Quantum Mechanics - I
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Lecture - 9
Introducing Quantum Optics

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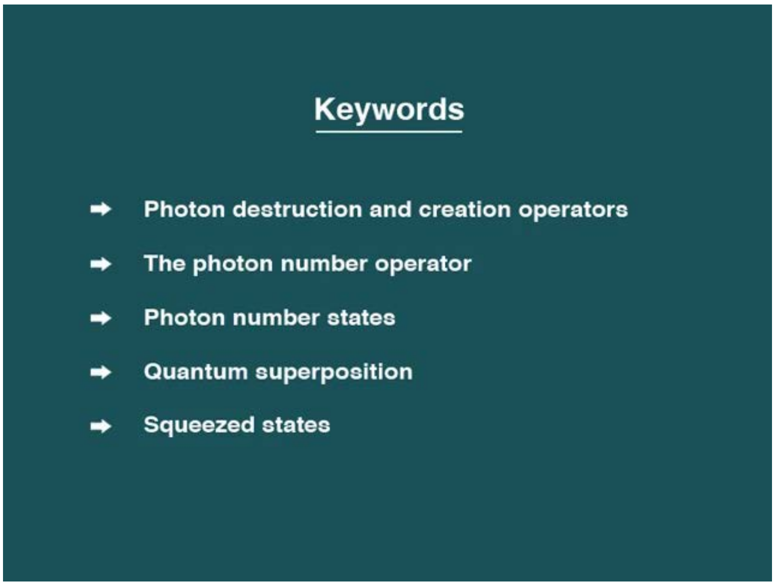

In the last lecture I gave an introduction to the linear harmonic oscillator. We showed that operators called a and a dagger can be constructed for the linear harmonic oscillator and these operators raise the energy levels.

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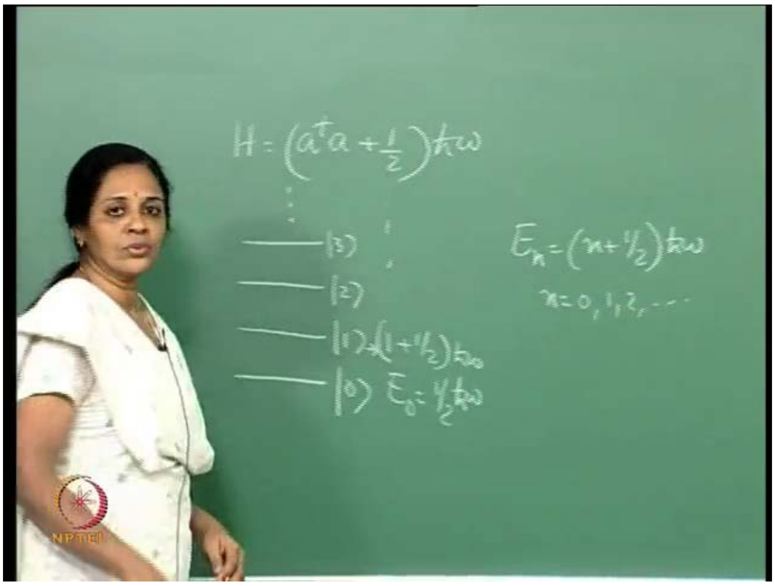

So basically I had the Hamiltonian for the oscillator given by a dagger a plus half h cross omega. The energy levels of the oscillator were equally spaced. The notation used, the ground state was ket 0, the 1st excited state was ket 1 and so on and that is an infinite set of energy levels. The energy itself was given by E 0 is half h cross omega which is a zero point energy. Then there is E 1 which is 1 plus half h cross omega and so on. In general E n is n plus half h cross omega, n taking values 0, 1, 2, 3.

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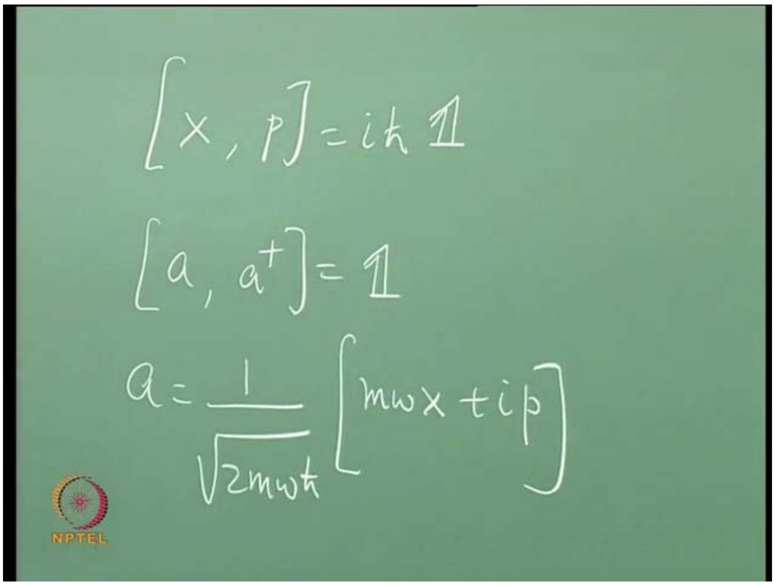

Basically, we started with the commutation relation. The commutator of X with P was i h cross identity, where P which is the linear momentum conjugate to X and identity are all operators. And this translated to a a dagger equals identity, where a itself was defined in terms of X and p as linear combinations, a dagger was the Hermitian conjugate of a.

This is the situation regarding the harmonic oscillator. It turns out that there is a 1 to 1 correspondence between the linear harmonic oscillator problem and quantum optics. That is the framework in which one discusses the quantized electromagnetic field, where we try to understand, various states of the quantized electromagnetic field or the radiation field. Describe and discuss the properties of the electromagnetic field.

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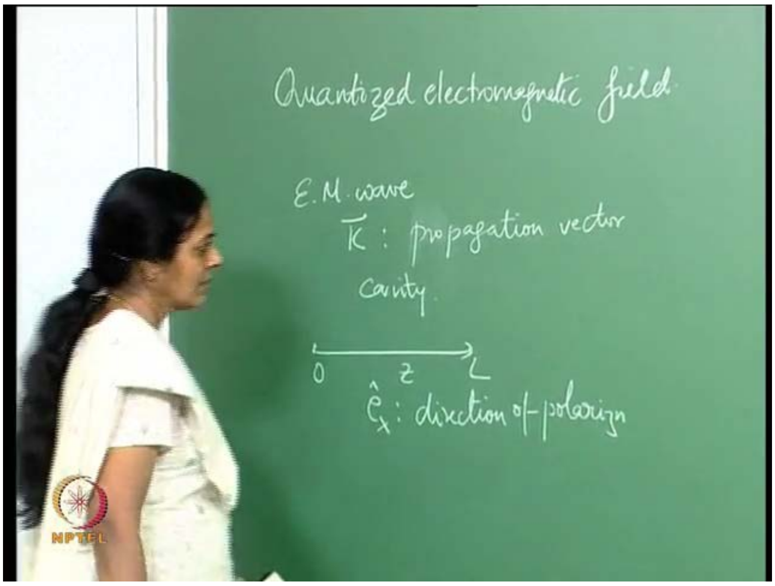

So, basically I will now consider the next problem, which is the problem of the quantized electromagnetic field, where one considers an electromagnetic wave which is quantized. K is the propagation vector. So, the direction of propagation will be chosen to be the z direction. I consider a simple case where this is the origin of coordinates and this is a cavity which extends in the z direction from zero to L.

