Select/Special Topics in Classical Mechanics

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Module No. # 01

Lecture No. # 04

Equations of Motion (iii)

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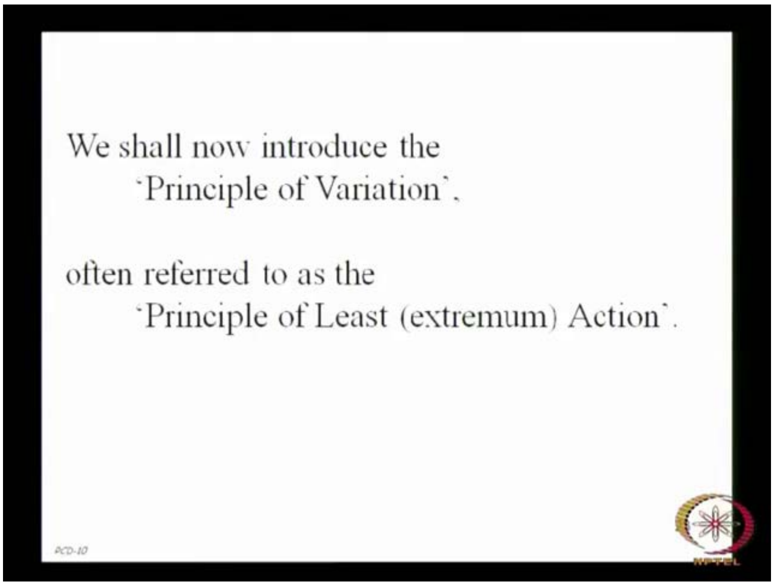

So, welcome again to the 4th lecture in this series of lectures on select topics in classical mechanics. This will be the third lecture on equations of motion, and what we learnt last time is that there is an alternative formulation of mechanics, not based on the principle of causality and determinism which is fundamental to the Newtonian formulation. So, this is an alternative formulation, this is based on what is called as the Principle of Variation or Principle of Least Action.

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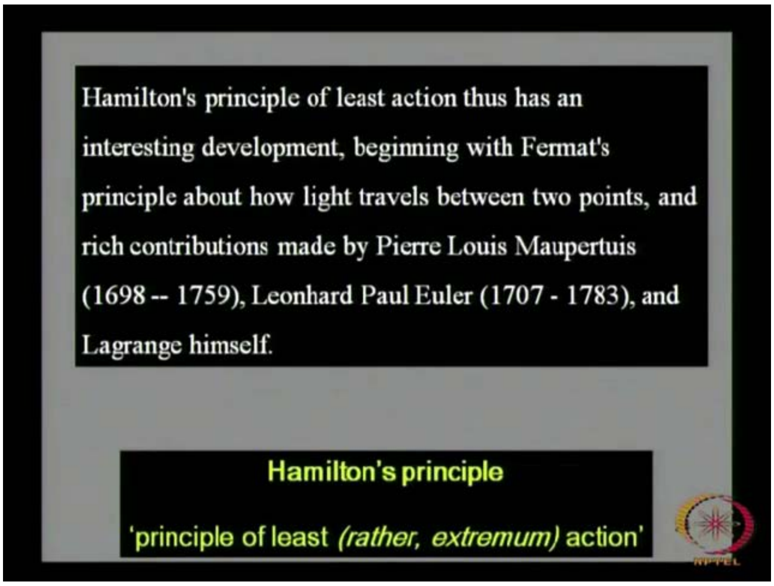

Now, this is also known as Hamilton's principle and this is named after Hamilton. This has a very interesting development. As I mentioned, it starts out with Fermat's principle about how light travels between two points, and we stated it in the last class and there were other contributions made by Maupertuis, Euler, Lagrange and various contributions. So, I will not be going through the entire historical development of the Principle of Variation.

But skip out of those steps and come straight to what is called as the Principle of Least action, and I will not comment very much on whether this is an extremum or a stationary point is. It a can it be a maximum these are somewhat deeper questions they require second order variations and we; obviously, cannot discuss them even before we discuss at least the first order variation. So, it is too early to talk about that.

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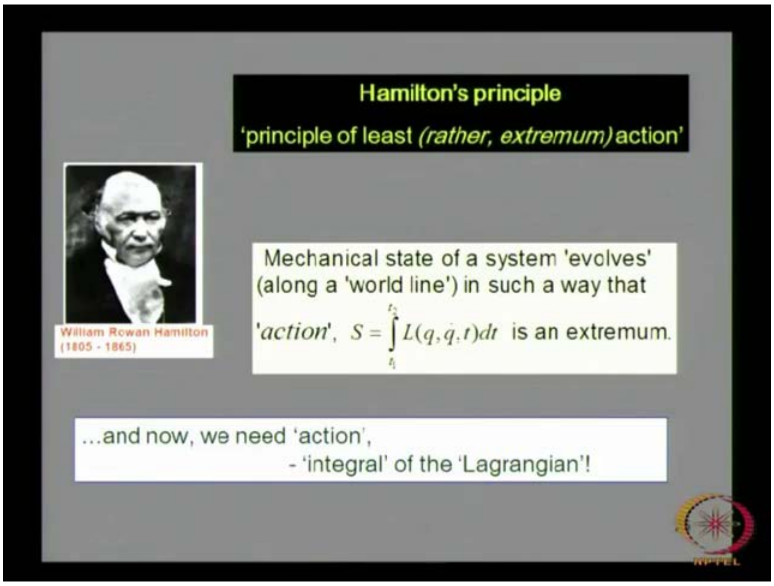

Now, this is the Principle of Least action. This was formulated and is known after William Hamilton. The statement is the following that the mechanical state of a system evolves in such a way that action is an extremum, an action is defined as this integral.

Now, we have stated this in terms of various unknown quantities. So, we are going to have to patiently wait to get a clarification of all of these issues. We have defined action as this integral, we have defined this integral of the Lagrangian which is a function of position, velocity and in general, it is a function of time as well, but we will be dealing with those situations in which the Lagrangian is not an explicit function of time, but I will explain this.

We have not told you what this Lagrangian is. We have only said that it is a certain function of q and q dot, but we have not told you what function it is. We are going to do that, but in anticipation of that, let us go ahead and state this principle and then as we discuss this further, what goes into this principle, which is the Lagrangian itself and its integral from t 1 to t 2; all these ideas will become clear as the discussion progresses.

So, we have stated the principle in terms of all these unknown quantities. We have also referred to the evolution of the system and this is evolution in a certain space. So, what kind of a space is it means, this is the space, this is the configuration space or this is the phase space, this can be the position velocity phase space, this can be the position momentum phase space, and the various possibilities that we can talk about, and in this phase space, the system will be described by a certain configuration some. If it is a single particle system with one degree of freedom, it will be described by a certain point in the phase space, and then it will get to another point in the phase space, and it could go from this point to this point either by going like this or by going like this or it could wander around, go outside this class room, come back, and on its way back it can visit Delhi, Bombay where not, anywhere. It could come back to Chennai, go back to Delhi and again come. So, it could even cross its own path. Does not matter.

