Selected Topics in Mathematical Physics
Prof. V Balakrishnan
Department of Physics
Indian Institute of Technology, Madras

Module - 01
Lecture – 01

So the first topic we will start with is complex analysis, a functions of single complex variable. And I am going to assume that all of you already have a little bit of preliminary information on this or knowledge about it. So, it will be more in the nature of re capitulation.

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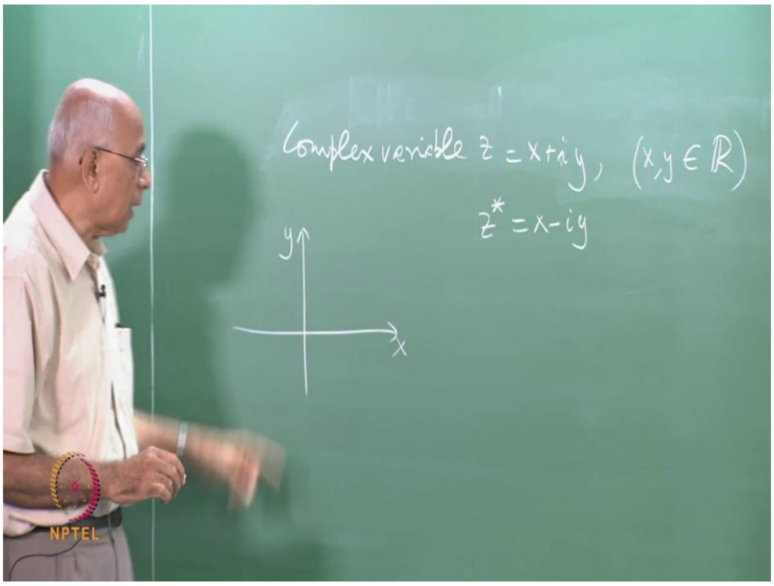

So, recall that we define a complex variable z as x plus i y, and x and y are elements of the real number line. Now this complex plane, the x y plain is called the complex plane. And we are going to talk about the functions which are analytic functions in a specific sense of this combination x plus i y. The reason this is emphasize is because x and y are linearly independent of each other, and of course, if you have arbitrary function of x and y, there is no reason it should be a function only a particular combination of x and y. In this case, it is a combination x plus i y.

Remember that this thing z star x minus i y is linearly independent of z. So, the whole thing about analytic functions is simply that you have a function of x and y which is the function of the combination x plus i y and does not involve the combination x minus i y.

So, that is roughly speaking what an analytic function is will make this idea much more precise, but before we do that I would like to introduce the idea of stereographic projections. Because on the x y plane, unlike the case of the real variable x by tend to plus infinity or minus infinity on this side, and likewise for y in the complex plane, there are actually infinite number of direction in which tend to infinity along any array, any direction whatsoever.

Now that is a little uncomfortable, because we seen to have many many points at infinity, so that speak. So, the standard trick is to try and put this idea of infinity on fourteen which is more less the same that of any finite point and that is done by compact define by this space. Another words, imagine this plane is a huge pan cake, and the new take pan cake lift it up and so it is ends together. So, it is comes a surface of a sphere, and this will then be or model for the complex plane, the extended the complex plane. To make this precise one does about called stereographic projection.

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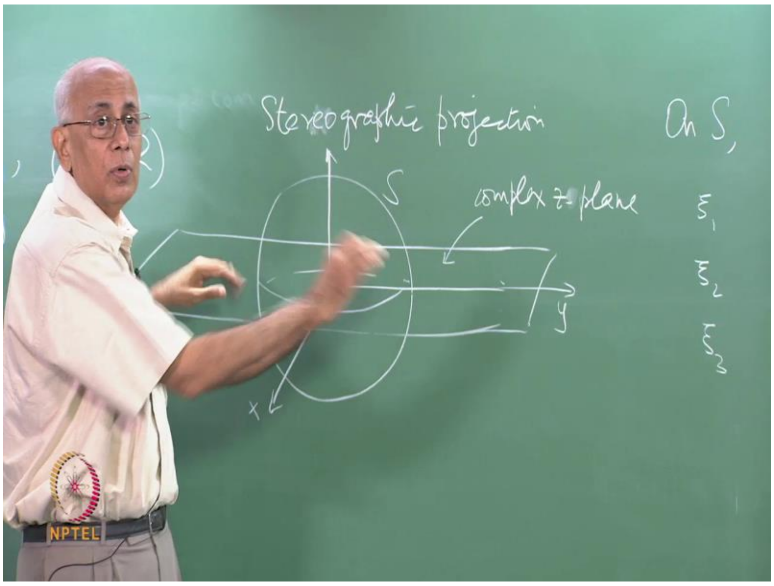