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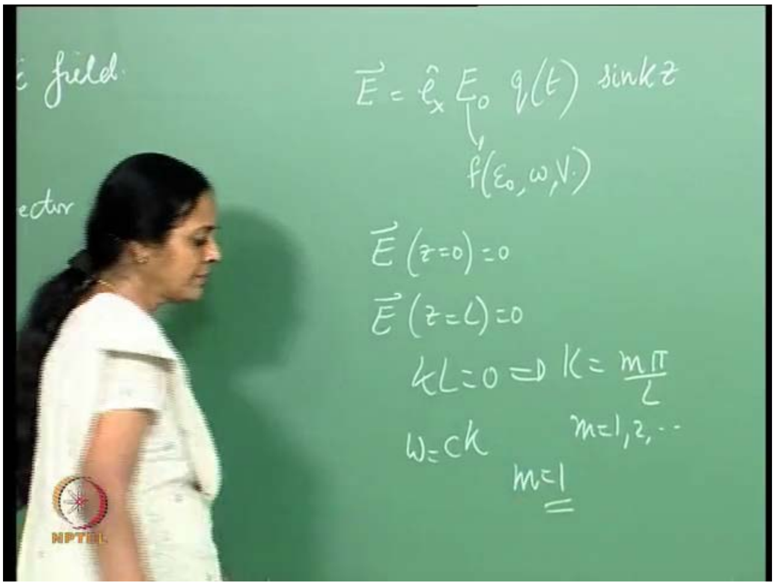

So the polarisation direction is the X direction, which basically means, that the electric field is along the X direction and can in general be written as e sub X being the unit vector along the x direction. E 0 now this object is simply a function of the permittivity, the frequency of the electromagnetic wave, the volume of the cavity and so on.

Some function of time because the electromagnetic field evolves in space time. And sin k z because of the boundary conditions, because the electromagnetic field at z is equal to 0 is 0 and at z is equal to L is also 0, which tells me that k L is equal to 0 and that quantizes K. So, K is m pi by L where m can take values 1, 2, 3 etcetera. The frequency omega is related to k through the standard relation omega is equal to C k, which means that omega is now C times pi by L or 2 C pi by L and so on. For simplicity, we choose m is equal to 1. This can be done without loss of generality because I am considering an electromagnetic wave of frequency omega, which is pi by L times C. (Refer Slide Time: 03:00)

Now, it turns out that the problem of the quantized electromagnetic field. This is a single mode field propagating along the z axis and so on. This has been chosen for simplicity. This problem has a one to one correspondence with the linear harmonic oscillator problem, and that is the mapping that we will see right now.

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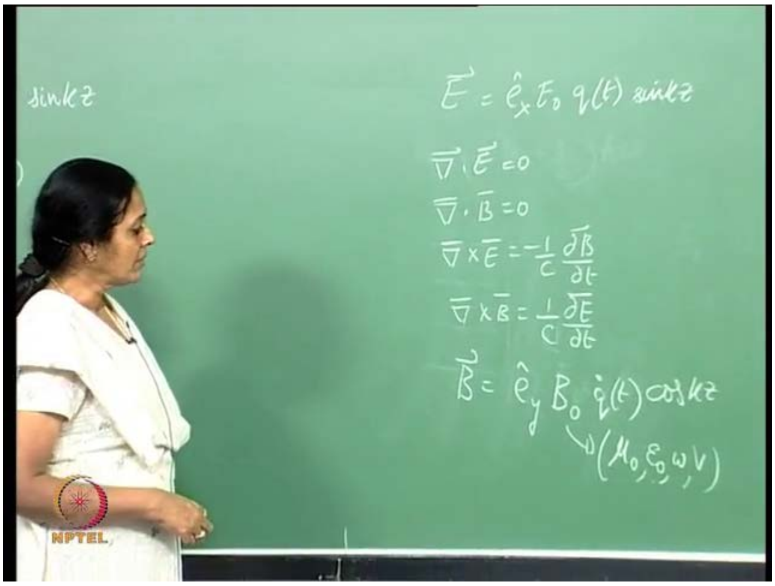

(Refer Slide Time: 04:06) So, in general I have the electromagnetic field E is e x E 0 q of t sin k z. I have the Maxwell's equations del dot E equal to 0 that is automatically satisfied here, because e is along the X axis and the function there is a function of z. Del dot B is equal to 0, del cross E is minus 1 by C delta B by delta t and then cross B is 1 by C delta E by delta t. I am assuming that there are no free charges or currents. So, these are the source free Maxwell equations.

Given this electric field, I can write down the magnetic field. Since the propagation direction is along the z axis and the electric field is along the X axis. The magnetic field is along the y axis. There is B 0 which is analogous to E 0 and it is clearly a function of the magnetic permeability epsilon 0 omega, the volume of the cavity and so on. This equation here tells me that delta E by delta t is related to the curl of B. So, the time dependence of B is going to bring about a q dot of t that sits here. And then it is clear that the space dependence of B comes from a del cross E, which means the derivative of the sin term will appear and therefore, that gives me a cos k z. So this is the way I write B.

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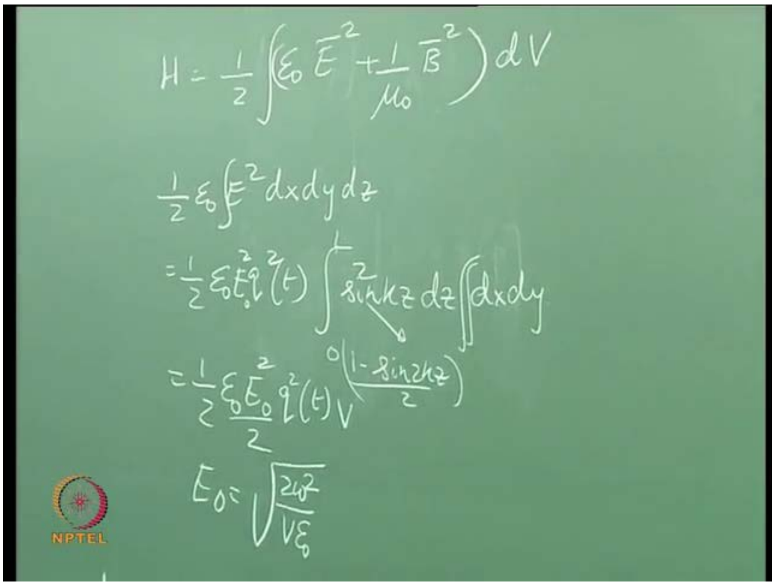

Now given this E and B, I have the Hamiltonian for the electromagnetic field. This is integrated over the entire volume d V. So, this part is the energy density and integrated over the entire volume gives me the total energy. Now I want to look at this term by term. So, let me look at the 1st term d V is simply d x d y d z that is a triple integral. So that is half epsilon zero I can pullout the q of t and the E naught, which is a constant. There is a sin k z d z whole square, there is an E naught square and a q square because there is an E square. Then of course, there is a d x d y the z integral goes from 0 to L.