So, there are all kinds of paths that you can think about, and this is what is called as alternative paths between two points. So, there is a start and a finish. Only thing that is not changing is the start and the finish, the system must start here and it must come here.

The question we are asking is how it gets from the start to the finish. And this question is answered by Hamilton's principle. So, this will obviously be applicable to any mechanical phenomenon. It could be applicable to a ball which is hit by a batsman in cricket. The ball undergoes it travels along a certain trajectory and you could ask that it started here, it finished there and how did it get there, and why did it take the particular path that it did, why did it not go by some other path.

And you could answer this invoking Newtonian mechanics in terms of f equal to ma and say that it was hit by a certain force, and this force caused acceleration, and then explain this. Now we are not going to do that, we will not invoke f equal to ma. We will not invoke force; we will not invoke the principle of causality. We will say that the ball goes from its start to the finish along a certain path in this phase space, and the path that it takes is sometimes called as the world line along which the object goes.

So, this is the idea of a world line. Then the system evolves along a world line in such a way, and this is the condition which must be satisfied, and on a satisfactory, when this condition is satisfied, a path is chosen by the system, it is a particular path along which this integral is an extremum.

Usually this extremum is a minimum. So, I will not distinguish too much; I will not comment too much about when it is a maximum, when it is an extremum, when it is a stationary point, these are matters these are important questions, subtle questions, but they require a consideration of the second derivatives which we obviously, cannot talk about unless we talk about the first derivatives and understand that scheme.

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So, action is defined as this integral along the world line of the Lagrangian, it is a time integral. The integration is carried over time. It is a definite integral from the start to the finish from time t 1 to t 2, and you know that the integral is the limit of a sum.

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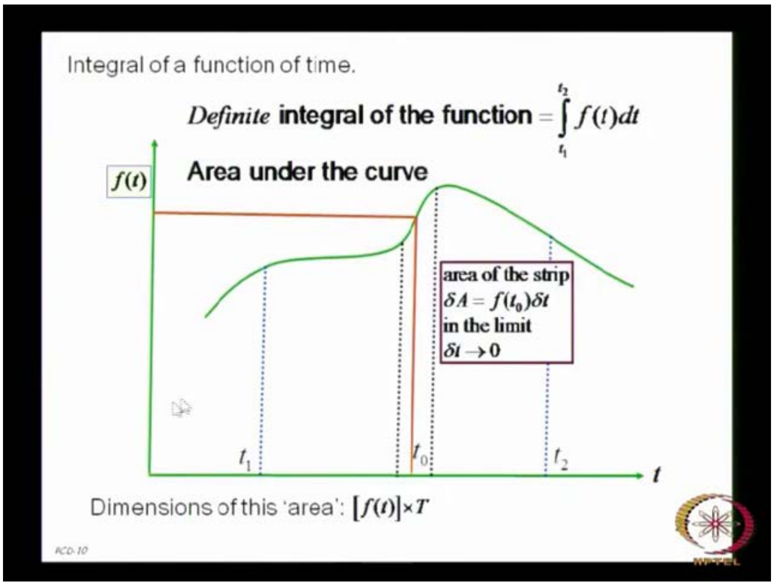

So, it has got a summation attribute, and we will recapitulate very quickly what an integral of a function is like. I am sure you would have studied this in your high school and other courses, but for completeness I will just comment on what an integral is.

So, you have got a certain function here, a function of time which is plotted against time, and you plot it in this range t 1 to t 2. If you consider a small interval around the point t equal to t 0, then if you look at the area under the curve, under this green curve in this little tiny strip and you consider this area in the limit that the time interval delta t becomes infinitesimally small, delta t going to 0, then you can easily see that the area of this strip will be the height of the strip multiplied by the width of the strip.

So, this area will be a product of the value of the function at t 0 multiplied by the width delta t in the limit delta t going to 0, and if you add up these areas all the way from t 1 to t 2, you get the integral of the function from t 1 to t 2. So, that is the idea of an integral. It is obviously, a summation. It is the net area under the curve, and it will have dimensions of whatever be the dimension of the function f times the dimension of the horizontal axis which is time. So, the dimension of this area will be f into t.

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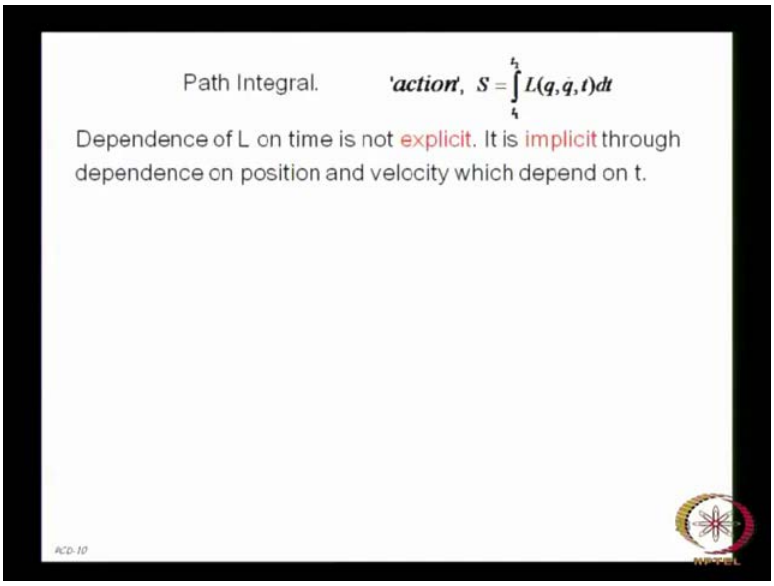

So, we are considering action as an integral, this is an integral of a certain function the function that we are talking about, the Lagrangian, and we are yet to define what the Lagrangian is.

So, we will begin an introduction to this Lagrangian. Then this Lagrangian is a certain function of the mechanical state of the system. So, it represents the mechanical state of the system. Typically this can be a function of the position, it can be a function of velocity, and in general it can be a function of time as well. So, the Lagrangian can be a function of position, velocity and time and there are two ways in which the Lagrangian can depend on time.

One is it can have an explicit dependence on time. Now when do you say that a certain quantity depends on time? It changes with time. So, if properties are different from the past to the present to the future, from yesterday to today to tomorrow, if the properties of nature are changing, then you have a system whose properties will be explicitly time dependent. But if you are looking at a certain object like you have got a certain ball which is rolling on this table, let us say, then the position of that ball which is rolling on this table depends on time. If I push this, it propagates. Their reflection and other things will determine the exact path, but at each instant of time, this object will have a different position.

And the position of this object is now a function of time. It is not that the laws of nature have changed from t 1 to t 2. When the object goes from here to here, the laws of physics have not changed. Whatever interactions are to be considered, those interactions are explicitly taken into account if you include all the interactions between the properties of the surface and how it interacts this, the perfection and everything is taken care of.