Stereographic projection which is you take a units sphere that is the equator and the plane of this equator is going to be complex plane - z plane. So, on this you have these axis this comes order the board this is what call the y direction in the x y plane. So, this is the complex plane, complex z plane. This is the units sphere, it is called unit radius; and on this sphere, on this plane, the coordinates of x and y x coming out here and y going that direction. And on this sphere, the units sphere I would like to have coordinates, the three coordinates, but they satisfied a constrained because the sums of the squares of the three coordinates will become unity, because it is a unit sphere.

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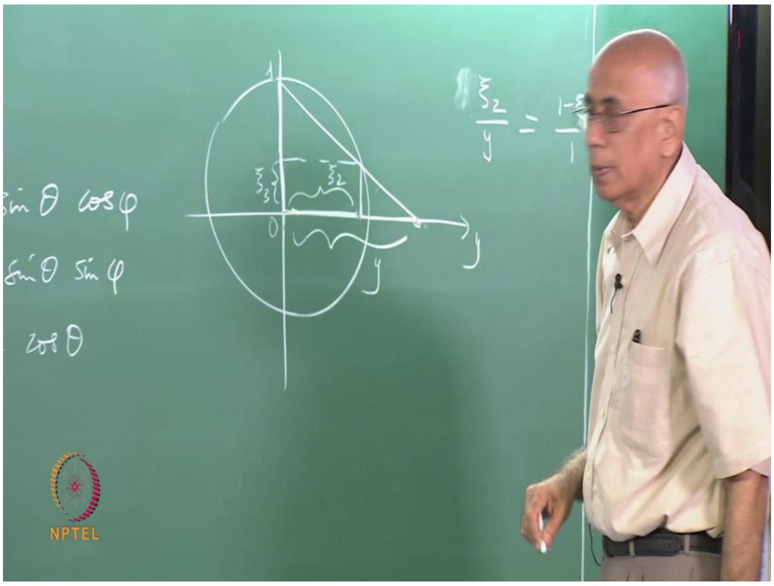

So, let us call though c ordinates psi one, psi two, psi three, so on that is sphere and this sphere I call s. On s, the coordinates are psi one, psi two, and psi three, where I want in the extraction psi two in the y direction and psi three what would have been in the z direction. But I am go to the z for the combination x plus I y in the third direction apper. Now what are these actually in terms of angular coordinates on this sphere. On that is sphere units sphere I can define polar angle theta, which is a goal attitude. And the longitude phi and as (( )) angle phi.

Then of course, psi one as we are all aware is a sin theta cos phi this is sine theta sine phi and this is cos theta. These are just spherical polar coordinates on this plane. Now they idea of stereography projection is that you take this point the north pole and this one is a south pole here and the other end that is the origin. You take this north pole and draw a line to any point on this plane like this for instance. This point here the line joining the point of projection to this point intersects this sphere s at one point and you associate this point, the point in the complex plane.

Now you can see that this is going to be a mapping from the complex plane to the surface of this sphere, because of the every point in the complex plane, there exist one point on this sphere. It is immediately clear here that all point lying unit circle in the complex plane are going to be mapped or on map from point in the southern hemisphere of this Riemann's sphere. And all point outside the unit circle like this for instance on mapped or map from points in the northern hemisphere. The equator of this sphere is the unit circle in the complex plane the circle on which mode z is equal to one.

The South Pole is mapped into the horizontal and the horizon in the complex plane is mapped into the south pole on the Riemann's sphere. And now what we need or equations which connect the coordinates psi one psi two psi three to the coordinates x and y on the plane remember there are only two independent coordinates because psi one square plus psi two square plus psi three square is equal to one. So, this is constrained between them. And just as you have x and y independent similarly two out of these three are independent coordinates. Now the coordinate the actual mapping from one to the other is very easy to see for hence example in the intersection in the psi three psi two plane.

