Select/Special Topics in 'Theory of Atomic Collisions and Spectroscopy'
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Lecture 39
Breit-Wigner Resonances

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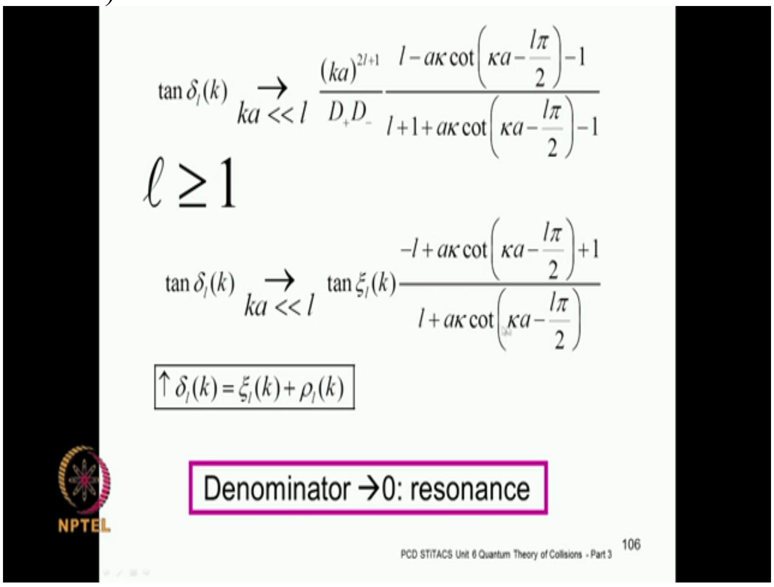

Greetings, we will discuss the Resonances further. The Fano Feshbach Resonances and then we will relate them to the, to the Fano parameters okay in the coming few classes. So, let me quickly recapitulate what we did toward the end that we had an expression for the tangent of the phase shift. And we express this phase shift as a sum of two parts: one corresponding to the impenetrable hard sphere component.

And the other is the one which would contain all the physics, all the Dynamics, which is coming from the actual scattering potential that we are dealing with. And you could write the phase shift as the sum of these two parts. And notice that when this denominator, when this denominator goes to zero, we would hit a resonance. So, that is the resonance condition.

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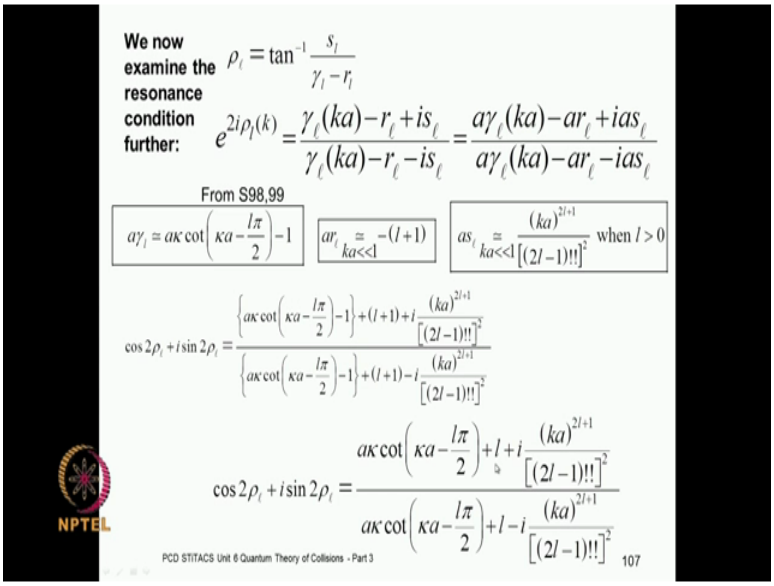

Now, this second phase shift which is not the impenetrable part phase shift, but the remaining residual part, which is the one in which we are really interested. This phase shift we had written in terms of the two parameters s and r. Each defined for every quantum number l orbital angular quantum number l. So, it is different for each partial wave. And then, gamma in this is the logarithmic derivative of the wave function at the boundary of the potential.

So, we had expressed it in terms of this r and s. And here I have multiplied and divided by a both the denominator and the numerator having a multiplied by a. So, this is the residual part of the phase shift. And we have the expressions for these terms already. And in terms of this the remaining phase shift, which is a to the 2i Rho, is now written in terms of these parameters; because we know what ar is.

So, ar is given over here, as is given over here. This goes as ka to the 2l + 1. So, this is i times as. So, that comes over here. So, you have only written this e to the 2i Rho l in terms of these, right hand side of these expressions. And we can simplify this relationship a little bit. So, this is the same expression as over here. So, there is no difference between these two expressions. It is the same one which has been simplified.

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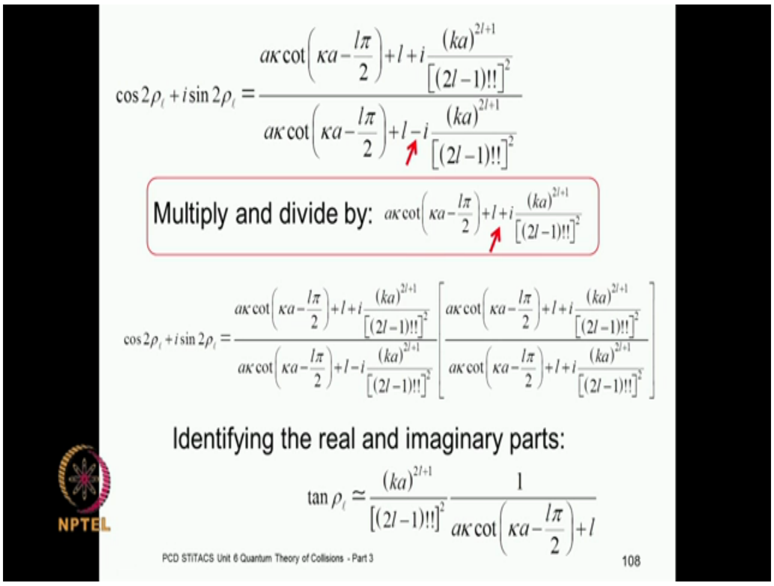

And now what does it give us? You have a minus sign here in the denominator. So, you have got in the denominator a complex number. And you multiply this ratio and also divide by the complex conjugate of the denominator okay, so that in the denominator, you will get the modulus square. That is the idea. So, you multiply and divide by the complex conjugate of the denominator. So that is what is done over here.

So, there is a little bit of analysis which is quite straightforward. I will not put the terms explicitly one by one and comment on every substitution. It is fairly straightforward. And now you have got a complex number the denominator is, now, just a modular square. So, it is a real number and the numerator will be a complex number.

And if you equate the real parts of the left hand side which is cosine twice Rho l with the real part of the right hand side and equate the imaginary part of the left hand side which is i sine 2 Rho l with the imaginary part of the right hand side.

And then take the ratio of sine2 2cos Rho l this is what you get okay. So, it is a straight forward substitution and this is what you get for the tangent of Rho in terms of this factor here. And we know that it is linked to the resonance condition.

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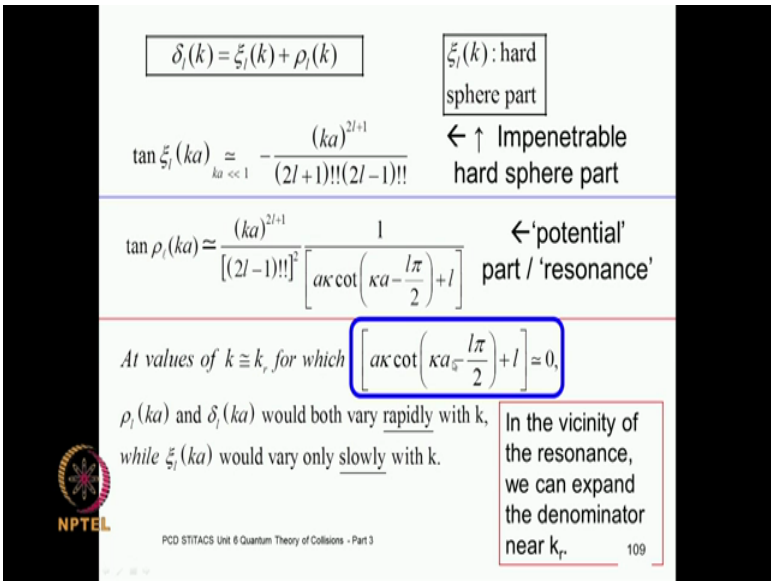

So, Xi is the hard sphere impenetrable part of the scattering. And we have already found that the hard sphere part has got this behaviour. This we obtained explicitly in one of our previous classes and we have been carrying this result with us. The remaining part which is tan of Rho is given by this, which we just obtained in the previous slide, right. So, we use this relation. And now, we recognize that when you hit a resonance, which is here.