And not because there are certain unspecified degrees of freedom, if there are unspecified degrees of freedom with which this object were to interact, then depending on how these unspecified degrees of freedom influence the dynamics, you would have a different evolution along a world line. So, we are talking about a time dependence of the Lagrangian which can be of different character, one can be because there are unspecified degrees of freedom in the system, whose interactions with what you think is the part of the system will change its time evolution, or else if you have taken everything into account there are no unknowns.

So, there cannot be any dissipation. I think this is a good juncture where we can talk about what is dissipation, what is frictional loss; when you talk about a frictional loss. What you mean by a frictional loss, energy is lost. Where does it go, where can it go? If a certain amount of energy is created at the beginning of the universe, it is going to get converted from one form to another to a third to whatever, but you and I have no mechanism to create energy nor do we have any mechanism to destroy it.

So, basic interactions in physics that you talk about must conserve the total energy. So, what is it that we talk about when we say that energy is not conserved? In certain processes like even in this process if you now take into account the details of the interaction between this watch and the surface and ask what is happening as it moves, why does it come to a halt, it is because the friction is stopping it, and then you will have to ask what is this friction? The friction is all of those minuscule of forces which operate between everything on this surface which rubs every part of the surface underneath this wristwatch. And those are the unspecified degrees of freedom. If I draw a free body diagram, just the forces on this watch, and if I were to include only the push by my hand, everything else becomes unspecified degrees of freedom.

But those interactions are also electromagnetic in nature. There are some additional attributes, quantum mechanics also plays the role, but that is the matter of detail. So, if you consider system in which all the degrees of freedom are included and there are no unspecified degrees of freedom. So, there are no losses which means that the properties of the system cannot change from yesterday to today to tomorrow. There will be no explicit time dependence.

However, the position of an object could be time dependent, and the Lagrangian will then have time dependence via it dependence on the position. So, the Lagrangian does not depend explicitly on time because all the degrees of freedom are included, the laws of nature are not changing from yesterday to today to tomorrow, but the Lagrangian depends on the position, and this position depends on time and therefore, the Lagrangian depends on time.

So, this is one mechanism of the Lagrangian having a time dependence. It could also depend on time because the Lagrangian depends on the velocity, and the velocity can depend on time. And whenever you have an object which is accelerated or decelerated which is the same thing with a negative sign, right, then again the Lagrangian could depend on time because the Lagrangian depends on the velocity and it is the velocity which depends on time. So, when you have a function which depends on an independent variable, time is the independent variable. It flows by itself. It flow is not controlled by anything else; it does not depend on anything else. So, it is an independent variable.

The Lagrangian is a dependent function. It depends on time not explicitly, but through its dependence on position and through its dependence on velocity and not directly. When you have such a description, you talk about what is called as an explicit dependence and what is called as an implicit dependence.

If it depends directly on time because the laws are different or because there are some unspecified degrees of freedom which will interact with the watch and change the system evolution, right, if that what were to happen, then the Lagrangian would have explicit time dependence. So, that is not the kind of system we are going to be working with. But it is possible in principles and there are ways to deal with such systems as well. So, the dependence of the Lagrangian on time is not explicit, it is implicit through the dependence on position and velocity which in turn depend on time.

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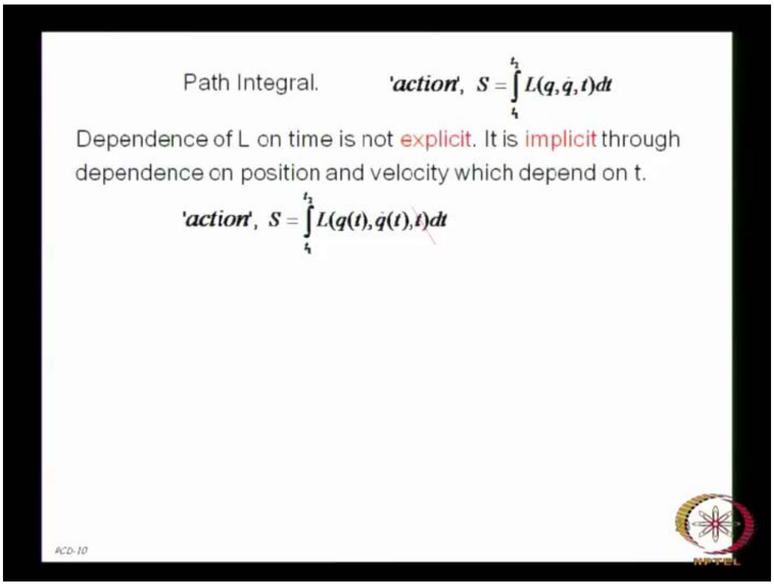

So, I have struck off t, there is a line crossing this t to remind me that I will not consider explicit time dependence. But implicit time dependence is explicitly shown by writing q as a function of time t. So, this is an implicit dependence on time, but shown explicitly by writing t as an argument of q and also writing t as an argument of q dot. We have still not told you what function it is. We have only said that it is the function of this and what function is it, how do you describe this function, is it a quadratic function, does it go as q squared, does it go as q cube, does it go as a polynomial function of q, a q plus b q squared plus c q cube; those things we have not answered as yet. We will of course, have to do that.

Obviously this system evolution cannot be shown on a two dimensional surface. In the previous figure when we had a time dependent function, let me go back to that again here in which we talked about the integral of a function. This could be plotted on a two dimensional surface, you can plot it on your paper or the blackboard or the screen of a laptop or whatever.

You can plot this on a two dimensional surface, but here you are talking about a time dependence which you cannot show on a two dimensional surface, but the meaning of area under the curve, now this is not an area in the two dimensional space. The idea of the integral being a limit of a sum, the idea of the integral being having an additive property, and this is important because one of the very important properties of action is that it has got this additive property and that is manifest from the fact that it is expressed as an integral and integral is essentially a sum. So, it has to have an additive property. So, all of those properties hold, but you cannot show it on a two dimensional surface.