Look at here for a minute then it looks like this and what we are doing is to mapping this fashion that was y this is y coordinate, but in the same direction on the sphere. You are calling it psi two and you are calling this coordinate two psi three and this is of course, one and this side that is the origin now I similarity of triangles it is immediately clear that this divided by the whole link that side is equal to divided by the length. So, it immediately follows that y w psi two divided by y that this forward that this side easy equal to on this side what should I write while this portion that is one minus psi three divided by one that is it right.

So, this is to this and this is to this entire, so that is the equation that immediately tells us right way that y equal to psi t one minus psi three. And if we done this plane here the intersection of this line, and this line plane form these to line then you immediately got x equal to psi one one minus psi three on the complex plane. You could go back and right psi one psi two etcetera in terms of these for all then you can ask what this of mapping actually look like easy to see that for instance x equal to psi one. So, for psi one, I write sine theta record for psi five and psi three one minus cross.

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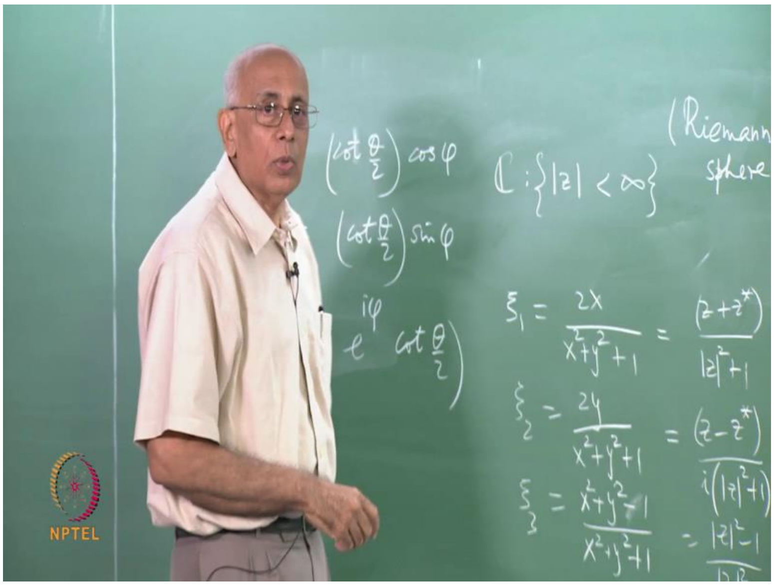

Then go to have angles then you immediately becomes clear. This is easy equal to cot greater than 2 cos phi y equal to cot seven phi. So, it means the z is equal e to the i phi cot theta over two, z start is e to the minus i phi cot theta over two, where theta and phi are the polar and (( )) angles on this spheres. By the way, this sphere is this thing here is called the Riemann sphere. You could ask what are the reverse mappings, well we need to exploit the fact that psi one square to plus psi square plus psi three square is equal to one. And then it hard to see that they reverse mapping so these to these to by the way I should complete this by writing this implies that z equal to psi one plus i z two over one minus psi three z star psi one minus psi three.

And what are the reverse mappings is not hard to see directly from here that psi one is two x divided by x square plus y square plus one; psi two is two y divided by x square plus y square plus one, and psi three is equal two x square plus y square minus one divided this one. Which could of course also to write as two x that is z plus z star divided by mod z whole square plus one, this is equal to z minus plus star over i times mod z square plus one, and this guy is mod z square minus one over mod z square plus one. So, that the one to one mapping between the complex plane and the Riemann's sphere. The question is what is the point n going to get mapped into under this map.

It is immediately geometrically clear immediately that as I get closer and closer here, the point in the complex plane is going on further and further away. So, it is evident that no matter which direction you approach N N, you are going to be hitting infinity along some ray in the complex plane that all those points are getting mapped on to N. So, I can call N the point at infinity. In fact, I call it at infinity; I denote it by infinity. And what this does is to bring infinity to start as which is similar to that of any finite point in the complex plane. So, very often I am going denote the complex plane by z, this is set of all the all is set the modulus of z is finite, and if I include the point at infinity and going to call it the extended complex plane. And very often, I am going to denote C tilt extended which includes the point at infinity. But the Riemann's sphere provides with the model for the extended complex plane, is it clear that it is a geometrically; obviously, infinity that this N now represent the point at infinity. So, that is why very often in complex analysis usually say just one point at infinity and tend to the different direction I do not care, but what I mean by that is this point here, this point N. So, this compact of this complex plane enable to extended to include the point at infinity puts it on the same put anything else. Then of course, began to calculus on a very rigorously without worrying about this infinity, what this infinity is.