This is the resonant condition that this denominator goes to 0. And this will happen at certain specific energies and corresponding specific momentum. And this is represented by the resonance momentum being given by k nearly equal to kr. So, k subscript r is the resonance momentum in units of h cross. So, h cross k is the momentum. So, this is the k value corresponding to that.

And when this happens, this phase shift Rho and as a result of this Rho also the phase shift delta, which is the net scattering phase shift okay, because in the detector, when you carry out measurements, you will see the net effect. So, you will be focusing your observations on the net phase shift which is the delta. But because Rho will change very rapidly in the vicinity of the resonance delta which is some of this Xi plus Rho will also change rapidly.

Out of these two pieces, Xi will be changing relatively, smoothly. But Rho will change rapidly, at the resonance okay. So, now we have got this denominator kappa and kappa includes the momentum, as you will remember. So, you can expand this factor, the resonance condition which is a function of k, in the vicinity of the resonance momentum kr. And you can expand this denominator near kr.

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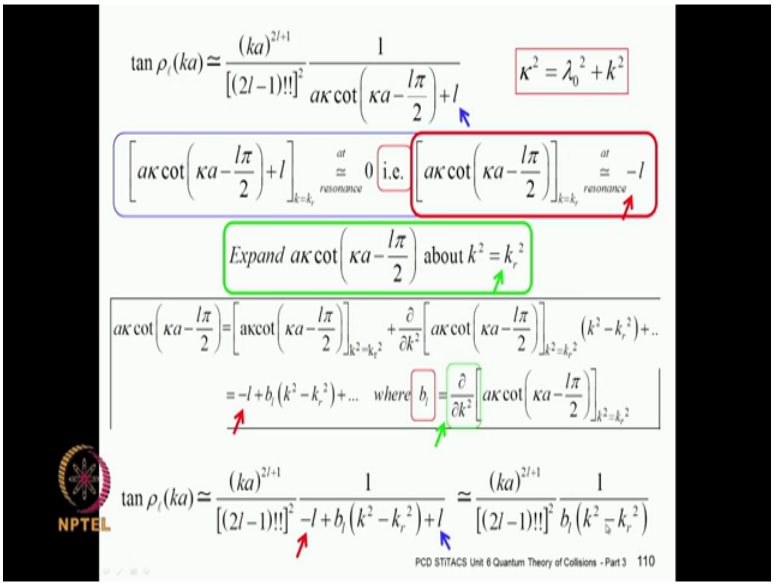

So, let us do that. So, kappa square is equal to lambda 0 square + k square. Lambda 0 square is the depth of the square well potential, the radial square well potential. And now, you have this as the resonance condition. That this factor, a kappa cotangent of this angle kappa a -l pi by 2, at the resonance is nearly equal to -l.

And we are now going to expand the left hand side of this about k square which is equal to k r square which is near the resonance momentum. So, let us expand it. So, you get the first term, let this is like a power series expansion. The next term will be the derivative, with respect to k square of this function multiplied by the difference in the independent parameter, which is k square -kr square.

Because you are expanding in the vicinity of k r square, right. And then, you will, of course, have the higher order terms; the third the third derivative and the cube of the difference and so on. But, in the vicinity of the resonance, if you are close enough that difference is small and then higher powers can be neglected. So, to, if you just retain the first two terms. The first two terms will be –l.

Because at resonance, this is equal to minus l, so it will. The first term, will, of course, be minus l. And then, the second term will be this derivative times the difference in the independent parameter, which is k square. So, this derivative which is the first order derivative is what I have written as bl, b l is the derivative with respect to k square of this function, which is a function of k square.

You can see that kappa square is lambda 0 square + k square, which is why this is the function of k square, ok. So, if here we are. Now in this denominator, you of course, have a +l over here, and you are picking a -l over here. So, this -l and this + l will cancel each other, in the denominator.

And then, you are left with this bl times k square minus kr square in the denominator bl, being the first order derivative of this function with respect to k square. So, here you have 1 over b l into k square - kr square. And then, you have got this k to the power 2l + 1 behaviour.
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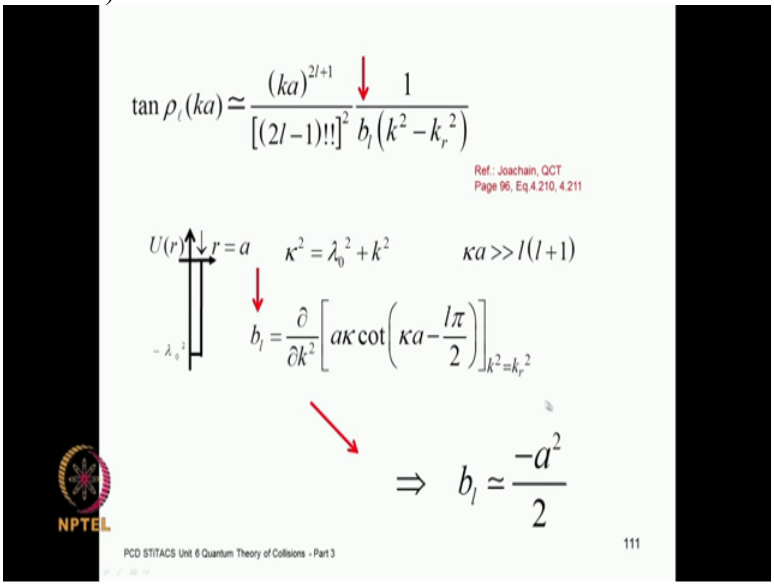

So, you need this bl. And for that all you need to do is to get this derivative which you can get quite easily and find the value of the derivative in the vicinity at the value of k square, when k square is equal to kr square. So, that result turns out to be - a square by 2. So, I will put this value of bl over here. And then, we will analyze the tangent of Rho.
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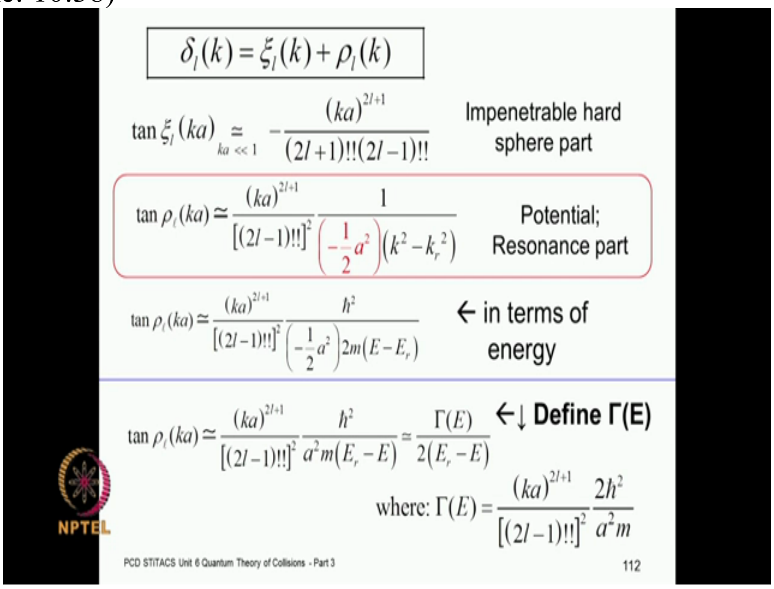

So, that value is not placed over here, which is minus half a square. And now, we had written in terms of k square, but h cross square k square by 2m being the energy, we can write this also in terms of energy, which is e square minus Er over here. But then, you have got this h cross squared over 2m. So, this same expression has been, now written in terms of the energy.

What we now do is to define a function gamma E which will be related to the width of the resonance as we have. In fact, in anticipation of this I had introduced this in one of the earlier classes as well. But here, we define it explicitly that this tangent of Rho, we, which we have obtained over here, okay.

We introduce a parameter gamma which contains important properties of the resonance. In fact, it will contain the width of the resonance and this is now defined. This is the definition of gamma that gamma is such a function, such that, if you divide by 2 times Er - E Er, being the resonance energy, you get the tangent of Rho.

So, the tangent of Rho, we have obtained independently which is given by this middle expression here. So, this has been obtained explicitly and if we write this as if it is a ratio of gamma over 2 divided by Er - E. Then, that gives us the definition of gamma.