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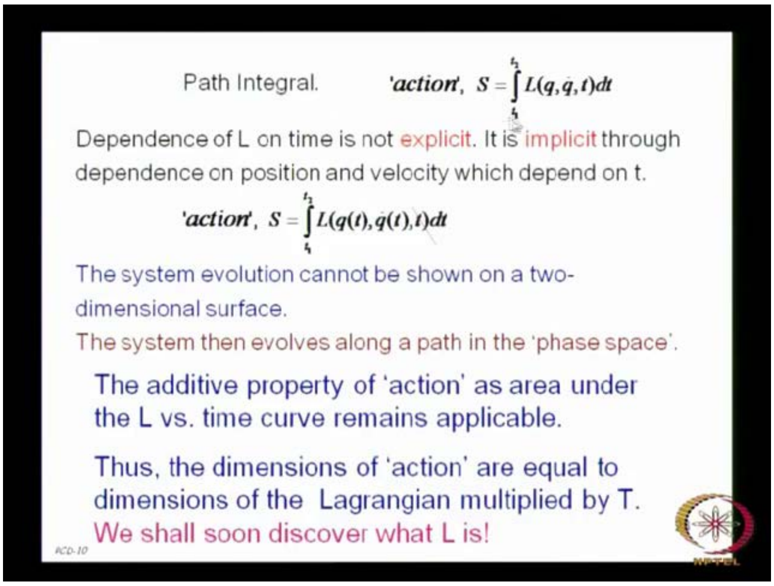

The system evolves along a path in a multi dimensional space which will be referred to as a phase space. What will be the physical dimension of action; action is this S, it is the result of this integration, S is equal to this. So, this is the meaning of action; action is not one of those things you see in movies like action movies especially the chinese action movies are I believe much more fun to watch than the hindi or tamil action movies. So, the action here is this integral, it does not come out of the chinese movies. So, this is action, this a definite integral of the Lagrangian from t 1 to t 2, and we are going to have to find out what this Lagrangian is.

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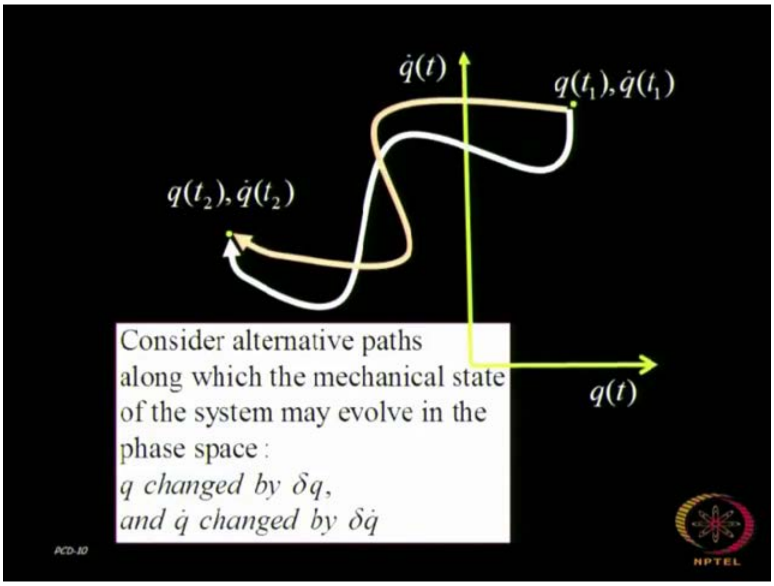

So, let us consider the evolution of the system and I consider a position velocity phase space. So, the x axis I have got the position which is a function of time, on the y axis I have got the velocity which is q dot which is also a function of time. I have got a certain start which is at time t1 at which the position is q at t 1, the velocity is q dot at t 1. So, this is the start point and this is the end point; this is the finish point, and this system travels along let us say this path which is just one of the many paths. You can ask me why did I draw this particular path. I would say that you do not like this path, I will draw some other path, and then you would ask me why did you draw other path. So, it is some arbitrary path to begin with.

What we want to find is; which is the path that the system will take to get to the finish from the start. Is it this path or some other path? If it is this particular path what is it that makes this system take a particular path? That is a question that you want to ask and when you answer this you would have solved the mechanical problem because this is exactly what a problem in mechanics is. How do you describe the system and how does it evolve with time. From time t 0 how does it get to a later time t, from time t 1 how does it get to t 2; this is evolution.

The question in mechanics is posed by asking how does a system evolve with time, it requires for its consideration, how do you first of all describe the mechanical system, we have agreed that the mechanical system is described by position and velocity in the present context rather by a function of position and velocity which is the Lagrangian.

We are yet to figure out how to express this Lagrangian as a function of q and q dot. But we are now learning to pose the question before we answer it precisely. So, the question is posed by asking which particular path will the system take to get to the point of finish, where its position and velocity will be respectively q at t 2 and q dot at t 2. It can take alternative paths. Here is another alternative path that I show in this figure. It could take paths, thousands of paths you know different any different path that you might think about, and these different paths can be considered conceptually in principle in your mind out of which the system will choose such a path along which if you evaluate the integral of the Lagrangian, the time integral of the Lagrangian from t 1 to t 2, then it must be an extremum more commonly the minimum.

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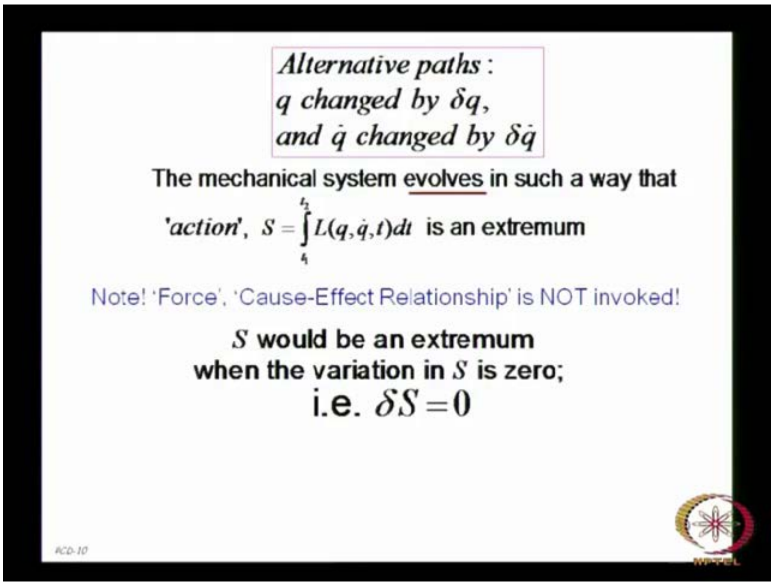

That is the answer to the mechanical problem that the system evolves in such a way that action is an extremum. You can consider alternative paths, and along these alternative paths, q would change say from q to delta q, q dot would change from q dot to q dot plus delta q dot, right, everything could change, but the action will be an extremum along the path which the system would choose to get to the finish from the start. And if you discover that path, your problem in mechanics is solved, you have something to celebrate.

We have not talked about the force. We have not talked about the cause effect relationship. We have not referred to the principle of causality. We have not said that the cricket ball is going to go there because it is hit by certain force. No. The cricket ball will go along such a trajectory in the phase space such that the time integral of the Lagrangian for this cricket ball would be an extremum. So, we make no reference to the cause effect relationship.

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If it is an extremum, as we saw in the figure, which I have on the board, then you have a variation would vanish, right, it is the ball is raised, let me draw that once again.