One could ask on this sphere, do you have a notion of distance on this sphere, was the many ways of defining distance on it, one of the them defining of great circle distances, now what the great circle distance between any two point on the surface of sphere. You put into the one of the point is the north pole, and then you look for the distance along the longitude to the second point and that is the (( )) distances, the shortest distance line on this sphere. You could also define distances in this case as if I have two points z one and z two in the complex plane, in the plane here. Then z 1 minus z 2 modulus is what I called this distance between these two points on the complex plane. On the other hand, I could ask what is the distance corresponding distance on this sphere, well it would be chordal distance between these two points. So, I draw chord through this hollow sphere from one point to the other, and calculate what that distance is when the two points on the complex plane on z 1 and z 2.

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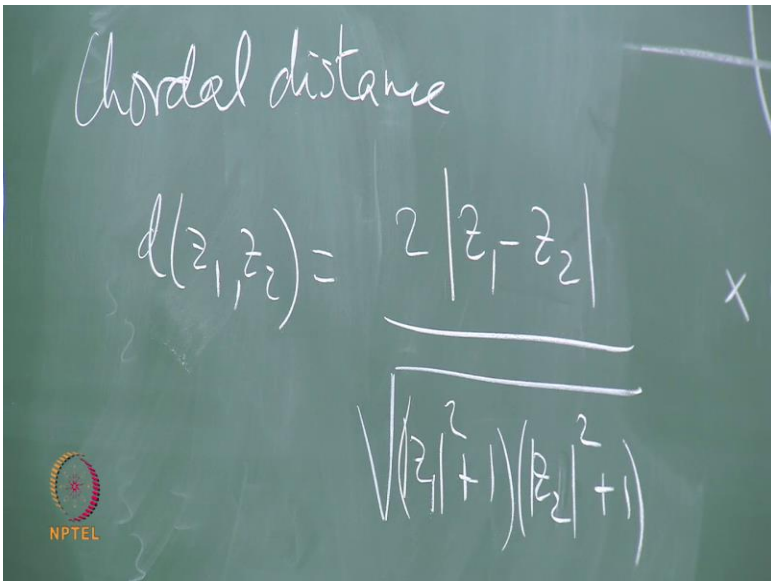

In little bit of algebra shows you that this distance chordal distance between z 1 and z 2, let us call d of z 1 z 2. This is equal to twice, turns out to be twice modulus z one minus z two divided by square root of mod z one square plus one z two square plus one, it turns out to be this quantity, a little bit of algebra. We can substitute this expressions and then you end up with this expression for the distance, notice the presence of these two denominators here. Now what is that do, that is makes the distance between any points on this sphere finite including point the point at infinity.

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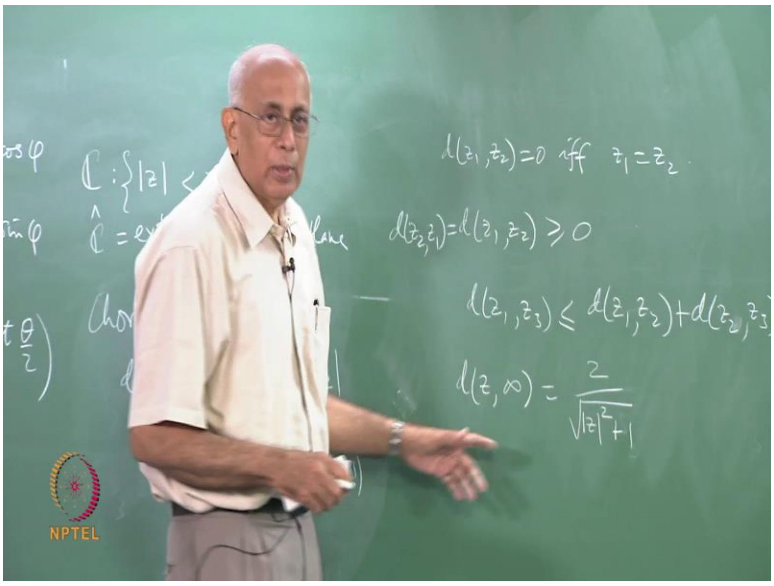