It is pretty clear that this object can be written in terms of sin 2 k z and 1 that pulls out a half. There is an integral d z d X d y d z which gives me a v and then there is an integral of sin 2 k z and that is between limit 0 and l, but we know that k is pi by L and therefore, it is just cos 2 k z between 0 and L and that quantity vanishes. So this is all that I have. It turns out I did not put down the constants there. It turns out that this E naught is really 2 omega square by V epsilon 0 square root, where V is the volume of the cavity and therefore, I can simplify this.

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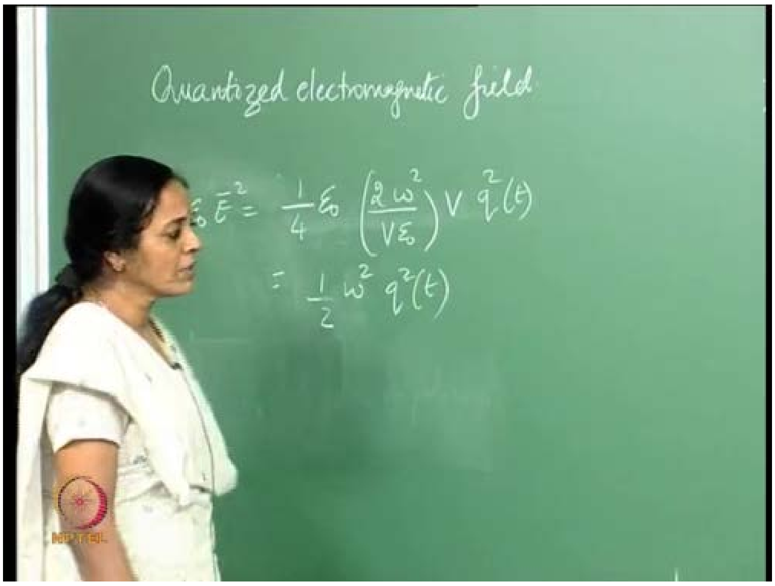

And I have half epsilon 0 E square has a half epsilon 0 so that is the quarter epsilon 0 E 0 square, which gives me a 2 omega square by V epsilon 0. (Refer Slide Time: 08:36) And there is a V by 2 which I pulled out due to the integration I have already absorbed the 2 there so that gives me a V and this object is just half omega square. But, there was also a q square of t so I have half omega square q square of t so that is the 1st term. Then I have the 2nd term which is an integral over b square d V. I can do it the same way that I did the integral over the 1st term.

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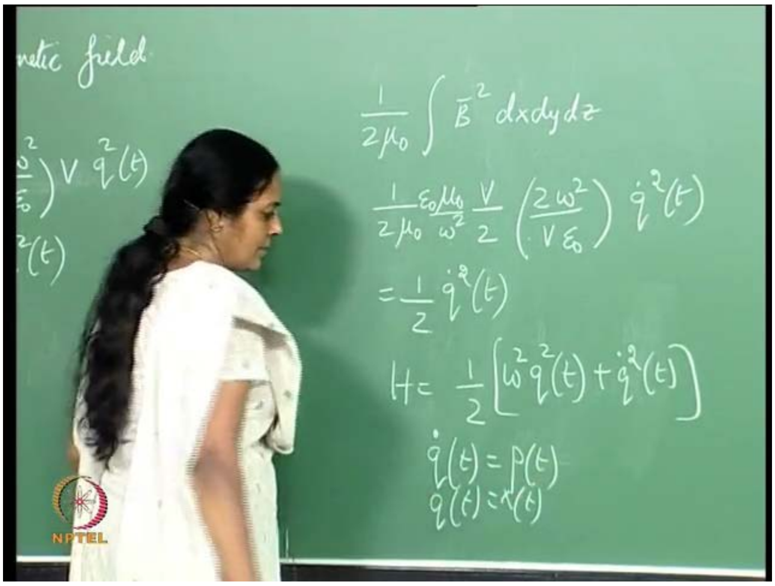

So, I need to consider 1 by 2 mu 0 integral B square d x d y d z. (Refer Slide Time: 06:21) Once more when I substitute for B, there is a q dot square of t, there is a cos square k z and can be written in terms of cos 2 k z. And using the limits of integration I find that that gives me a V by 2 can substitute for B naught explicitly, and once that is done times 2 omega square by V epsilon 0 there is an epsilon 0 mu 0 out here.

This turns out to be with the q dot square of t. So the mu zeroes cancel out, the epsilon zeroes cancel out, the V cancels out. There is an omega square here which too cancels out and I will be left with a q dot square of t. And therefore, the Hamiltonian can be written as half. The total Hamiltonian is half omega square q square of t plus q dot square of t. And this is how one maps the problem of the electromagnetic field, the quantized electromagnetic field to the harmonic oscillator, because I will now re-label q dot of t as p of t and q of t as x of t.

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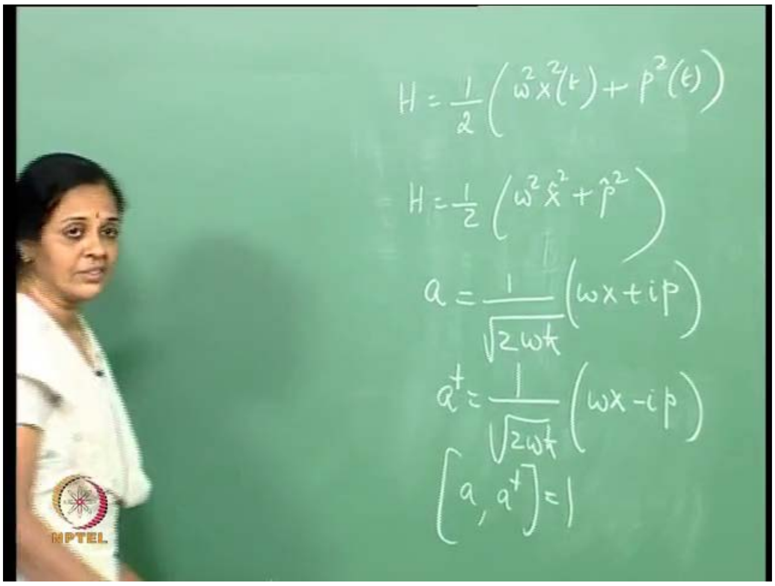

And I have the Hamiltonian given by half omega square x square of t plus p square of t. Suppose, we did not consider the time dependence at all and decide only to look at that part of x of t and p of t, which do not have time dependence. I simply have this object because I am not interested in the dynamics that is a simple case that I am taking up for the moment. And since this is the Hamiltonian for the electromagnetic field and you can trace this term back to the electric field e square and this term to the magnetic field b square and the electric and the magnetic field are the observables. The operators in this case are x and p, not to be confused with position and linear momentum because this has to do with the electric field and this has to do with the magnetic field. The notation has been used essentially to show that there is a one to one correspondence between the quantized electromagnetic field and the linear harmonic oscillator.