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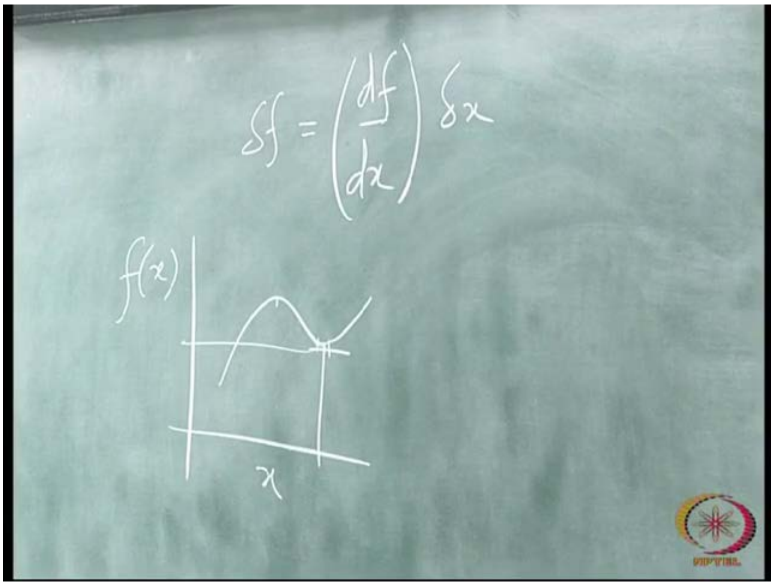

So, if you have an extremum either here or a minimum over here, then what you are plotting is f of x against x, delta f at two neighboring points over here is the rate of change of the function with respect to x multiplied by delta x for the two points.

If this is an extremum, this derivative vanishes, and delta f therefore, goes to 0. That is what you have in the screen. But if S is an extremum, then delta S must go to 0. It is the statement of the extremum.

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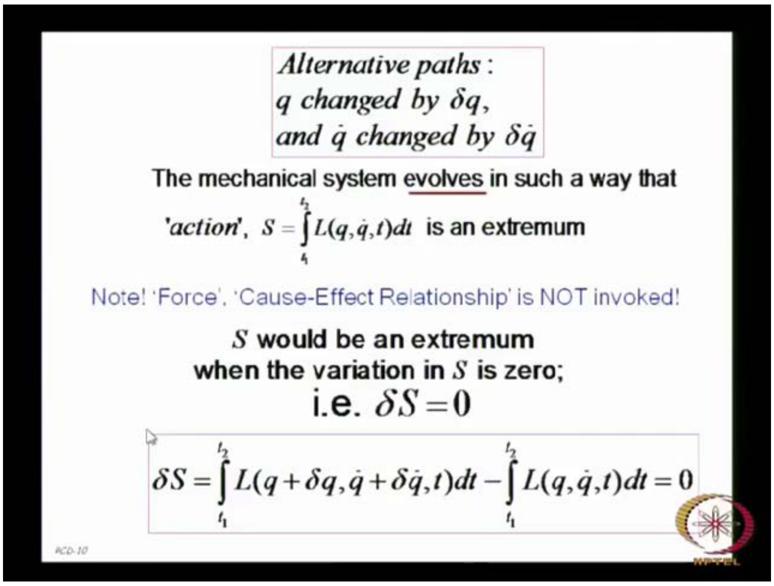

And now let us write it more fully because we know what S is. S is this integral of the Lagrangian. So, delta S is the value of this integral considered for an alternative path which is another point in our one dimensional example here. So, you consider an alternative path which is different from the one that you were talking about, which was our reference path, and the difference is indicated by writing the argument as q plus delta q, and the velocity argument as q dot plus delta q dot.

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So, if you consider this time integral for an alternative path which goes for which the arguments position arguments are q plus delta q, and velocity arguments are q dot plus delta q dot, and subtract from this action, the action along the path that the system would actually take which is this integral L q q dot t; that is the reference path. That is the path that the system would actually take, no matter what it turns out to be…

How do you know that that this system takes exactly the path?

This is what the principle of least action tells us.

So, we're sure that…

We do not derive it from anything else. This is the fundamental principle. What we do is we make no attempt to prove that this principle emerges from anything else which is more fundamental than this. What we do is to state that this is the fundamental principle, this will tell us how a system will evolve the time, and we subject it to all kinds of test. If you find no exception to its predictions, then it becomes the law of nature. It is just like the fundamental principles in Newtonian mechanics. Like the first law and the second law, we did not deduce it from anything else.

Why is acceleration proportional to the force? How do we know that it holds? We know it because we learnt it from Newton's insight. Newton recognized the fact that equilibrium can be changed, and if and when it does change, the change manifests as acceleration which is directly proportional to the cause. So, here is a principle which was not deduced from anything, but it came out of an insight. We can test it.

We can look at any situation in which equilibrium is changing, and we can ask has the equilibrium changed, has the momentum changed, and if it has changed, at what rate does it change, what is dp by dt. This dp by dt will be mass times the acceleration mass times dv by dt which is mass time acceleration right. Acceleration you can measure, you can take your measurement devices, you can actually carry out a measurement, right with a speedometer for example, you measure the speed at t 1, measure the speed at t 2, take the difference, divide it by delta t, take the limit delta t going to 0, you can actually measure. It is a measurable thing. So, you can test it. You do not deduce it from anything else.

So, the foundation principle in Newtonian mechanics is the principle of causality. Over here, the foundation principle is the principle of least action. You do not deduce it from anything, but you use it to deduce everything else in mechanics. Whatever else is happening in the mechanical universe, whatever motion you see, you can ask, is this observation explicable in terms of the principle of least action. If the answer is yes, it makes this principle a law of nature. If you find an exception, you must say that, no it does not describe everything, and it cannot be a law of nature. It cannot be a universal law of nature because you are…, in physics what you are doing is you are looking for universal laws of nature. Something which can be applied to everything, right.

So, you do not have to formulate new laws to explain why apples fall from the trees, or why various objects have whatever trajectories they will finally have, why is that water chooses a particular path while running down a hill. There are so many alternative paths that the water can take, why does it take a particular path. If you can answer this using one principle, then you have a law of physics. You have a law of nature. The principle of variation is that law, ok.

So, the question of how do we know that this holds can be answered only by testing it. Not by deriving it from anything else, but you derive everything else in mechanics from this principle because it is a fundamental principle. So, this is at the foundation of mechanics, and this is the alternative scheme that I talked about that you this alternative scheme is then inspired by the principle of variation rather than the principle of causality or the cause-effect relationship.

So, this is the principle of variation, and now we consider action along two alternative paths indicated by the difference in the arguments of the Lagrangian. So, here in the first term, I have q plus delta q and q dot plus delta q dot, and in the second term the arguments of the Lagrangian are q and q dot, and what we see from what I just showed on the board is that, at an extremum, since at two points in the vicinity of that extremum the change in the function is 0, because the change in the function is equal to the derivative of that function multiplied by the difference in the independent variable, right.

Delta f is df by dx times delta x, and df by dS if it is an extremum; either a maximum or a minimum, then delta f must have vanish. So, this delta S, the S the function we are talking about is action, this is the integral, this is the definition of action, and this delta S must be equal to 0. So, 0 is at the end of this equation, right. So, over here, you have delta S equal to 0, and in the intermediate step, we have written delta S explicitly as integrals over two alternative paths. Now, this is a re-statement of the principle of principle of variation, right. We are not proving it. We are using it.