Because we can see this satisfied all the properties you need of a distance function. For instance d of z one z two is equal to zero if one only if z one equal to z two is no other way this distance can vanish. And then we know that the distance d f z one, z two in general is non-negative of this from here. And the distance from any point to any other point satisfied the triangle inequality, z one, z three is less than equal to d of z one, z two plus d of z two, z three. And this is also well d of z two z one is equal to this, it is symmetric under the interchange of these two points. So, this distance as defined here satisfied all the requirement that you make of what do you called at distance function. So, it is respectable distance function satisfied in triangle inequality and so on.

Now what is it means to say that the distance to the point at infinity well d of z infinity equal to so z one equal to z and z two equal to infinity. There of course, in the limit then z goes to infinity this is all the contributes that cancels again the z two here and immediately get two dived by square root of mod z square plus one. So, what is the distance between the origin and the point at infinity, the chordal distance, it is two, that is in fact the chordal distance between the south pole and the north pole on this sphere, that is the diameter of the sphere. So, this thing here is extremely useful the idea of this chordal distance and one can make lot of progress using this. We cannot to do much more with this, but simply to point out here that there exist such a distance, notion of distance and it got an interesting structure.

Student: (( ))

Professor: We do, but I am the talk about it problems set something like that, it is useful.

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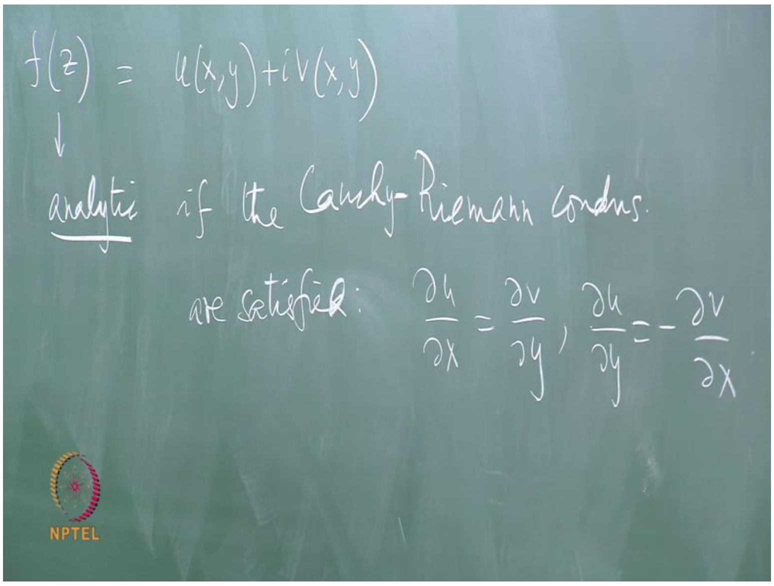

Now let us get to analytic functions. Let see what is mean by analytic functions. So, there is roughly speaking and analytic function in some region. So, we have a function f of x and y (( )), function of z is an analytic function and going to use this term analytic, and later on I will qualified in certain ways. It is analytic in some region in this complex plane. If it is satisfied couple of relations, if this f z is u of x, y plus i times of x, y; where u and v are the real and imaginary parts of this function. When this is analytic, if the Quasi-Riemann's conditions are satisfied, I am not going to prove Quasi-Riemann's conditions series, it is fairly straightforward. But I would like to tell me recall what this conditions are, and then we try to interpret them.

Analytic if the Quasi-Riemann conditions are satisfied. What are this conditions, the conditions are delta u over delta x is equal to delta v over delta y delta u over delta y minus delta v over delta x. So, right away you see that an analytic function must have partial derivatives, first order partial derivatives, it is first order, with respect to both x and y both the real and imaginary parts. So, that is a pre requesting you need those partial derivatives and there should be continues and that small as all that you need for a function to be analytic. But what it really means these conditions is that this thing here, as I said is not a function of other linearly independent combination. It is a function of x and y, but it is a function of combination x plus i y with no reference to x minus i y.

So, we could say that an analytic function something for which delta f over delta z star equal to zero, no dependence on z star at all. So, it is by the way, we know that we have to tell me what is x in terms of z and z star, it is this y is z minus z star over z I, just z is x plus i y and z star is x minus i y and invert those these are the other inverse relations. Now what would this implies, now I can take this z star write it as x minus i y and then use change rule of differentiation. So, this would imply that delta f over x minus i delta f over delta y equal to zero.