So now, given this I can define an operator a as I did earlier. This is an operator shows x and shows p except that this has to do with the electric field and that with the magnetic field, and that is an operator which is a linear combination of objects that are pertaining to the electric and the magnetic fields. And a dagger is simply the Hermitian conjugate of this. So, I have simply renamed things I have gone from x and p to a and a dagger and it is easy to check that a a dagger is the identity operator. Except that, now I have to give an interpretation for a and a dagger and that is rather easily done in this context.

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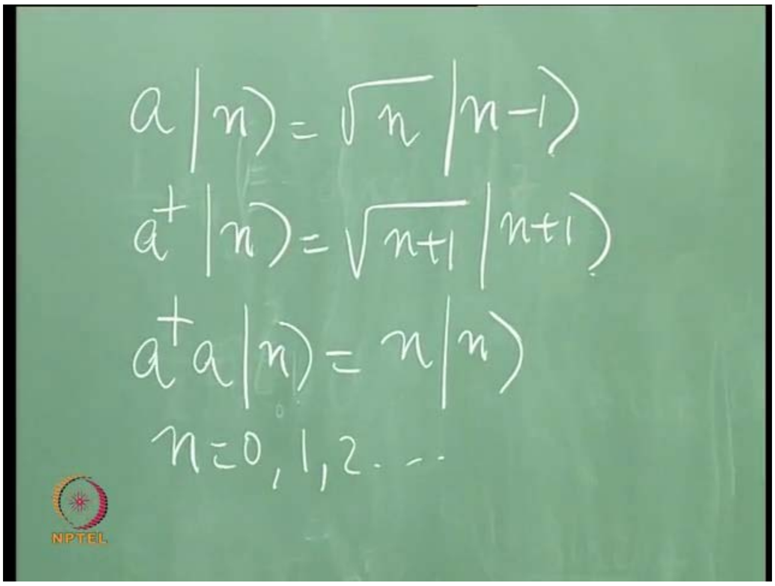

Because, as I pointed out in the case of the linear harmonic oscillator, a lowers levels the same algebra of the oscillator is carried over to the context, that we are discussing at present and therefore, all this is true.

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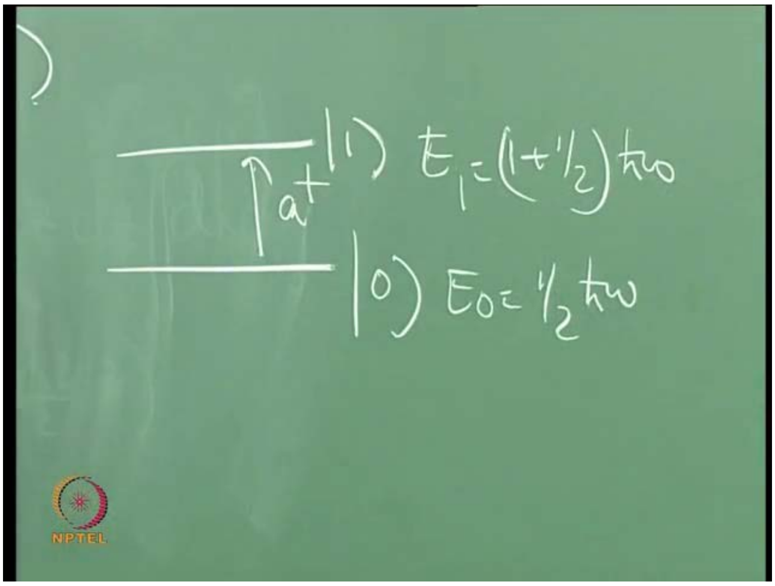

The point is the following: and we know from the problem of the harmonic oscillator, that if I started with n is equal to 0 and when to n is equal to 1 through an a dagger that increases the energy by 1 h cross omega. I also know that a quantum of light the photon carries energy h cross omega. And therefore, a dagger would be the operator that creates a photon pumping in energy h cross omega into the system, a is the photon destruction operator which destroys photons, n is no longer a label n is a number of photons. That is the crucial difference between the harmonic oscillator problem and the problem that we are discussing now. (Refer Slide Time: 16:39)

Because, in the case of the linear harmonic oscillator there was a single linear harmonic oscillator, n was merely a label and you moved from ket 0 to ket 1 to ket 2 which are excited states of a single linear harmonic oscillator. Here, we have made a leap in our understanding. We are now discussing a problem where n stands for the number of photons. Because, if I repeat a dagger n times if I repeatedly operate a dagger n times on a state it is going to add energy n h cross omega and therefore, creates n photons. So n is no longer a label in this problem n refers to the number of photons.

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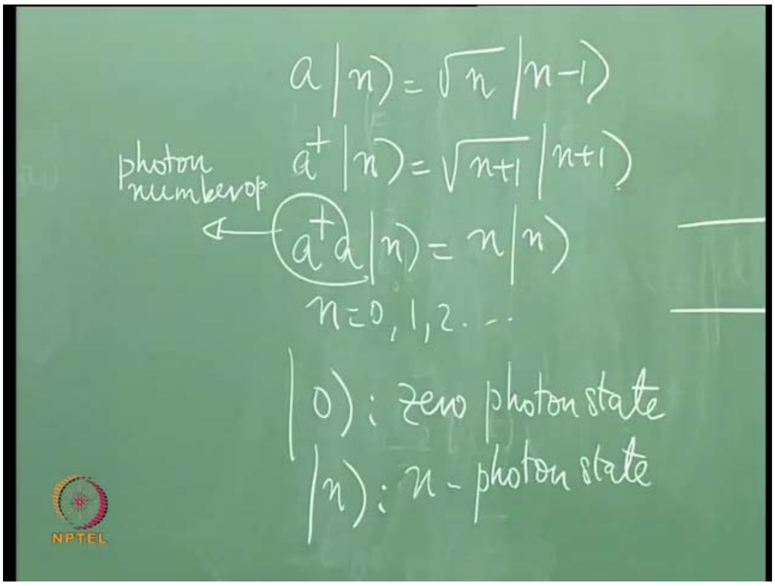

So, ket 0 would correspond to the 0 photon state and this is a crucial difference between the harmonic oscillator problem and here. In the quantized electromagnetic field, because while the algebra is the same and the mathematical framework is the same the interpretations are different.

So this is a zero photon state, this is the n photon state. And as you go from the 0 photon state to 1 photon state, there is an increase in energy h cross omega. The more the number of photons that are added mathematically represented by repeated application of a dagger on the original state keep adding more photons and therefore, more energy is pumped into this system of photons. So, what is this operator? This operator would be the photon number operator. It counts the number of photons in the state. So it is the state with n photons and that is what is pulled out here. So, this is the photon number operator a dagger is the photon creation operator and a is a photon destruction operator. It is well worth passing at this point and getting the identity straight.