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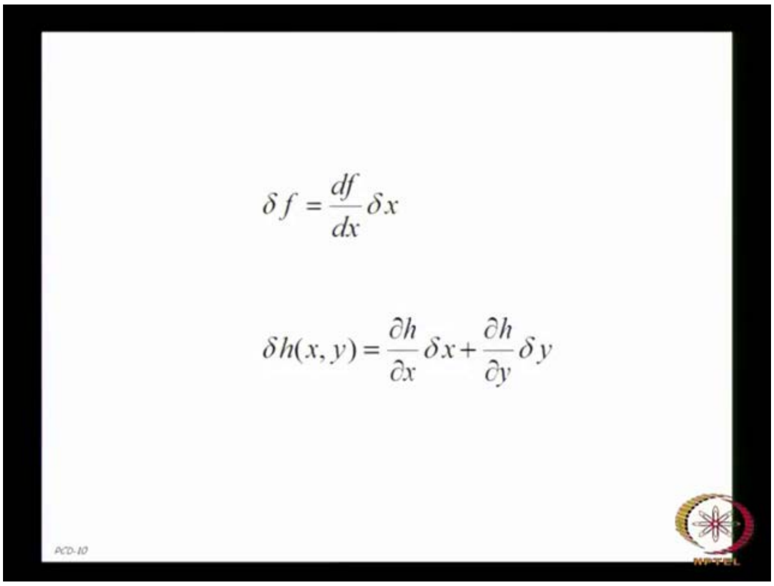

Now here, mind you, this is the idea that we have used that the change in function is the derivative of the function multiplied by the change in the independent variable, but here, we are talking about a function which depends not just depend on one variable, but on two. So, you have got a function h which is a function of two variables x and y.

So, the change delta h will be expressed by a very similar expression, and I am writing this because some of the students sometimes do not have sufficient experience with partial derivatives, and here you are going to have to use partial derivatives because you have got a function not just of a single variable, but a function of several variables. We already have two to begin with. We have position, and we have the velocity, right. So, you have two functions, two independent degrees of freedom to talk about.

So, the change delta h will be given by the partial derivatives of h with respect to x multiplied by delta x plus the partial derivatives of h with respect to y times delta y.

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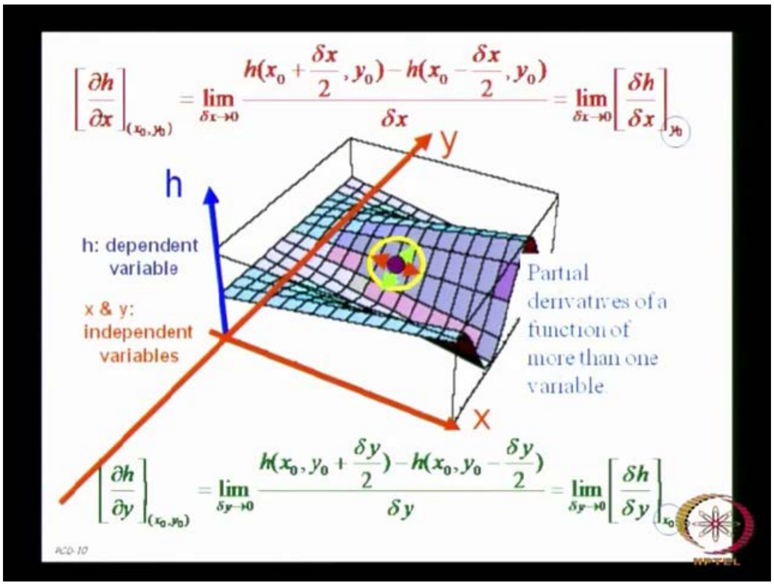

And this let me remind you that if you have here is a function which is the height of a surface, which has got some which is a warped surface, and the height of this surface above a flat bottom is different at different coordinates x and y. So, it is therefore, a function of two variables. And the partial derivative of the height with respect to x is given by this ratio in the limit delta x going to 0 determined at a constant y 0, and the partial derivative with respect to y is likewise determined by this ratio in which you take the difference with respect to changes in the y variable at constant x 0. So, here y0 is held constant, and x 0 is held constant. So, these are partial derivatives.

We must use these partial derivatives in the context of the Lagrangian because we have to take the change in the Lagrangian, but the Lagrangian can be changed because of changes in q, it can also change because of changes in q dot. So, you must take the partial derivative del L by del q, and also the partial derivative del L by del q dot, right. So, both will be involved.

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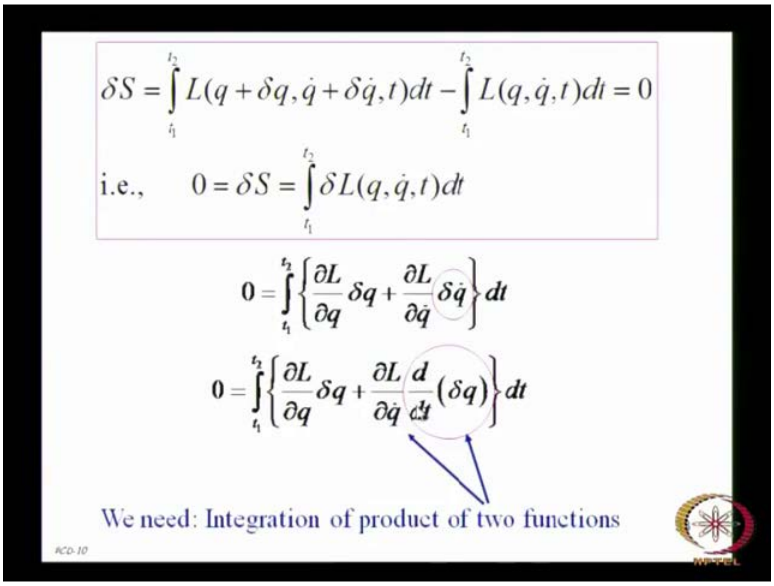

So, let us do that. So, this is the kind of difference equation that we get; delta S is the difference in the values of these two action integrals, and delta L, I have combined these two terms, this integrant minus this integrant is this delta L, right. And this delta L will be the partial derivative of L with respect to q times delta q plus the partial derivative of L with respect to the velocity times delta q dot. And this dt is just the time integration that we must keep track of.

What is written in this beautiful bracket this curly bracket is the exact analog of this expression. Here that a function of two variables is expressed as the partial derivative of the function with respect to one variable times the change in that independent variable plus the partial derivative of that function with respect to the other variable times the change in the other variable.

So, this very function is what comes over here, that this delta L is del L by del q times delta q plus del L by del q dot times delta q dot and then you have this dt which stands out over here, and the definite integral from t 1 to t 2. So, this is a restatement with explicit reference of the partial derivatives.