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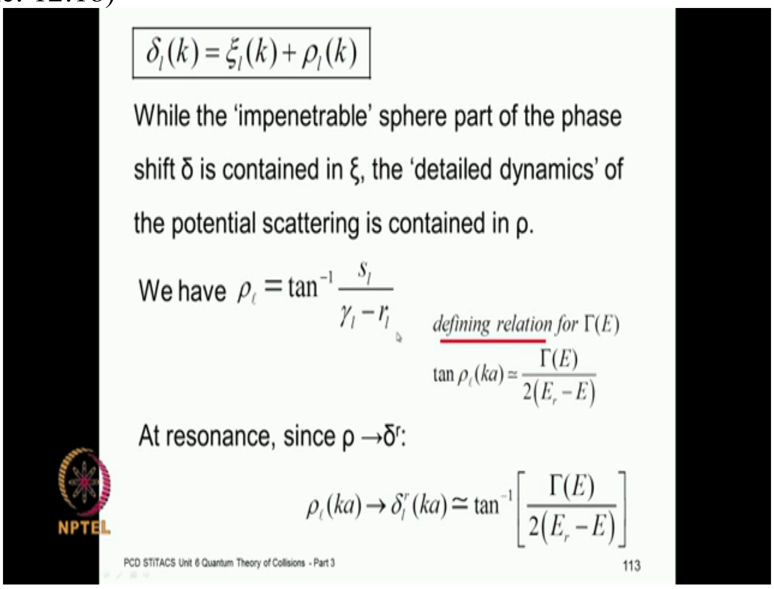

This will be the resonance width as you will see. So, this is the slowly varying part of the phase shift Rho, is what contains the dynamics. We have written the ratio in terms of s and r this is how we had obtained the expression for the residual phase shift Rho. And we have now defined gamma as this ratio. So, both of these are expressions for tangents of Rho.

This is tan Rho is given by the ratio s over gamma - r. This is tan of Rho, the same quantity which is defined as a ratio of gamma over 2Er - E. So, you can rewrite the expressions in terms of gamma instead of s and r. So, here because at the resonance I will indicate the resonance phase shift the residual phase shift as delta with a superscript r.

So, this delta superscript r is, nothing but the remaining phase shift other than the impenetrable hard sphere phase shift that remaining part of the phase shift, add resonance, we are using a specific symbol to represent this, which is delta with a superscript r and at the resonance Rho, is now given by Delta superscript r, which is tan inverse of gamma over 2Er - E, by the definition of gamma that we have introduced.

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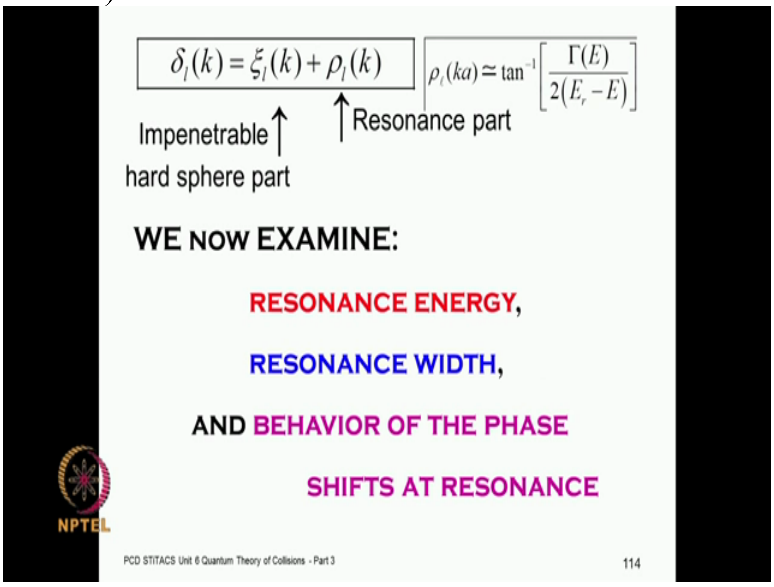

What we will do now is to examine the resonance energy, resonance width and the behaviour of the phase shifts at the resonance, okay.

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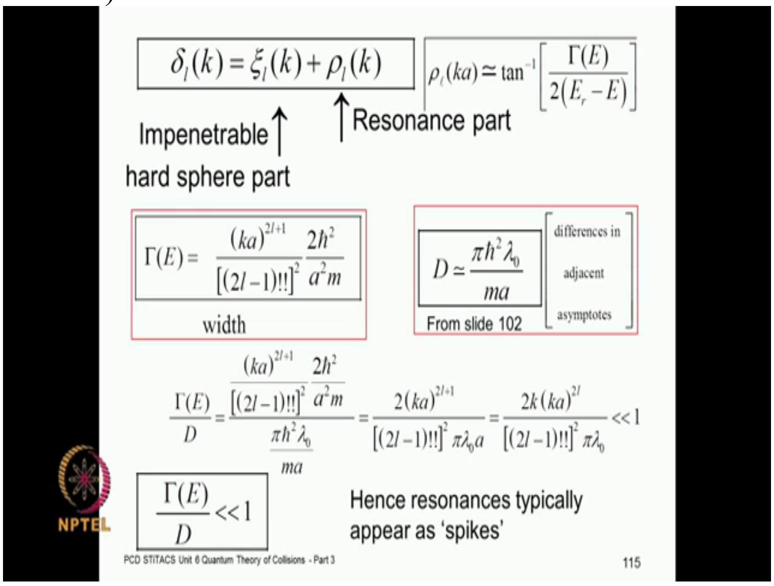

So, here you have got the resonance width which is defined according to the previous relation. And then, in our previous class, we had obtained the difference in the adjacent asymptotes. So, you will remember this expression from the previous class. It is there on slide number 102. So, now if you take the ratio of these two gamma over D, okay.

You find that this ratio is a rather small number, okay. Now, what it means, is that the widths are small compared to the energy spacing between the resonances which is why the resonances will appear as spikes, okay, because the energy spacing between the resonances is given by D capital D okay.

Which is the spacing between the adjacent neighbouring asymptotes; we obtain the expression for D explicitly all. All we are doing now is to take the ratio and the resonances typically appear as spikes.

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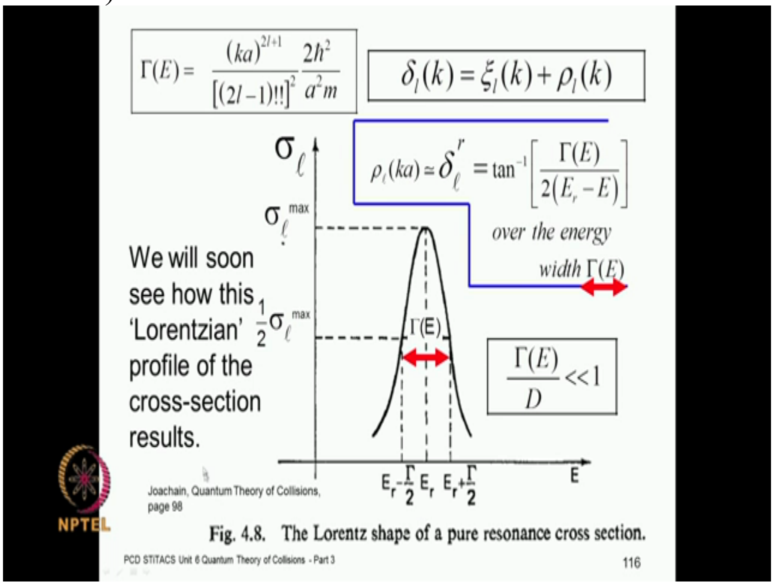

So, here is an example and here you have the resonance phase shift which is given by tan inverse of this ratio. And in this width the scattering cross section will rise sharply go to a maximum at the resonance energy and then it will come down. So, this is a rather pure resonance most of the resonances we meet in atomic physics they are not pure resonances. But, I will tell you why, they are not pure.

But this is a pure Lorentz's kind of Resonance. So, so this is the resonance cross-section figure. This is from Joechain's book on page 98 okay. And this is the behavior we will get. And we will now discuss how we obtain this Lorentzian and shape.

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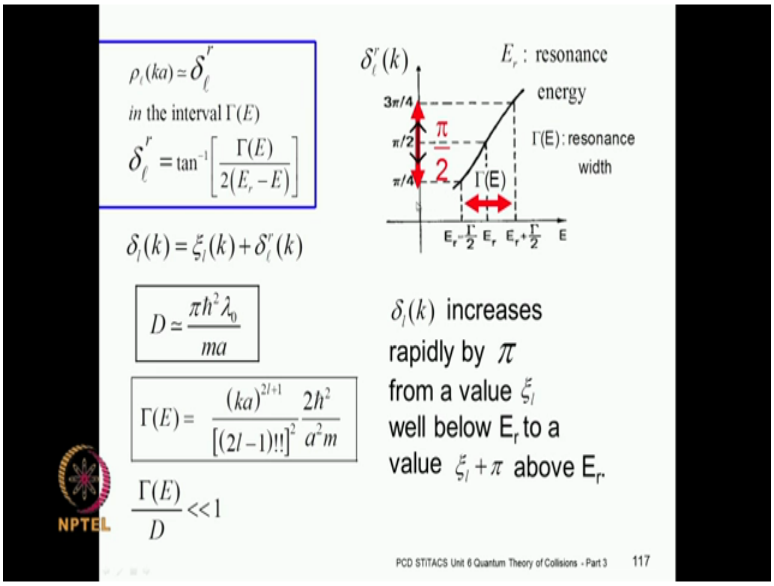

What is happening is that in the vicinity of the Resonance, the phase shift as I mentioned in our previous class. It goes through a change in pi very rapidly through pi by 2. So, from here to here, very close to the resonance it changes rapidly through pi by 2. So, it goes from pi by 4 to 3pi by 4, the difference here is pi by 2, okay.