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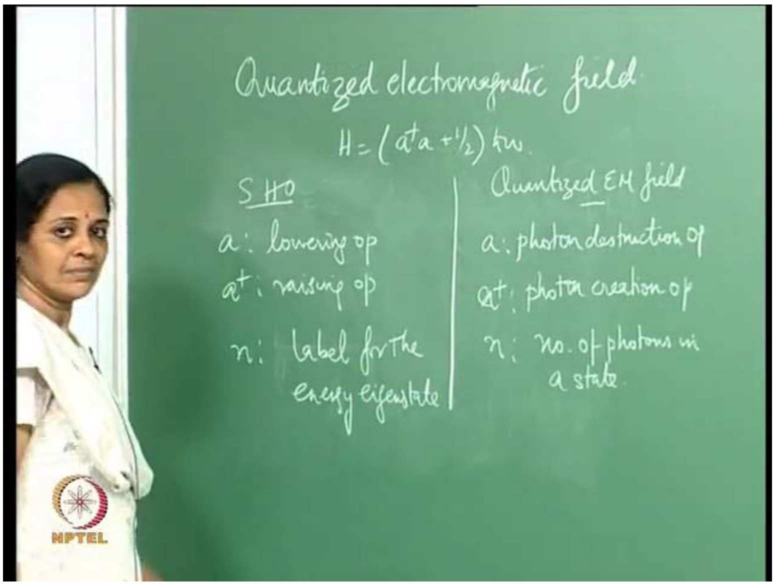

In the case of the simple harmonic oscillator a was the lowering operator which took us from one energy level to another, a dagger was the raising operator and n was merely a label for the states. Now, in the case of the quantized electromagnetic field, a is the photon destruction operator, a dagger is the photon creation operator, n is the number of photons in the given state. The Hamiltonian in both cases is simply a dagger a plus half h cross omega. So this is the mapping that we are interested in. When I spoke about the harmonic oscillator, I discussed the uncertainty principle in that context and showed that the ground state of the oscillator was a minimum uncertainty state.

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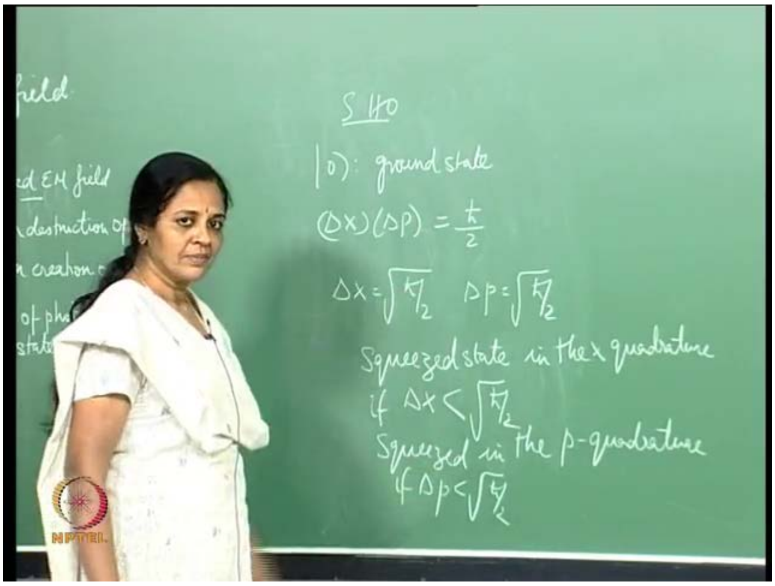

So, let me recapitulate in the case of the simple harmonic oscillator the ground state of the oscillator satisfied delta X delta P is equal to h cross by 2 not only that delta X was root of h cross by 2 and delta P was root of h cross by 2. In the case of the oscillator, if m equals 1 and omega equals 1 it was not only a minimum uncertainty state, it was a state where delta X was equal to delta P. The general uncertainty relationship tells us that delta X delta P is greater than or equal to h cross by 2 and certainly for the excited states of the oscillator the inequality holds. Delta X delta p will be greater than h cross by 2.

Now, we say that a state is a squeezed state in the X quadrature if you wish, if delta X is less than root of h cross by 2, it is a squeezed state in the p quadrature. If, delta P is less than root of h cross by 2. Now it is clear that since the uncertainty principle holds if it is squeezed in the X quadrature, if there is a state which is squeezed in the X quadrature the corresponding delta p will be large such that delta X, delta P is greater than h cross by 2. If it is squeezed in the P quadrature the corresponding delta X value will be large so that there is a compensation, and delta X delta P is greater than h cross by 2.

You cannot have a state where both delta X and delta P are less than root of h cross by 2 because that violates the uncertainty principle. So, let us look at various states in the context of optics, quantum optics or the quantized electromagnetic field. The problem of the radiation field in other words the problem of the light quanta. So, specifically we will consider various superpositions of photon number states ket n. It will illustrate things, it will illustrate the property of squeezing, (Refer Slide Time: 21:46) it will also illustrate a very important aspect of quantum mechanics and that is the power of quantum superposition.

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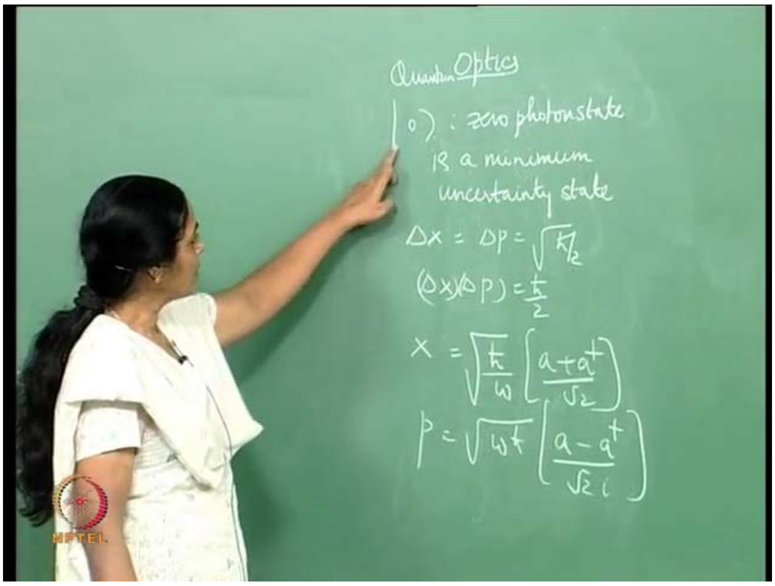

So, first of all without much ado we can say that in the case of quantum optics the zero photon state it is a minimum uncertainty state. We have already established this in the case of the oscillator and all that we have done is a mapping from that problem to this problem. So, in the zero photon state certainly delta X is equal to delta P is equal to root of h cross by 2 and delta X delta P is equal to h cross by 2. You could do this by writing X in terms of a and a dagger, X was root of h cross by omega a plus a dagger by root 2.

In the case of the harmonic oscillator there was also a mass sitting there and as I pointed out earlier root of h cross by m omega has a dimensions of length. Now there is no m, but we do have a root of h cross by omega that gives us X and P is root of omega h cross a minus a dagger by root 2 i.