And now you put f equal to u plus iv and equate real and imaginary parts, and you get the Quasi-Riemann's condition. So, quick way of asking what is an analytic function is a you see this function is nice and smooth has first order derivatives and so on. And then check whether it has any dependence on z star or not. And if does not and you can expressed clearly as a function of z, and you say it is analytic. Check these conditions out each time.

Is this analytic function? I want even call it function f or z, just call it f, is that analytic function. No, because you see by the way this immediately tells you that you cannot have a function it is analytical in certain region. If in that entire region, it has no imaginary part at all, it is purely really or if it is purely imaginary with the real part been identically zero, or the imaginary part being identically zero in a whole region, you cannot be analytic. And that is indeed true, because x z plus z star over two, it involves z star of course, so it cannot be analytic function. How about y, not analytic function?

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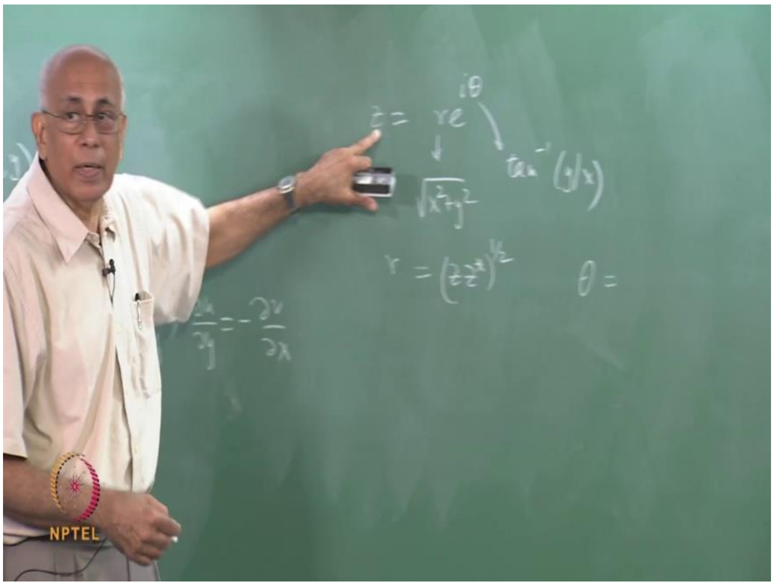

How about you know sometimes you write this as z r e to the i theta by the way, this theta is not the same as that, so it is get rid of this, you write it on polar form in this fashion. Where r is equal to square root of x square plus y square and this theta is tan inverse y over x, write in this form. So, tell me is r analytically function how do write r in terms of the z and z star, r square is x square plus y square. So, it is z z star. So, this is r is equal to mod z that is not an analytically function, it is got this dependence here.

What about theta, is theta an analytic function, tan inverse y over x. Now you write y and x in terms of this. How do I write theta in terms of and z star way z star. Well z star is r e to the minus i theta. So, if I divide z by z star, the r cancel out and I get e to the i theta. So, this is equal to one over two i log z over z star, does that is involves z star. So, can it be analytic function, no not analytic function; x is not, y is not, r is not, theta is not and so on. So, analytic function have to have very specific structure, not every function is analytic, is this analytic function?

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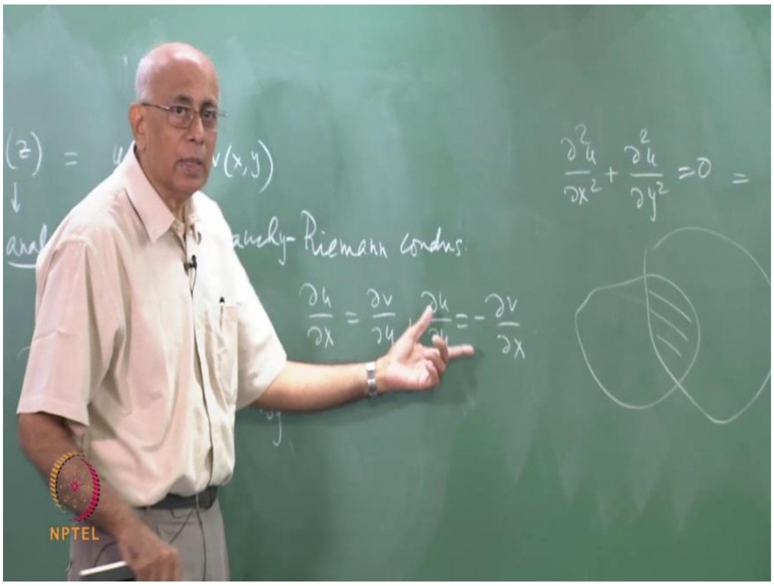