But from an energy which is somewhat below the resonance to an energy which is somewhat above the resonance, the total phase shift is a quarter of a pi here and a quarter of a pi there. So, the total phase shift is pi, when you sweep the resonance, when you go across the resonance. But right at the resonance, in the immediate vicinity of the resonance, within what you call as the width of the resonance.

That is where the phase shift changes very rapidly through pi by 2. And the these resonances appear as spikes as sharp variations in the cross section because the widths of these resonances is somewhat small compared to the spacing between the resonances which is given by the distance between the adjacent asymptotes which we discussed in the previous class.

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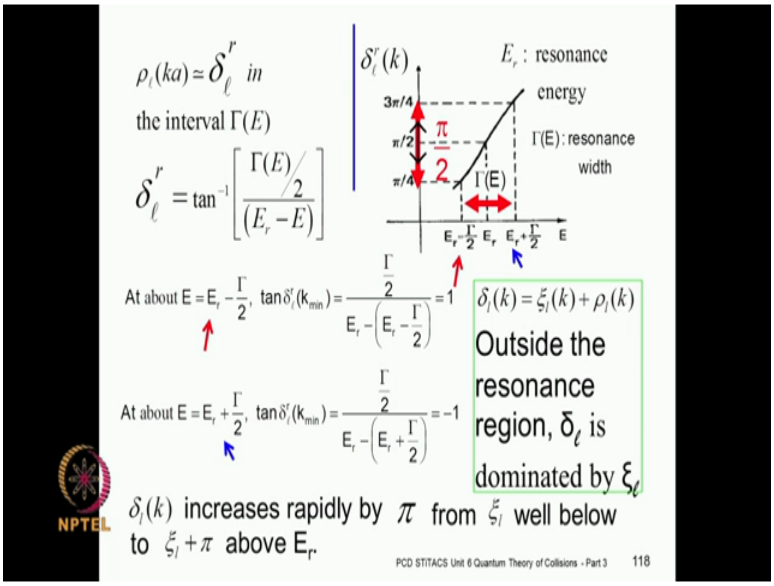

Now, let us examine this figure a little closely. Let us look at this particular energy. This is resonance energy minus half gamma. So, this is the gamma width, okay. You, there are two points of significant interest which is half gamma below the resonance and half gamma above the resonance okay. These two points are of significance. So, let us look at what happens at this energy which is half gamma below the resonance value.

And at this energy, if you look at this expression okay, you will have E = Er - gamma by 2. So, I put this E = Er - gamma by 2. This Er and this Er cancel, the gamma 2 divided by gamma 2 becomes +1, okay. So, the value of the tangent of the phase shift becomes +1 at Er - gamma by 2. And these are somewhat significant points. What happens at the upper value which is Er + gamma 2? So, this is Er + gamma 2.

This is the upper value over here. And over here, if you put this E to be Er + gamma by 2, Er cancels Er. But now, you have got gamma 2 divided by this -gamma by 2. And the ratio becomes -1. So, at these two points, the tangent of the phase shift, over here the tangent of the phase shift is plus 1. And the tangent of the phase shift at this point is minus 1. So, this will play a significant role in our analysis as you will see.

So, outside the resonance region, of course, the most of the phase shift is coming from the impenetrable part. So, that is the one which is dominating the region outside the resonance. But in the resonance region, the phase shift delta superscript r or what is our Rho makes an important contribution.

And in the resonance region, the net phase shift will go through pi as you go from well below the resonance which is here, too well about which is here. So, you have got pi by 2, variation in the immediate vicinity of the resonance and a quarter of a pi below that and a quarter of a pi above that. So, the total variation the phase shift is pi.

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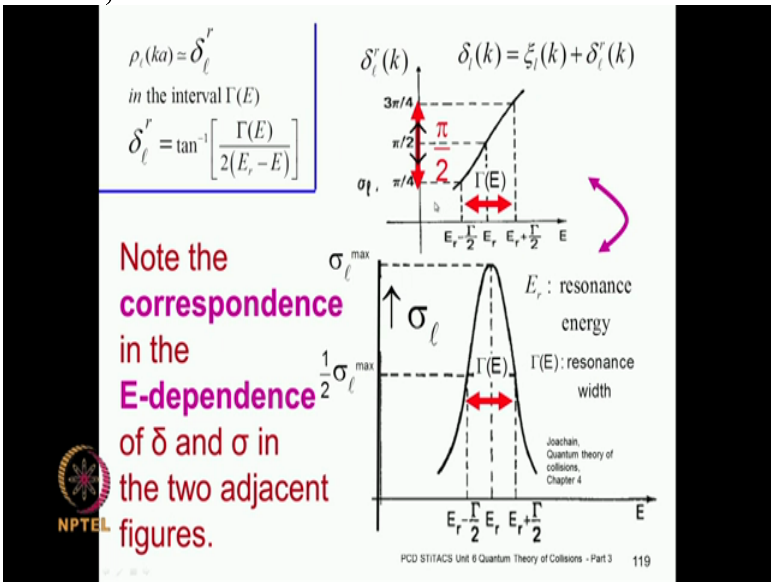

Now observe the correspondence between these two figures. Both are plotted here in the upper figure. I have got the phase shift which is the resonance phase shift delta superscript r and in the lower figure, I have got the scattering cross section. Now, both have been plotted against energy. And the energy horizontal axis is more or less, you know, it is it has the same scaling.

So, the point Er - gamma by 2 for this figure comes right above the point Er - gamma by 2 for this figure. The point for Er + gamma by 2 comes right above the point for Er- E, Er + gamma by 2 for the lower figure. So, this is the correspondence as the phase shift goes through pi by 2 in this immediate vicinity, in the width gamma.

This is what is called as half gamma okay, the cross section which is maximum over here, at these points; the cross section becomes half of that. So this is the scattering cross section. This is the maximum. And this is half the scattering cross section. So, which is why, this gamma is sometimes called as the full width at half maximum okay f, w, h, m is what you will often read in literature.

That full width, it is the full width right from the lowest point of this range to the highest point of this range, which is what, which is why, it is called as the full width. At half maximum, because this is where the cross section at these two points if half of this for a pure Lorentzian resonance, okay. But the pure Lorentzian resonance is a special case. There are many other complex features which I will be discussing.

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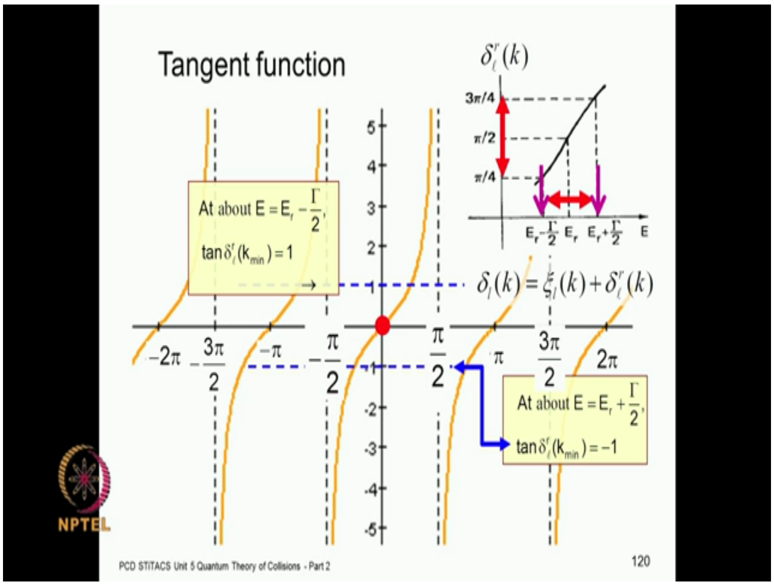

So, now you we will study the tangent function, which all of you are familiar with from your high school days. So, this is the usual tangent function. This is the zero of the angle, on the x-axis, the y-axis is the tangent function. So, this is the zero of the tangent function. And then, you have got pi by 2, pi, 3pi by 2, 2pi and so on.

And then, on the negative angles, you have got -pi by 2, -pi, -3pi by 2 and so on. Now, what did we find that at Er - gamma by 2, the tangent of the phase shift was equal to +1. So, this is this horizontal line blue line corresponds to tangent of the phase shift equal to +1. And you know that this is where the significant part of the resonance where the phase shift changes rapidly through pi by 2 okay.