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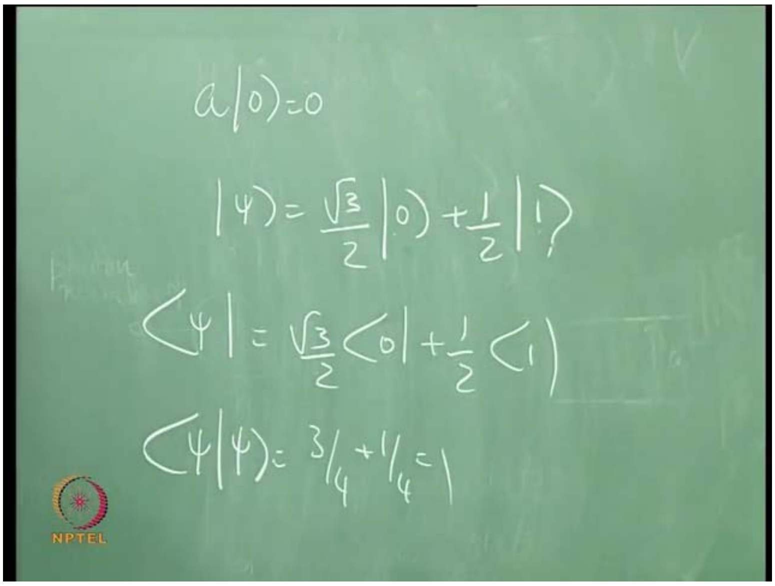

So, given this I can find out delta X and delta p in the 0 photon state, where I already know that a on ket 0 is equal to 0. And, this is established in precisely the same manner as we established that the ground state of a simple harmonic oscillator is a minimum uncertainty state. Now, let us look at a superposition of photon number states. Let us consider this state. I consider a superposition of the 0 photon state and the 1 photon state. The 1 photon state was got by acting a dagger on ket 0, a dagger on ket 0 gives me ket 1 with the root 1 in front as a coefficient. This is a normalised state because bra psi it has been constructed such that ket psi bra psi is a 3 quarter plus 1 quarter and that is a 1. Let us compute delta X and delta P in this state psi.

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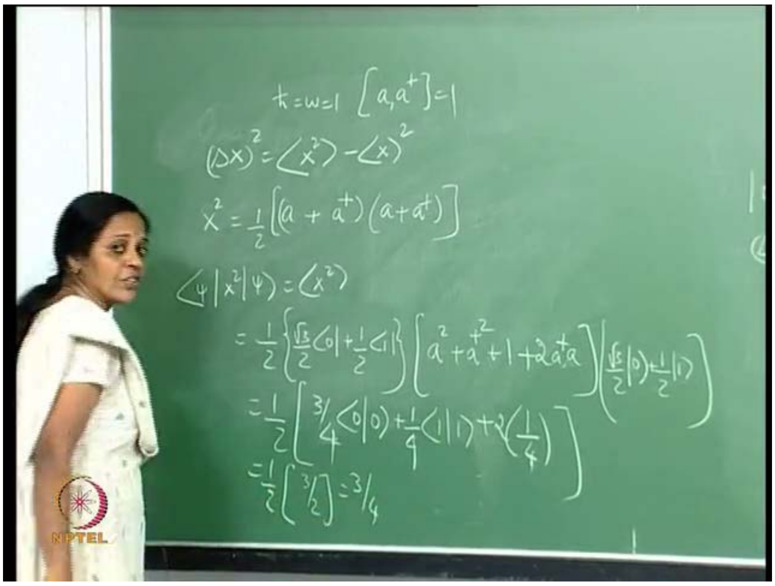

So, first of all delta X square is expectation value of X square in the state minus expectation value of X the whole square. So, I start with X square. Suppose, I set h cross and omega equal to 1 for convenience. We can always put it back for the sake of dimensions. This is essentially a plus a dagger, producted with itself with a half there. I need to find psi X square psi. This is expectation value of X square.

Now this object is half. The half is put here when I expand out this operator. I get an a square plus a dagger square. Then I have an a a dagger plus a dagger a. I have the commutation relation and therefore, a a dagger is 1 plus a dagger a. So this gives me a 1 plus 2 a dagger a. There was already an a dagger a and the a a dagger contributes another because of that commutation relation. And then on the other side I have root 3 by 2 ket 0 plus half ket 1. So this is the object that I have to compute.

Let us look at the first term. Now, a square a acting on ket 0 is 0 so a square simply destroys this and therefore, the inner product with this does not contribute does not contribute with this either. So, there is no contribution from a square. Look at a dagger square, a dagger acting on ket 0 gives me ket 1 apart from some constant number multiplying with it. Repeat it twice and that takes it to ket 2. Now, ket 2 is orthogonal to both ket 1 and ket 0 and therefore, there is no contribution from this term.

Similarly, when a dagger square acts on ket 1 takes it through ket 2 to ket 3 and again by the orthogonality property, it does not make a contribution. There is a contribution from here. This is the identity operator and therefore, from the first term I get 3 by 2 and from the second term I get a quarter 3 by 4 and from the second term I get a quarter. So this is simply the identity and this is what comes out of that by way of contribution.

But then there is a contribution from 2 a dagger a. Now a dagger a picks up an Eigen value 0 when it acts on this state. So the overall contribution from the inner product of ket 0 with bra 0 is 0, but a dagger a here gives me a 1 and there is already a quarter, because of these two and that is what I have. Now, this can be simplified that is a half that is a 3 quarter plus 1 quarter which is a 1 and there is a half, so expectation X square in this state is 3 by 4 that is a 3 quarter.

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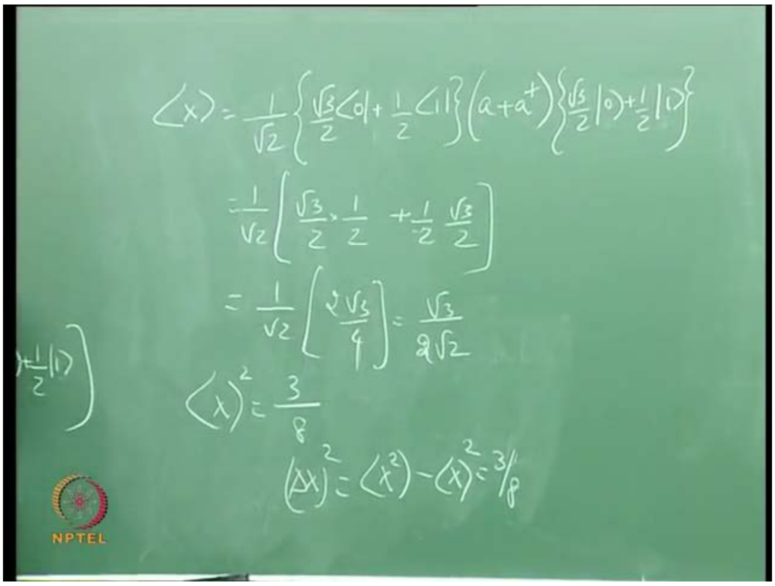

So, let us look at expectation X the whole square. I repeat the performance. So, expectation X would simply give me 1 by root 2 root 3 by 2 bra 0 plus 1 by 2 bra 1 a plus a dagger, that is expectation X apart from a root 2 root 3 by 2 ket 0 plus half ket 1 and this is what I need to compute. I work as before, I have an overall factor 1 by root 2 a destroys 0 ket 0 and therefore, it does not contribute here. On the other hand, a acting on ket 1 gives me ket 0. So, the inner product of this state with a acting on ket 1 makes a contribution. I pick up an overall constant root 3 by 2 from here and a half from there and since a acting on ket 1 is ket 0 this is all I have from this term.