Let us go ahead give you few more examples as we go along, but one consequence or what this equation tell us one consequence is immediately that d two u over d x two plus del square u over delta y square equal to zero and that is also true for v. So, in the region in which some function f of z with real part – u, and imaginary part- v, in the region in which this function is analytic, the real part and the imaginary part separately satisfied Laplace equation in two dimensions. What do you call the function, which satisfy

Laplace equation.

Students: Harmonic Function

Professor: Harmonic functions, so the real and imaginary parts of analytic function are harmonic functions. Now this set of equations is telling you in some sense that if you give me real part, the imaginary part is more less determinant by in principle solving the Quasi-Riemann's conditions. So, what an analytic function is like this. Suppose u is harmonic in this region in the complex plane, and v is harmonic in this region in the complex plane. When u plus i v is guaranteed to an analytic function in the intersection of these two regions.

In that intersection region, both u and v are harmonic functions, and they can be real and imaginary parts of an analytic function. So, in that sense, if you solve the Quasi-Riemann's conditions given u, we can find v, but you have to specify the region, you have to specify the region in which things are analytic. And now let me jump a little bit and explain that if a function is analytic in the whole of the finite complex plane, it is called an entire function.

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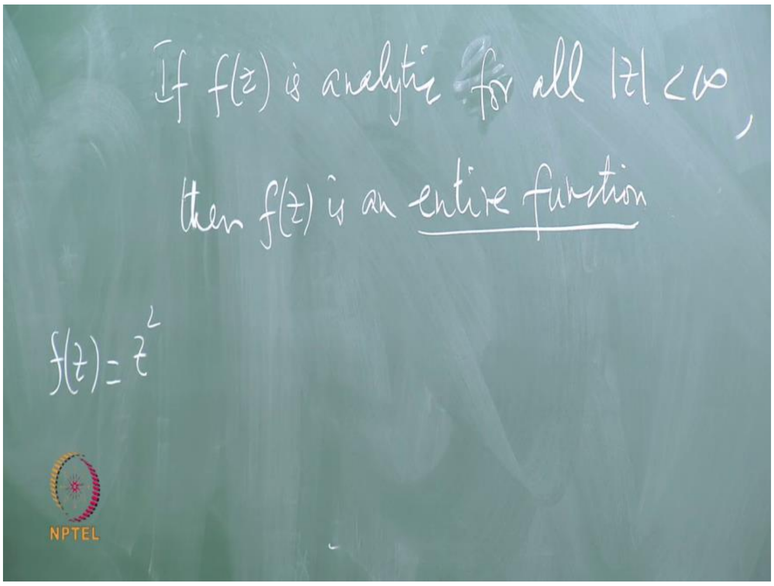

These conditions are satisfied in the whole of complex plane for all mod z less than infinity then f of z is an entire function. Lots of examples of a entire functions, how about this, z an entire function, yes or no.

Students: Yes

Professor: Yes, it is satisfied the Quasi-Riemann's condition for all z, all finite z. How about this, do you think it satisfies? Well you have to write z x plus i y, but more simply this function is differentiable and it is clear that does not involve z star. So, it is very much analytic you can in principally write x plus i y whole squared and then you differentiate the real part and imaginary part, see and check it is satisfied Quasi-Riemann's, they certainly do, so that is an entire function. How about this is that an entire function?

Students: Yes

Professor: Yes, certainly. How about f of z equal to p n of z some polynomial of degree n is that and entire function?

Students: Yes

Professor: Arbitrary polynomial, yes, certainly it is an entire functions. How about f of z equal to e to the power z. Do you think that is an entire function?

Students: (( ))

Professor: Well, we need to write the definition of e to the z in terms of power series. And then ask whether that series is valid for all z or not. Is that z true or false, what is the definition of e to the z by the way.