Now look at this variation delta q dot. What is this, what is delta? Delta is making a reference to the alternative paths, right. And this process is completely independent of the time dependence of that property. So, delta q dot; this is variation with respect to alternative paths. d by dt is measuring the rate of change with respect to changes in time.

These are two independent operations, and they can be carried out in any order because they are completely independent of each other. One does not affect the other. Therefore, delta q dot what you have in this circle here is exactly equal to d by d t of delta q because these are two independent operations.

So, this offers a little bit of simplification, and now we have to handle this particular expression here. So, we will do it, but when we do so, we have to carry out an integration from t 1 to t 2 of the first term which is del L by del q delta q, and over here in the second term, these integrals are of course, additive. So, in the second term, we have to integrate a product of two functions; one is this del L by del q dot, and the other is d by d t of delta q. So, we have to use the relationship for integral of a product of two functions.

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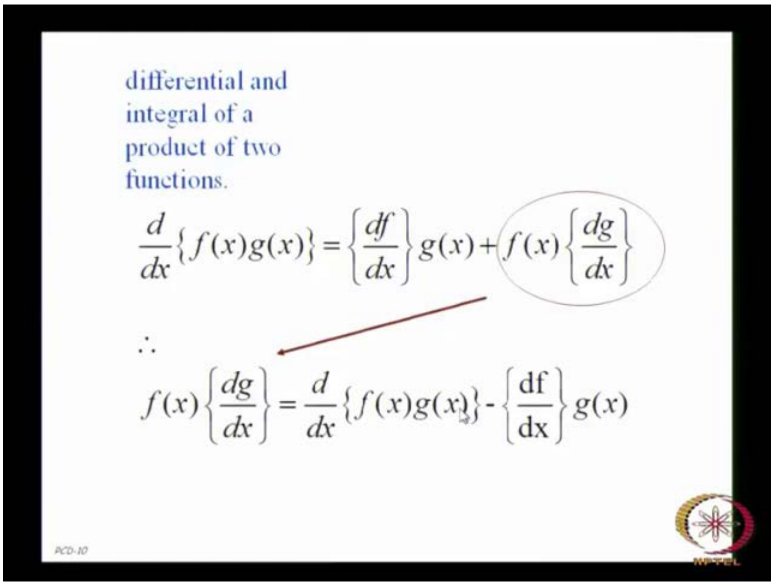

And this is very easy to do because you know that the integration and differentiation are inverse operations. If you take the derivative of a product of two functions, so, you have got a function f and a function g, and you take their product and take the derivative of this products of these two functions, then it is df by dx times gx plus fx times dg by dx, and you write this term on the left, bring this term to the right, you get fx dg by dx equal to d by dx of this term minus of this df by dx times gx. So, it is the same written by re-adjusting the terms moving them to the left and right and keeping track of the sign.

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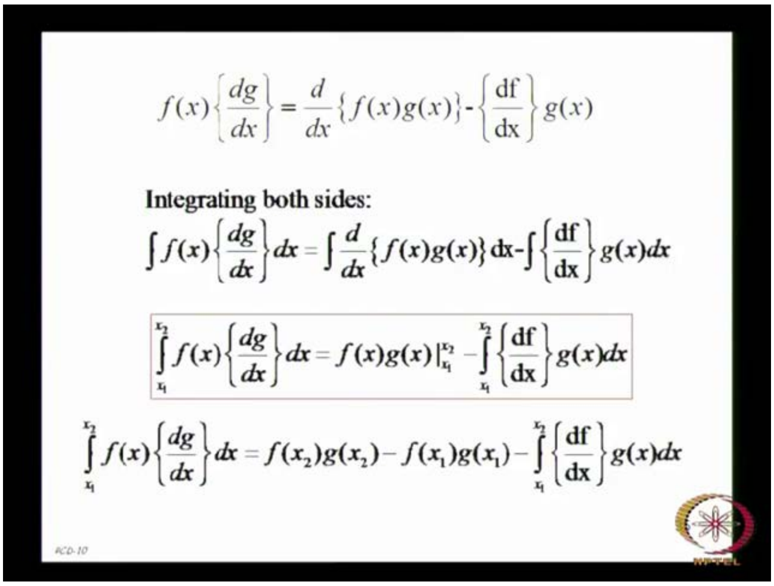

So, this is what we have. And now, if you integrate both the sides, the integral of the left hand side is equal to the integral of the right hand side which will consist of a sum of integration over these two terms, and the minus term kept track of over here, but over here we have a very fortunate situation because you get the integral of a total derivative. And these are inverse operations. So, you get the integration of the total derivative will give you just the term whose total derivative whose integral you are determining. So, you get fx gx to be determined between the limits x 2 and x 1. So, all you will have to do is to subtract the product fx gx at x 1 from what it is at x 2. So, this is really what gives us the very famous and well known formula for integral of a product of two functions.

So, try not to just mug up the formula, but get it from first principles; it is very easy. So, this is the result now. This is the integral of a product of two functions; f of x is the first function, dg by dx is the second function, and you take the difference of the fx and gx which is the first function, this is the integral of the second, integral of dg by dx is g of x and you take the difference x 2 minus x 1. So, difference of fx gx at x 2 minus the corresponding product at x 1. So, that is gives you the first 2 terms, and then you have the integral of the remaining term, right. So, that is how the formula comes and many students know it by heart, which is not always necessary because you can very easily get it from first principles.

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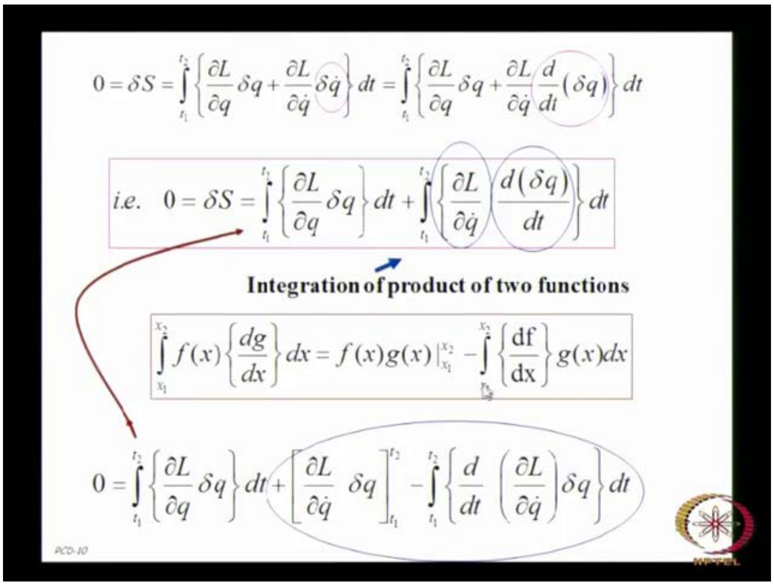

So, here you are the actual integration that we have to do is over here. This is where we met the integral of a product of two functions, and we already have the formula with us which is this. So, we just apply this formula to this situation in which this will be interpreted as the first function, and d by d t of delta q as the second function.