Now look at a dagger, a dagger acting on ket 0 takes it to ket 1, so there is a contribution from here. There is no other contribution, because when a dagger acts on ket 1 it takes it to k 2 which is orthogonal to both of them. So, this object is twice root 3 by 4 and therefore, that just gives me a root 3 2 root 2, I need expectation X the whole square that is 3 by 8 and therefore, delta X the whole square is expectation X square minus expectation X the whole square, which is 3 by 8.

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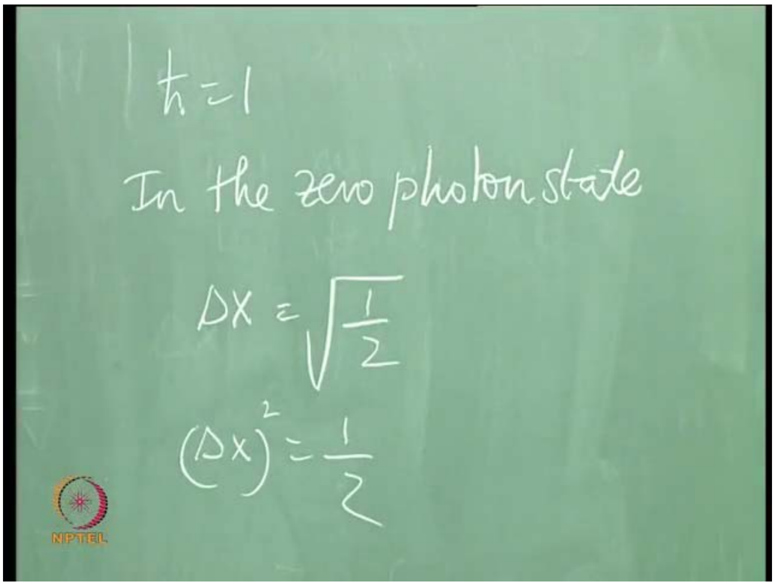

What happens in the ground state? Delta X the whole square when I set h cross equal to 1. If h cross is equal to 1, which is what I have been working with for simplicity. In the ground state or in the zero photon state delta X is root of 1 by 2 and therefore, the variance is half. So that is the ground state and here I have a state where the variance is less than half. So it is squeezed in the X quadrature. I would like to check that it is not squeezed in the p quadrature, because that would violate the uncertainty principle.

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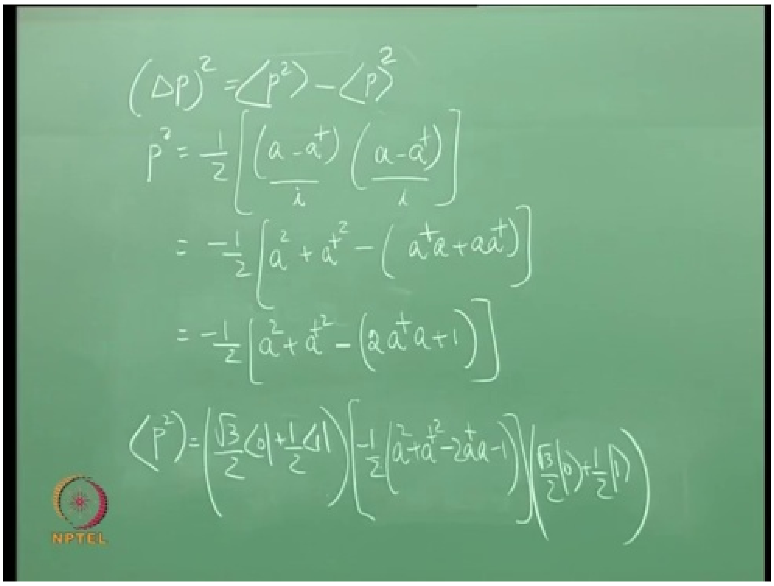

So, let us work out delta p the whole square. So delta p the whole square is expectation p square minus expectation of p the whole square. Now p square is again 1 1 half of a minus a dagger by i a minus a dagger by i. That is the minus half a square plus a dagger square minus of a dagger a plus a a dagger, as before I use the commutation relation. Now we will write this as minus half a square plus a dagger square minus 2 a dagger a plus 1. It is clear, that when I sandwich p square between the state of relevance here. Expectation p square is simply going to be root 3 by 2 bra 0 plus half bra 1. I repeat what I did earlier times half a square plus a dagger square minus 2 a dagger a plus 1 and then there is a ket root 3 by 2 ket 0 plus half ket 1. As before a square and a dagger square do not contribute, but a dagger a does and the identity operator there does and we can simplify this.

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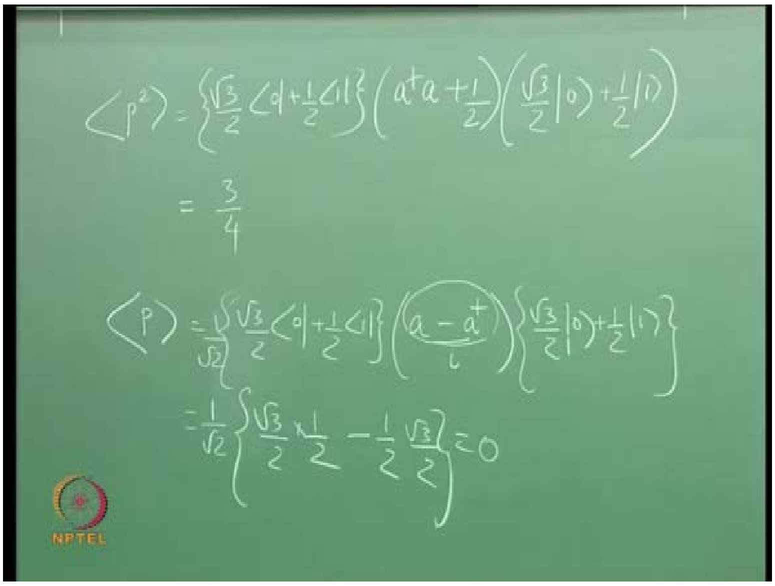

So, what is expectation P square in this state as before a square and a dagger square do not contribute, and we nearly have expectation p square the state itself is given here. And, there is an a dagger a plus 1, there was an overall half and that comes there. We started with the a dagger a and therefore, it is just divided by 2 and the ket is root 3 by 2 ket 0 plus half ket 1,but I know the answer to this. This has been done earlier by us, that is 3 quarters. But now look at expectation P in this state, so the state itself once more I sandwich the operator in this manner. I have a 1 by root 2 is an overall factor. So, essentially one looks at the action of a minus a dagger on this. Of course, a acting on ket 0 simply gives me 0 there is no contribution. They acts on ket 1 brings it down to ket o so there is a contribution from here which is a root 3 by 2 from ket 0 from bra 0 here and a half from there.