Students: (( ))

Professor: So, for what z is this valid.

Students: Z less than one

Professor: All finite z, mod z less than infinity all finite. So, e to the z an entire function.

Students: Yes

Professor: Yes, e to the minus z?

Students: Yes

Professor: It all does minus want to the n, it does not do anything. How about e to the z plus e to the minus z divided by two.

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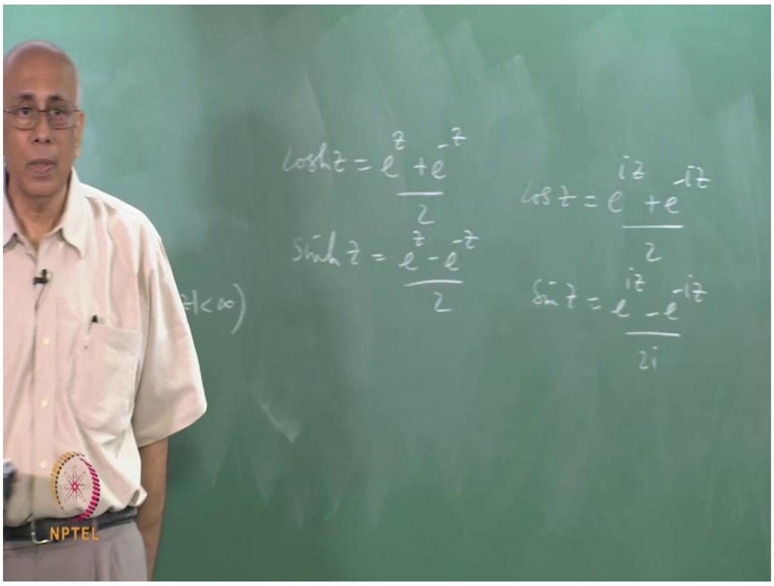

Student: Yes

Professor: Yes. So, what do you call that by the way cos that is entire. So, e to the z, e to the minus z cos z, what about sine hyperbolic z is that only entire function yes yes indeed. It is only the odd part of these series and the other things is an even part. So, the entire function is odd part or even part, it all entire functions. How about sine z or cos z is that an entire function, can you write it as exponential,

Student: z plus e to the minus z…

Professor: e to the i z plus e to the minus i z over two. And sin z, z is the arbitrary complex number by the way. Sin and sin hyperbolic, cos and cos hyperbolic, there are same function, I mean essentially there are analytic continuation of each other. So, regard all these function are defined by power series and as the argument of the power series being a complex variable say that is a correct way to define, look at all these function. All these are entire functions, everything is an entire function.

How about tan z that's that is not minute clear to that all not clear the (( )) entire functions, certainly singular at some point (( )), so that is not entire. How about z to negative power.

Student: Zero (( ))

Professor: The loop at infinity right at zero, z equal to zero. So, those are not entire functions, polynomials yes, but rational functions no. Those you have where the function (( )). Now this theorem called (( )) theorem, which says that if a function has no singularity everywhere it is analytic everywhere in the extended complex plane at all points on the Riemann sphere then it is must necessarily be a constant is no other function at all.

So, if you have a function that is entire and at infinity, it is not singular, but continues to obey the Quasi-Riemann conditions then that function must be a constant, to real constant. So, what is the conclusion immediately for all these functions for all these normal entire function these are not constants the functions as we have said not trivial functions. But the entire, so the conclusion is they cannot satisfy the Quasi-Riemann conditions at the point at infinity. They must be singular at infinity; it must be singularly infinity. So, all these functions that is why very careful to write mod z less than infinity, because at infinity there are singularities. And will come to what kind of singularities they are etcetera. In these cases, there all what is called essential singularities.

Now the same thing goes to for this. This is a monomial, you have an arbitrary polynomial here, but these things blow up at the point at infinity they have singularities at infinity, those would be called poles of various kinds. So, we will classify this singularities quickly, but the fact is that you cannot have an entire function which continues to be analytic at infinity without being the constant, as to be a constant. But these are lots of examples here all these power series etcetera could all entire functions, but they are blow up at infinity. They have (( )) behavior at infinity. Now before we go on to the behavior of power series there is something else I want to mention and that it is to ask what is the derivative of an analytic function, that is one more way of looking at the Quasi-Riemann's conditions.