That is the onset of this significant part of the resonance. So, here the tangent of the, this phase shift is equal to 1. So, I draw a horizontal line for tan theta = 1 where theta is the resonance phase shift delta superscript r. Likewise here you have -1. When do you have -1? This is when the energy is Er + half width of at full maximum, right.

So, this is this is a full maximum with this is where the scattering cross section becomes half and at this point. You have got the tangent of the phase shift becomes -1. So, here is this horizontal line corresponds to tan theta = -1. So, now, let us look at this point over here. Now, this point is the intersection of the tangent function. This yellow line or orange colour or whatever colour it is amber maybe.

So, this colour, this tangent function intersects tan theta = 1, this blue line at this point which is right below the cursor that you can see in the figure, right. So, this is where the tangent function intercepts tan theta = 1. So, this is where it has got a value 1. And if you go to pi by

2, angle above it, because from here to here, as you go to the point to the energy, when the energy becomes half gamma above the resonance energy, right.

So, at this energy the phase shift would have changed from here to pi by 2. So, you take a step pi by 2 to the right and you come to this point. This is where you have the -1, right. So, this is the picture that emerges and it tells us how the phase shift or rather how the tangent of the phase shift changes across the resonance. The phase shift itself changes as tan inverse of this function.

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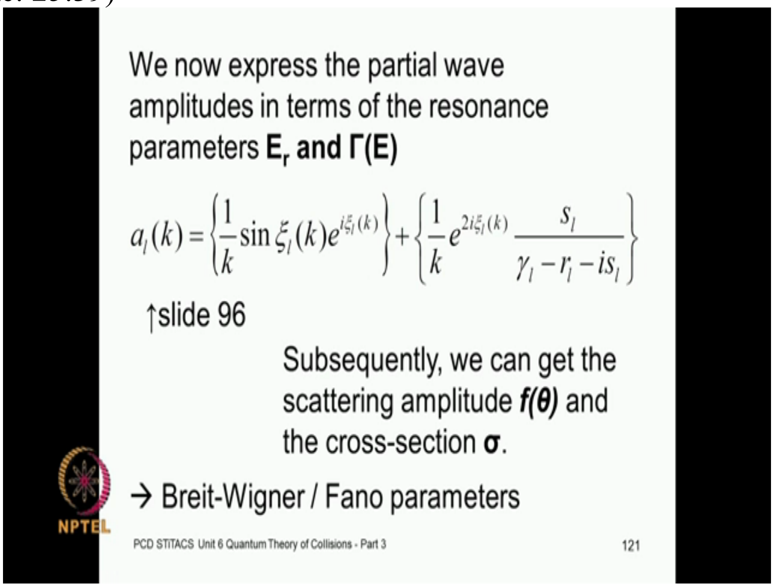

What we will do now is to express the partial wave amplitude in terms of the resonance parameters, because now, we have introduced these resonance parameters. Er is the resonance energy, gamma is the resonance width and in terms of these, we will analyze the partial wave amplitude which we have obtained explicitly in terms of these two parts.

So, Xi is the part coming from the impenetrable hard sphere part. And then there are other factors which were introduced in terms of the parameters r and s and of course, the logarithmic derivative of the wave function. And once we get the partial wave amplitude we can then get the scattering amplitude.

And once you have the scattering amplitude, you can get the scattering cross section itself which is a major quantity of interest. And this will lead us to the Breit Wigner formula and subsequently to parameterization of the resonance profiles in terms of what are famously known as a Fano shape parameters Queuing and epsilon which I will be defining in the coming few classes.

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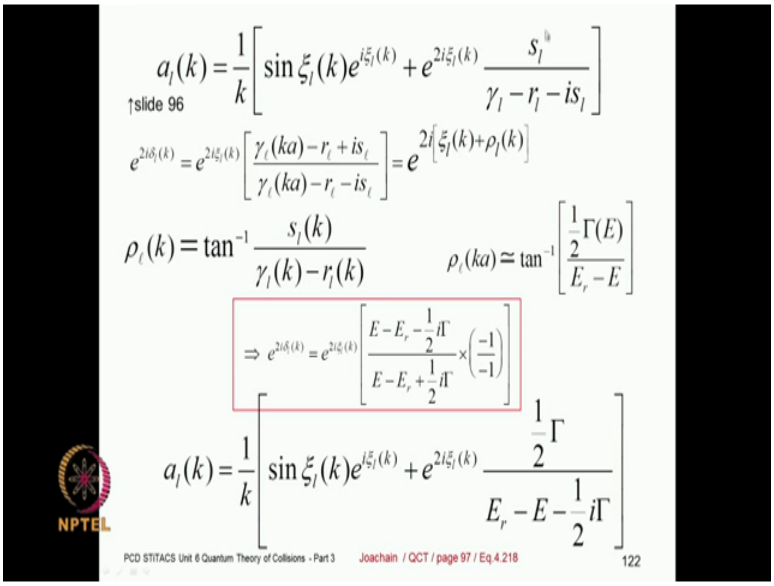

So, here, this is the partial wave amplitude we have this separation of the phase shift to Delta in 2 Xi + 2 Rho okay. Rho itself was introduced in terms of s and r and the logarithmic derivative of the radial function. But now, we have defined gamma such that tangent of Rho is equal to this ratio which is half gamma divided by Er - E. So, now you can write using these two relations.

Because these are the corresponding ratios, both describe the same angle Rho, but here, this angle Rho is described in terms of ratio involving s gamma. And over here, this ratio the Rho angle is defined in terms of the resonance energy parameters gamma and the resonance energy.

So, instead of these quantities, you get the corresponding quantities from here instead of r and s. So, now you get E - Er - half i gamma. It looks as if the sign were reversed but that is only because both the numerator and denominator have been multiplied by -1. But otherwise it is essentially the same factor, okay. So, now you can write this partial wave amplitude instead of in terms of s and r and gamma.

Instead of these three, we define it in terms of this gamma this upper case gamma is different from this. This is the logarithmic derivative of the radial function. This gamma is the resonance width, okay. They have the same names, but different symbols. So, this gamma is the resonance width and in terms of the resonance width you, now have this expression for the partial wave amplitude. So, here you are.

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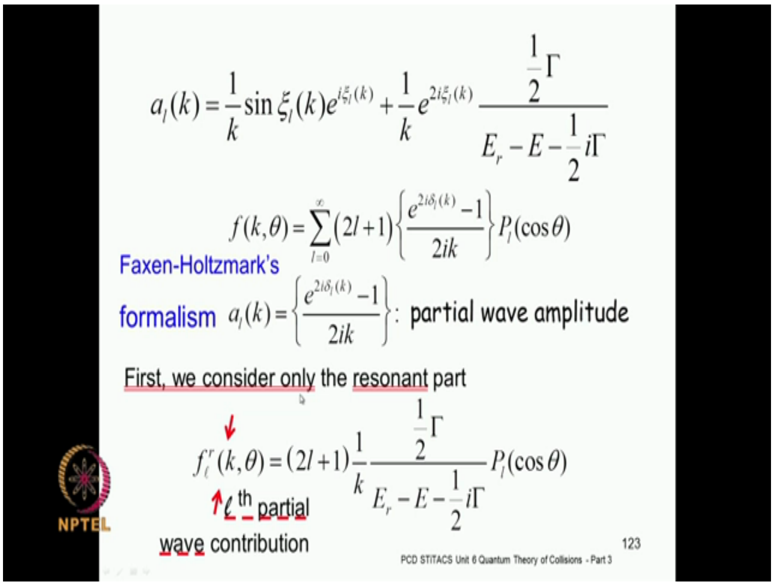

And we can put this in the Faxen and Holtzmark's equation for scattering, get the scattering amplitude and we can go ahead and get the scattering cross section which will give us the Breit Wigner formula. So, for the time being, let us focus on only the resonance part. So, we will pretend that the non-resonance part is not of importance, or it is zero okay.

So, let us first consider only the resonance part. This is what, will give us the expression, for what is called is the pure Breit Wagner formula, okay. So, this is the leading this is going to lead us to the pure Breit Wagner expression. So, for this in the expression, for the scattering amplitude, this is coming from the resonance part for the pure resonance part.

So, you have this 2l +1 coming here, okay. And then what is this alk? This is the partial wave amplitude but the partial wave amplitude is given by the sum of these two terms of which we pretend that the impenetrable part Xi, is not of importance. So, the whole first term does not matter. This term e to the 2 Xi becomes unity if you take Xi to be 0, right because that is not of any importance over here okay.

So, this is the impenetrable part which is not of importance for a pure resonance and then you are left with only this one over k which comes over here. And then, this ratio half gamma over Er - E -half i gamma which is here and then you have got this Legendre Polynomial pl cos theta. So, now, you have got the expression for the scattering amplitude, okay.

