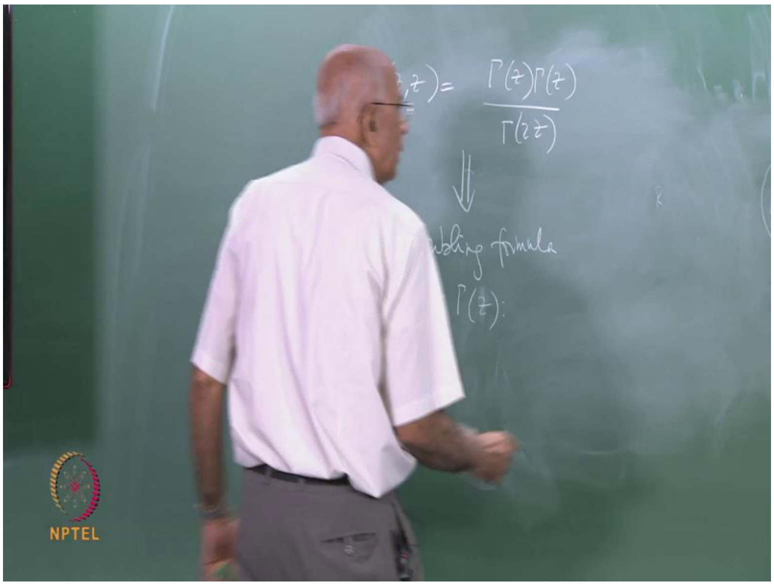


You end up with the for in relation beta of z z equal to gamma of z gamma of z over gamma of two z. So, bring this to that side, and play around with this integral here the explicit integral, and you can divide define you can derive a formula which is call the doubling formula. So, this will lead to a formula call for the gamma function doubling or duplication formula I do not remember, which for gamma of z.

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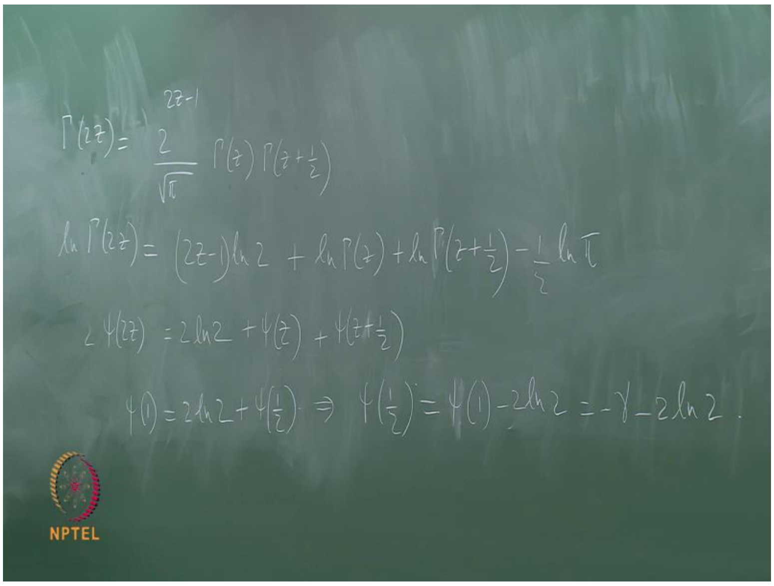


and it reaches follows says gamma of twice z equal to two the power two z minus one over gamma of half, which is root by not surprising, because you will get this playing around with this beta function, and then there is a gamma of z gamma of z plus half a very very useful formula imply this this actually a formula of a gamma n z n times this which is as a product of gamma functions of smaller argument, but this is the most commonly used formula here most useful one very appears very, very frequencies. So, whenever you have these expressions involving sterling approximate sterling formula, and statistical mechanics this formula is very, very useful to simplify these. So, notice in particular that if a gamma is an integer, then this the factorial that is the factorial here natural number, and then this here will have a square root of pi on the right hand side which comes from this formula.

Here explicitly again you can play the same game you can take log derivatives on both sides you take logs on both sides you get log gamma of two z equal to two z minus one o sorry two to the two z minus one times what I differentiate this with respect to z what do I get. What's the derivative of two to the two z minus one well you can write this as two to the two z minus one can be written as e to the two z minus one log two write. So, you differentiate it, and then you get the same factor up in the numerator, and then you must differentiate two z minus one log two, and if you do that you get a two log two get that term. So, let us first take let us take log first, and then easier two z minus one log two plus log gamma of z plus log gamma z plus a half minus half log pi.

Now I differentiate etcetera. So, you going to get on this psi you going to get a psi of two z one over this the here, and and I differentiate this respect to this, and there is going to be a factor two I say that is equal to two log two log two plus psi of z plus psi of z plus half. So, this is going to help us find things like psi of half three half's five halfs etcetera we already found psi of all the integers by this recursion relation, and we can find psi of half three half's etcetera, and there is an extra two log two setting here. So, for instance put z equal to half immediately what you get I want to make sure that the answer comes out right if you put z equal to half you get two. So, let get two psi of one equal two log two plus psi of half plus psi of one. So, this it was a way have this.

So, this implies at psi of half equal to psi of one minus two log two, and what psi of one, we already discovered what it was.

Minus gamma.

Minus gamma. So, this equal to minus gamma minus two log two check the sins I am not hundred percent sure about this sins I am just doing this at the top of my head, but just check out the sins here make sure it is right. So, you can now find psi half three half's five half's etcetera their all going to be similar to this things added, but, then you going to have this minus gamma, and then an two log two there are lots, and lots of other identities there are other integral representations also for one over gamma of z etcetera we will try it we might come across one of them little later on when we do multiple valued functions I will give an integral representation for the gamma.

For gamma of z, and also far one over gamma of z the latter being in entire function alright. So, much for this now what is next there are several identities between the beta function, and the gamma function involving substitution of variables changes of variables etcetera some of which I will put down in the problem sheets, and hand over to you for you to try out. So, that is about all I wanted to say about these functions we will come back to these functions over, and over again in particular this relation we had between the psi of z plus one minus psi is z is pi cot pi z we will came back to that in different context when we talk about legendre functions, because as a connection of this formula with the corresponding formula in the case of legendre functions I will recall this formula to you at that time.