So, from this integral, I have written this term exactly as it is, which is t 1 to t 2 del L by del q delta q and dt. So, I have not changed the first term at all, and with respect to this second term over here, I will make use of the integral of a product of two functions, and we have that formula with us. So, that gives us corresponding to this fx gx for fx we have del L by del q dot which is here, and for gx we have the delta q over here, right. So, this is the d by dt of delta q. This is the integral of the second.

So, this is what we have and then we should take the difference in this value, subtract the value of this product at t1 from what it is at time t2, and then we have the last term which is which comes with the minus sign and you must take the time integral of d by dt of del L by del q dot delta q which is this. Are we all together? Good.

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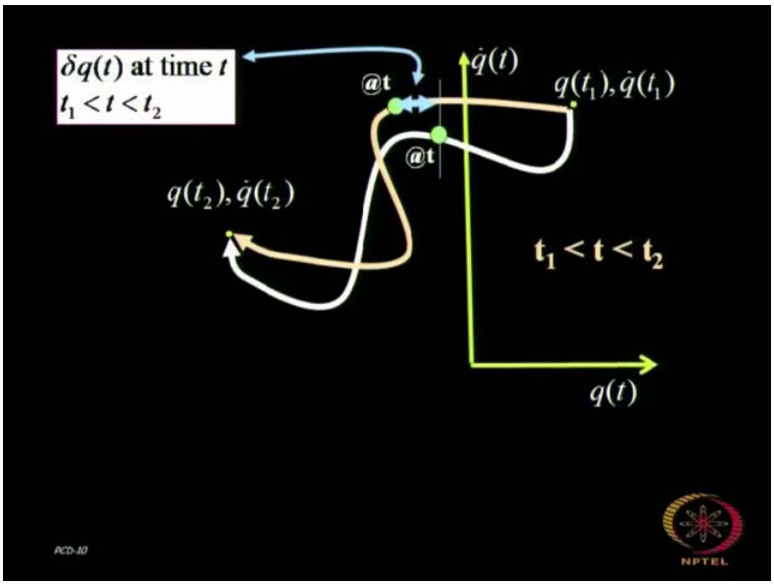

So, now let us see this. Now these are the alternative paths that we were considering. You have one path, you can think of alternative paths. You can think of many very many different paths, and what is changing is that as the system, if it were to go along this track all along this track, and these are different world lines, if you might say. So, these are different world lines along which the system is evolving, and at a certain instant of time, let us say that if the system were to go by the lower path over here, at a certain instant of time t which is between the start and the finish.

At an intermediate time t, if the system is at this instant of at this position here, if these are its the system description in terms of its position and velocity. So, the position is given by the component of this point along the x axis, the velocity is given by the component of this point along the y axis. So, that is the state of the mechanical system right, and at that instant of time, if it were to go by an alternative path, it would be here, if it were at the same instant of time.

So, if you look at the corresponding horizontal distance which is the distance along the position axis. On the horizontal axis, you have got position right. So, delta q is actually the distance between the horizontal distances between these two points. This is the delta q at time t where this time is some intermediate time between t 1 and t 2.

And what can be the value of delta q at the start. Now we know what delta q is. Delta q is the horizontal difference between the positions along alternative paths. There can be millions of alternative paths that you might want to consider. Hundreds, we have shown only two on this figure. You can consider hundreds, but no matter how many you consider, delta q at the start will be 0. And it will be 0 also at the finish, right. So, that is one thing we know, and I think I am going to stop here today, and we will resume the discussion from this point in the next class. I will be happy to take some questions.

Preethi signaled to me that it is time to stop. Yes

Is there a specific definition for a Lagrangian?

Yes. We are very close to it. I kept telling you that we have stated a principle namely the principle of variation. We stated it by saying that this tells us how a system evolves from the start to the finish, and the criterion is that action which is the time integral of the Lagrangian is an extremum. So, we stated a principle without defining what the Lagrangian is. Except that, the Lagrangian is a function of q and q dot, but what kind of a function it is? We have not stated as yet. We all very close to doing that, we will probably be doing it in the next class.

We will like to do it in such a manner that it will not conflict with Newtonian mechanics because we have some experience with Newtonian mechanics, we know it works. We are looking for an alternative development of mechanics, but it would be terrible if it would be in conflict with Newtonian mechanics right because you know Newtonian mechanics works.

So, we will make sure that there is no conflict with Newtonian mechanics and Lagrangian must be defined in a manner so that it will not have any conflict with Newtonian mechanics, and it will then give us the correct exact function which will describe L as the function of q and q dot. So, we are just about to hit it. Yes

Now that Newtonian mechanics is already you know defined and well established, what is the necessity for another system?

There are very many advantages. There are very many advantages, and I am not going to mention these advantages in terms of any particular order, but first of all, you get different angles at looking at the mechanical evolution of a system. You get different insight. You get an insight which Newtonian mechanics does not. Newtonian mechanics does provide an insight which is a different kind of insight, but you develop a different kind of insight using the principle of variation which is always nice because you understand things much better. But more importantly, you will find that in this alternative formulation, you make no use of force, and quantities which are fundamental to this are things like potential energy, kinetic energy, the potential, and these are the quantities which are of greater importance when you deal with physics at different levels. For example, if you were to go over to quantum theory, you cannot adopt the notion of force easily into quantum theory. It does not work.

In quantum theory, you make use of the Schrodinger equation or there are alternate ways of developing quantum theory from the principle of uncertainty or from the from the point view of the De-Broglie Schrodinger equation etcetera, right. And in those formulations, you can begin with the Lagrangian, the Hamiltonian and develop the Schrodinger equation. So, Lagrangian mechanics and the Hamiltonian mechanics does give you a platform to begin the quantum formulation. Not only it does, it is necessary.

You do not write the Schrodinger equation without writing the Hamiltonian for this system, and you do not write the Hamiltonian for the system without writing the classical Hamiltonian, and you do not write the classical Hamiltonian without writing the classical Lagrangian. So, you go from the classical Lagrangian to the classical Hamiltonian and then quantize it, then you get quantum theory.

So, you cannot really bypass this. Newtonian mechanics will never get you there because there is no analog of force in quantum theory. But of course, Newtonian mechanics has certain limitations, and the Lagrangian and Hamiltonian mechanics also has certain limitations, but I will comment on it in the next class. If there is any short question, I will take it.

Why does delta q have to be the horizontal distance?

Oh because q is measured along the horizontal axis in this diagram. What else can it be.

You are plotting these are two independent variables. Position and velocity are independently required to specify the state of a system. The state of the system is specified by a point in the position-velocity phase space. These are independent of each other. The velocity is not a function of position. This start is represented by the state of the system at the start at time t1, right. So, what is sketched in this diagram along the horizontal axis is q which is the horizontal axis. So, what else can it be?