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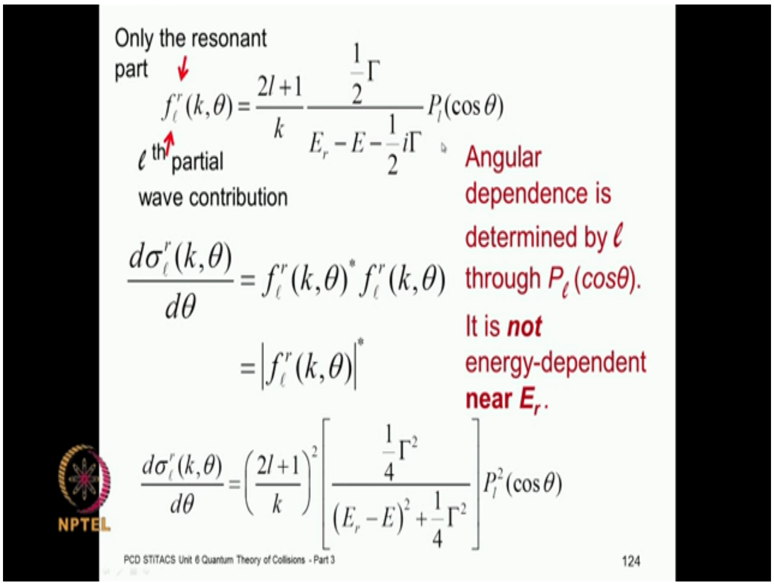

So, this is the, only the pure resonance part of the scattering amplitude. What is the differential cross section? It is given by the modular square of the scattering amplitude. It is f star f. So, it is a modular square of this. You already have this expression which is a complex number. So, you take this factor multiplied by its complex conjugate and you get the differential cross section to be given by the square of this 2l + 1 over k whole square.

And this is the modular square of this complex number which is one-fourth of gamma square Er - E square + gamma square over 4. And then you have got this pl cos theta. Notice that the angular dependence is determined only by this Legendre polynomial. There is nothing over here in the remaining factors which depends on angle, okay.

So, whatever angular dependence the scattering cross section has, the differential scattering cross section has, is coming only from the dependence of the Legendre polynomial on the angle because Pl for every value of l has bought a different dependence on theta which is given by Pl cos theta right. So, this is the only thing which determines the angular dependence.

And then, if you are very close to the energy when Er and E are almost equal, when energy is almost the same as Er. So, this will also vanish. And then, you have got 14 gamma square by 14 gamma square. And then, the energy dependence is also lost. There is no energy dependence because all the energy dependence is coming from, from here. So, all the angular dependence is determined essentially by l through Pl cos theta.

And this angular dependence becomes independent of energy because when you are close to resonance that factor also disappears.

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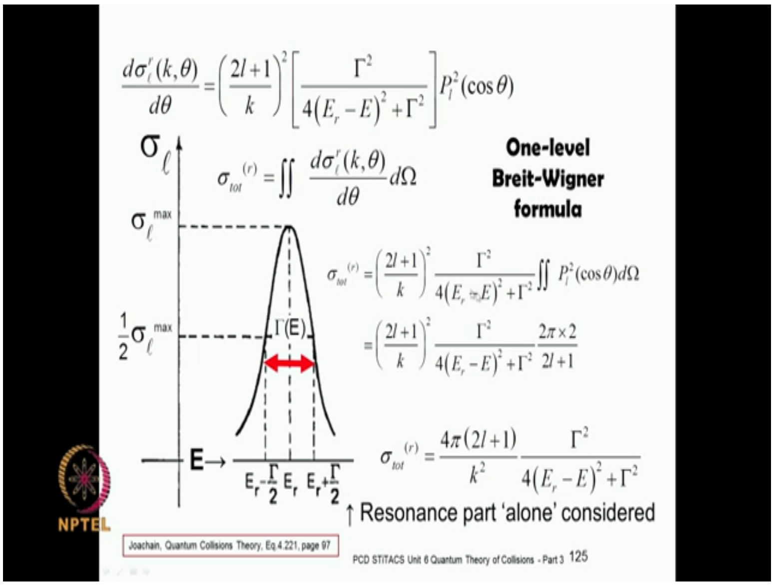

So, this is the picture that is emerging. And this gives us the Breit Wigner relation because this is a pure resonance. You, have the differential cross section given over here. You can get the total cross section which will, given by the Breit Wigner formula or what is often called as the one level Breit Wigner formula.

The total cross section you get simply by integrating the differential section over all the angles. So, all you have to do is to integrate Pl square cos theta over all the angles, right. So, when you do that, you get 2 pi times 2 over 12 +1 coming from this integration. And now, this 2l +1 will cancel one of these 12 + 1you have got a square of this over here. Now, you get the total cross section which is a resonance cross section.

This is just the resonance part we have ignored the residual part which is coming from the background. And what we get from this is, this gamma square over 4 Er - E square + gamma square. Sometimes you divide both the numerator and the denominator both by 4. And that is often a form, in which, you see this formula because then, you can write it as terms of, in terms of, square of the half width which is gamma over 2 square.

So, if you divide the numerator by 4 and divide the denominator also by 4 then, you will, this 4 will go away. But then, you will get the square of gamma by 2 here. So, that is often the form in which you will meet the Breit Wagner formula. So, in this expression, this is hm, we have considered only the pure resonance part or the resonant part alone has been considered. And this formula is what is known as the Breit Wigner, one level formula.

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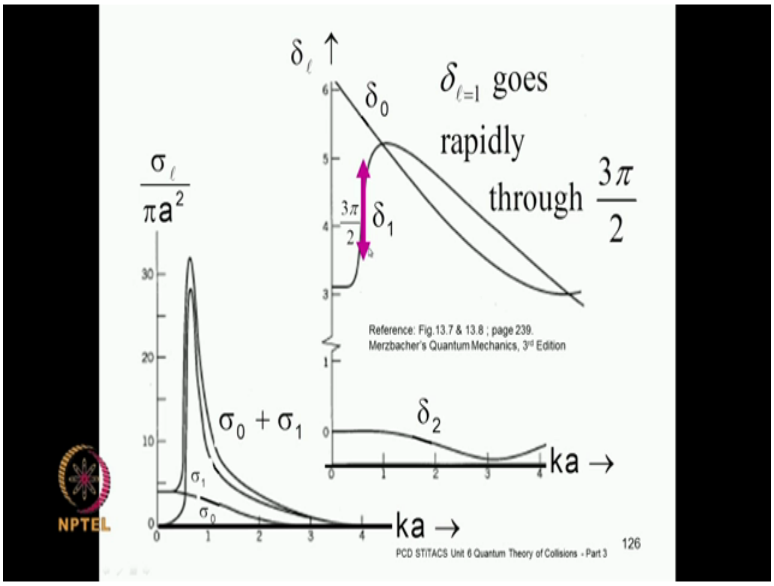

Now, here is an example from Merzbacher's book of scattering phase shift. You will remember that the scattering phase shift can go rapidly through pi by 2 or 3pi by 2 or 5pi by 2 and so on. So, here is an example which Merzbacher gives, in which, the scattering phase shift goes through 3pi by 2 for l = 1, partial wave. So, that is the P wave . The s-wave phase shift is varying smoothly. This is delta 0, the s-wave phase shift for l = 0.

This wave changes smoothly but the l =1, the p-wave phase shift in this narrow region it changes rapidly through 3pi by 2.So, here the angle phase shift is plotted in units of pi. So, this is about this from 4 to 5 is 1 unit of pi. So, this is about 3pi by 2. So, that is the difference of which the phase shift is changing and as the phase shift changes through 3pi by 2, the cross section in the l = 1 partial wave, sigma 1.

So, Sigma 1 goes, it goes through a significant change over here. Sigma 0 is changing somewhat smoothly. The total goes quite a bit up. And it is one example of a scattering cross section which is very nearly Lorentzian okay. But you will notice that it does have departure from a Lorentzian shape, because they are there is a background. There is contribution from delta 0 which is coming from the s wave scattering.

So, the resonance, the Lorentzian resonance formula, will give you the pure Lorentzian shape only for a particular l, partial wave. But there may be contributions to the scattering from some of the other partial waves. And the actual cross section profile may differ because of this. But then, of course, there are other reasons because there is a mixing which I will now comment on.

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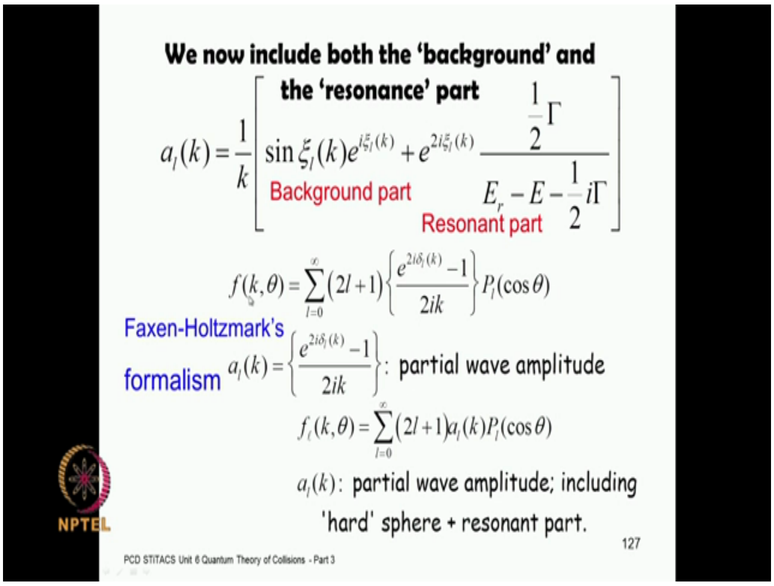

So, the mixing comes because, as we know, the partial wave amplitude is given by the sum of these two terms. And this term is not the only 1. So, in our previous discussion, we ignored the background phase shift. We set Xi = 0, as Xi = 0 also got rid of this term which is because this gives us e to the 0, okay.

So, this is the only thing that we talked about, which is, what gave us the pure Breit Wigner formula. And what we really should be doing is to consider both the background part as well as the resonance part. So, when you do that and then put this expression for the partial wave amplitude consisting of both the background part and the resonance part, then you will get a different expression for the scattering amplitude.

And for a different expression for a differential cross-section and for the total cross section as well. So, now we will the actual amplitude that should go in the expression for the scattering amplitude is the complete partial wave amplitude which includes not just the hard sphere part. But it also includes the resonance part. So, both the parts have to be included.

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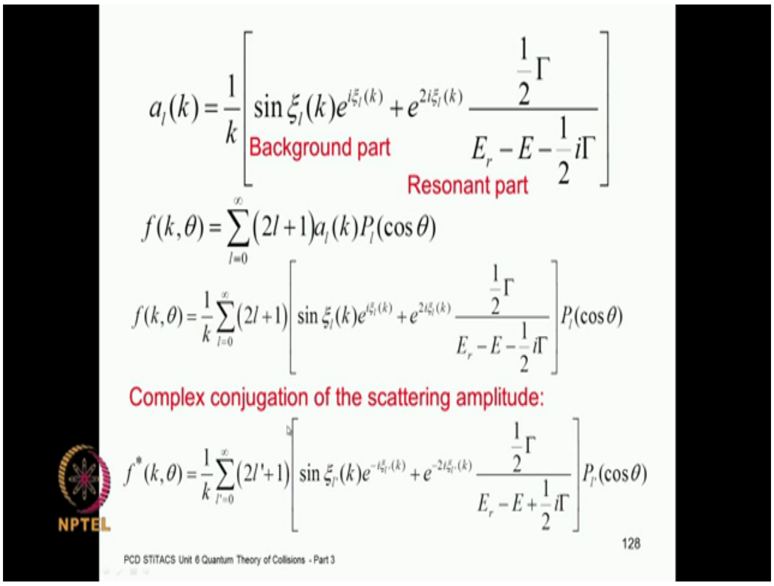

So, let us do that. So, these are the two parts and we should put them in the expression for the scattering amplitude. So, now we write the scattering amplitude as this is an infinite sum over all the partial waves, okay. Sorry. So, this is an infinite sum over all the partial waves. And here, this amplitude now, has both the parts.

The background part and the resonance part, there is a 1 over k which has been written outside the summation sign. And then, there is this Legendre polynomial which comes here. Now, what do we need? We need the complex conjugation. So, we write the complex conjugate of this expression here.

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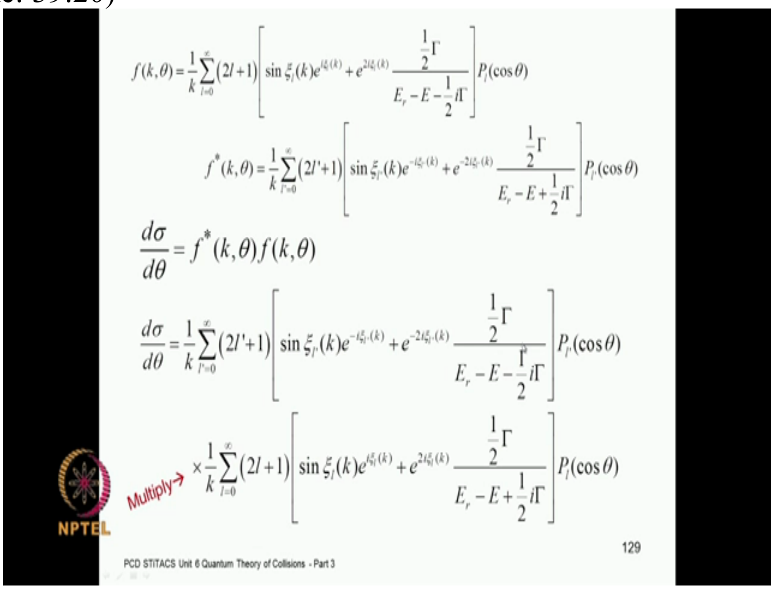

So, now we have got the scattering amplitude. We have got the complex conjugate of that. If you multiply the two, you get the modular square and you get the differential cross section. So, let us multiply these two terms. You have to be careful that here, you have got infinite sum, over the partial wave. So, l goes from 0 to infinity but it does so even in the expression for the complex conjugate.

So I have used a different dummy index over here, which is, l prime going from 0 to infinity; because it is an independent sum okay, although it is a sum over all the partial waves. So, you have got l prime going from 0 to infinity. So, this expression is the same as this except for complex conjugation and for a different dummy index.

So, now you multiply these two. You multiply the scattering amplitude by its complex conjugate. So, here is the scattering amplitude multiplied by the complex conjugate okay. So, you can work out this multiplication step by step. It is quite easy to do.

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$$\frac{d\sigma}{d\theta}=\frac{1}{k}\sum_{l'=0}^{\infty}(2l'+1)\left[\sin\xi_{l'}(k)e^{-i\xi_{l'}(k)}+e^{-2i\xi_{l'}(k)}\frac{\frac{1}{2}\Gamma}{E_r-E-\frac{1}{2}i\Gamma}\right]P_{l'}(\cos\theta)$$

$$\times\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\left[\sin\xi_{l}(k)e^{i\xi_{l}(k)}+e^{2i\xi_{l}(k)}\frac{\frac{1}{2}\Gamma}{E_r-E+\frac{1}{2}i\Gamma}\right]P_{l}(\cos\theta)$$

$$\sigma_{tot}=2\pi\int_0^{\pi}\frac{d\sigma}{d\theta}\sin\theta d\theta$$

$$\sigma_{tot}=\frac{2\pi}{k^2}\left\{\begin{array}{l}\sum_{l'=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)(2l'+1)\left[\sin\xi_{l'}(k)e^{-i\xi_{l'}(k)}+e^{-2i\xi_{l'}(k)}\frac{\frac{1}{2}\Gamma}{E_r-E-\frac{1}{2}i\Gamma}\right]\\ \times\left[\sin\xi_{l}(k)e^{i\xi_{l}(k)}+e^{2i\xi_{l}(k)}\frac{\frac{1}{2}\Gamma}{E_r-E+\frac{1}{2}i\Gamma}\right]\end{array}\right\}\int_0^{\pi}P_{l'}(\cos\theta)P_{l}(\cos\theta)\sin\theta d\theta$$

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So, these are the terms. So, you will get cross terms, this term multiplied by this. Then, this term multiplied by this. Then, you will have this term multiplying this and this term multiplying this, right. Now, notice that when you have this term multiplying this, you will have e to the -2iXi multiplying e to the 2i Xi, but notice there is an l prime here and there is an l here. But if l were equal to l prime, they would cancel each other.

But then, you also have to take the integration over the whole space 2pi is coming from the integration over the azimuthal angle Phi okay. Because there is an axial symmetry and then you are left with a residual integration over pi going from 0 to pi in the radial, in the spherical polar coordinate system.

So, now to get the total cross section, you will have to integrate this. And the only angular dependence is coming from the Legendre polynomials over here. So, now you have got a double summation 1 over l prime coming from here, 1 over l coming from here. So, you have got this double summation 2l prime + 1 multiplied by 2l +1, which is here, these two factors multiplying each other.

And then you have the angle integration 2pi is taken care of over here. The 1 over k, the 1 over k, give you a k square in the denominator. And then, you have got this integration of the Legendre polynomials. And now, you can use the Orthogonality relation for the Legendre polynomial okay.

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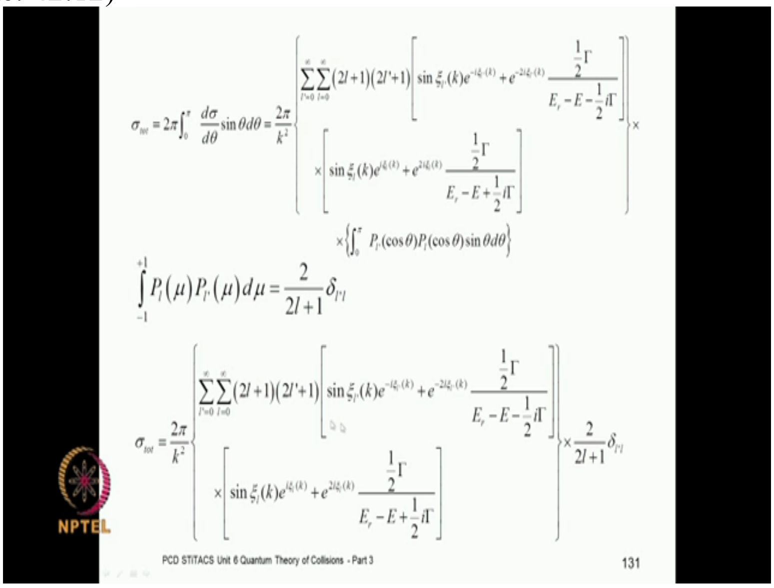

So, now if you use the orthogonality relationship you will get a Kronecker delta, delta l prime l. And then, you can contract the double summation so that summation over only one index will survive. And now, you have this double summation. But then, you have got this Kronecker delta which is coming from the orthogonality of the Legendre polynomials 2 over 2l + 1.

And then, when l prime becomes equal to l, this 2l + 1 will cancel one of these two which will both become equal when l prime = l, okay. So, it is a very simple straightforward simplification of this relation. They look big it would take a while to write it in your notebook on, or on the board. But you can see the physics at a glance as we just go through this, right.

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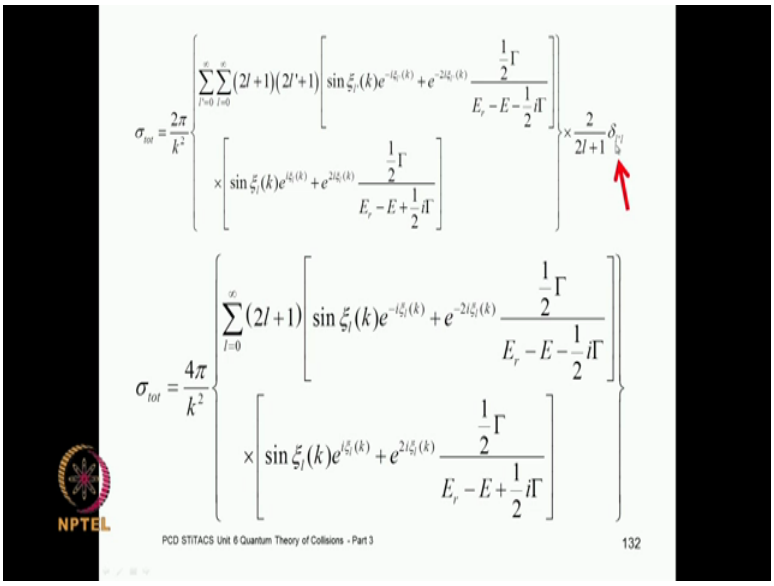

So, now after contracting over this Kronecker Delta, you are now left with only one single summation which is over l going from 0 through infinity. There is only 1 to l plus 1 factor because this to l plus 1 has killed one of these two. And now you have got these four terms sine Xi. Then, e to the minus 2i Xi coming from here, then this sine Xi is coming from here.

And then, this term and now, when these to multiply each other, both have the same partial wave quantum number l and e to the -2i Xi, will multiply e to the +2i Xi, giving you a factor of unity. So, in this cross product, the Xi will disappear.

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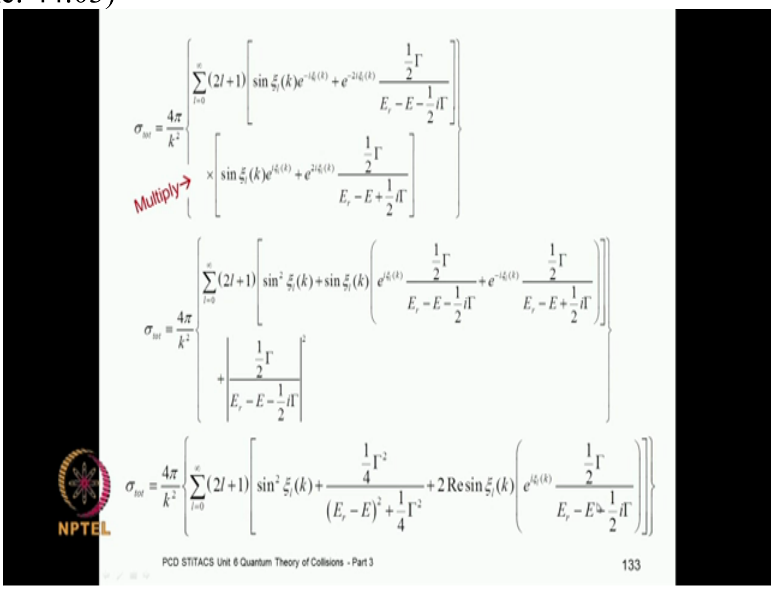

So, let us go ahead and carry out that multiplication. So, you have got all of those terms. So, in the last term, the Xi has disappeared. It is just the square of these two terms, okay. The modular square of these two terms because these are complex conjugates of each other, okay. And now, when you take the cross terms, you will have sine Xi times this. And here, you have got sine Xi times this.

So, sine Xi comes as common. And then, you have got this factor multiplied by e to the + i Xi and over here you have got - iXi. So, you have got these two terms. And the sum of these two is nothing but twice the real part of this complex number okay. So, now we have got a very simple relationship in which we have got the pure Breit Wigner part.

Then, you have got the background part. And then you have got the interference part also. So, the complete expression is now developed from this analysis for the total cross section.

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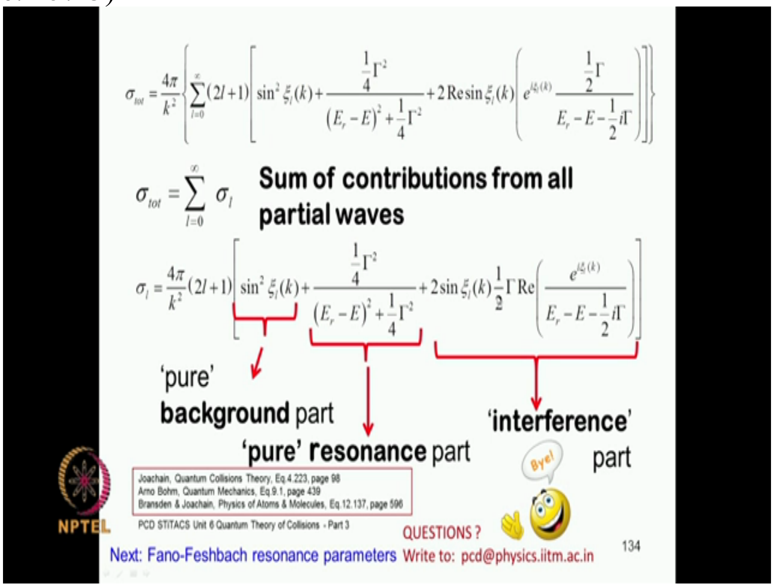

So, let us recognize these terms explicitly. So, you have got the total cross section, which is given by, this sum over infinite partial waves. We do know that we rarely have to go to very high partial waves. We just usually have to go to only a few partial waves. And in these, in this infinite sum, only few terms will be contributing in this infinite partial wave. And then, you have got a sine squares Xi, which is coming from the background.

And then you have a sum over partial wave of all quantum numbers l going from 0 to infinity. You have got all of these terms in the lth partial wave. So, this is a focus on the scattering cross section contributed by a particular partial wave with quantum number l. So, Sigma l is given by this. Of which, this is the background part.

This is the pure background okay. It then, you have got the pure resonance part. This is what we call is a pure Breit Wigner part. And then, there is an interference part which is what gives us the Breit Wigner extended formula which includes the pure resonance part on the background. So, in the next class, we will discuss the Fano resonance parameters.

In the next unit, we have I believe, 4 classes in the next unit. So, with this we conclude Unit number 6. And then, we will have unit number 7, in which, we will have 4 classes. In which I will introduce the Fano shape parameters. Is there any question?