Now look at a dagger, a dagger acts on ket 0 takes it to ket 1, so there is a contribution. But the coefficients are half from here and a root 3 by 2 from there. There is no contribution when a dagger acts on ket 1, because it takes it to ket 2 and I do not have ket 2 on this side, bra 2 on this side so this 0. So the crucial thing to note is that unlike the case of expectation X where I had a plus sign there and therefore, there was a contribution.

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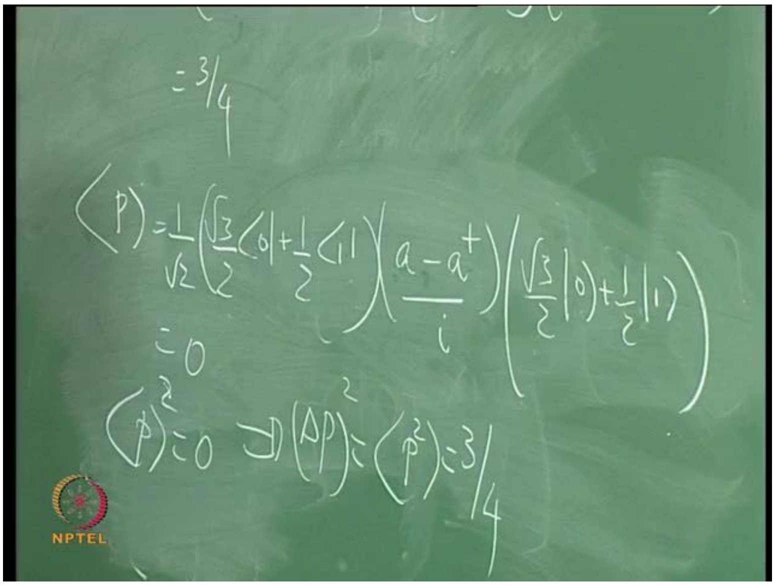

In the case of the expectation P, the individual contributions cancel out and I get a 0 expectation P the whole square is 0, that implies that delta P the whole square is simply expectation P square which is good news, because delta P the whole square is greater than half. (Refer Slide Time: 34:09) And therefore, in this state that I have given which is the superposition of the 0 photon state and the 1 photon state that is squeezing in the quadrature, and there is no squeezing in the p quadrature as expected. And the product delta X delta P is greater than root of h cross by 2. So it is not a minimum uncertainty state, but certainly there is squeezing.

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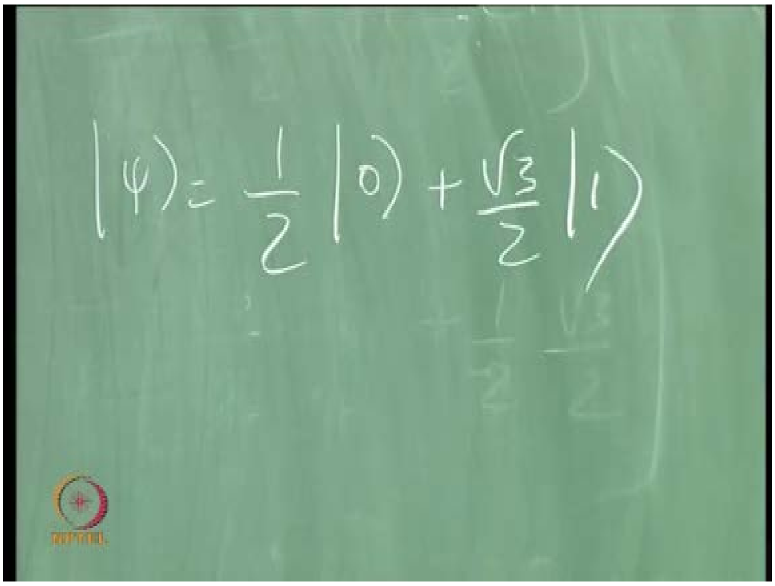

Now, it is clear that I could have changed the coefficients and suppose I constructed a state which is given by half ket 0 plus root 3 by 2 ket 1 it will be squeezed in the p quadrature and not squeezed in the X quadrature. So, this is the power of quantum superposition. I have constructed a state which shows squeezing and squeezed states are very important in optics. The squeezed vacuum is a very important non classical state. There is no analogue of squeezing in classical physics and for that matter there is no analogue of the uncertainty principle in classical physics either. And therefore, a squeezed state is an example of a non classical state. Quantum optics abounds a non classical states of light. To illustrate the power of quantum superposition I want to show another example, not in the context of squeezing, but in a very different context.

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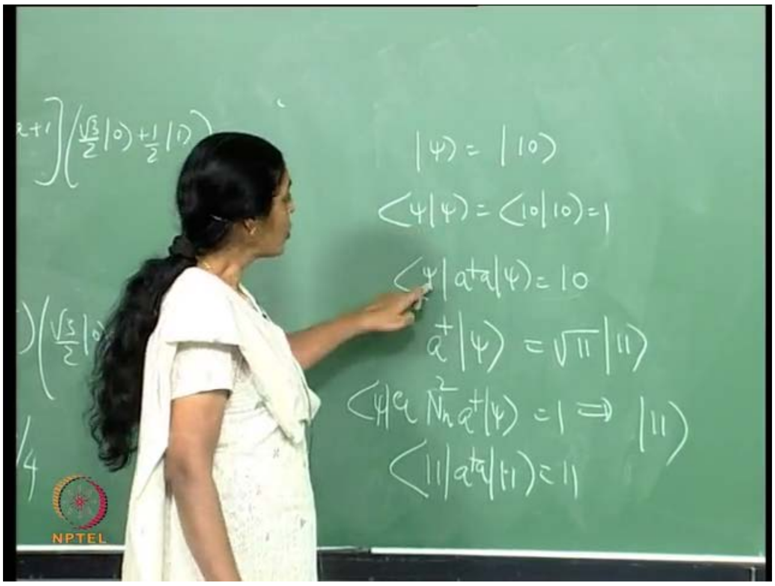

Consider this state the normalised 10 photon state. So, it is clear this is a 10 photon state. In my notation it is represented in this manner, n is equal to 10 and this state is normalised. So what is the mean photon number in this state? The mean photon number is clearly 10. Because, a dagger a acting on ket 10 is 10 ket 10 and that inner product becomes 1. So the mean photon number is 10. Now, let me take this state psi and act on it by a dagger. In other words, i will add a photon to the state. So a dagger on ket psi is root 11 ket 11.

I have to remember that a dagger on a state ket n is root of n plus 1 ket n plus 1. This is not a normalised state, and we need to consider normalised states, because otherwise the probabilistic interpretation fails the total probability must be 1. And therefore, to normalise the state I have the normalised state should satisfy, if this is a normalisation factor the bra which is a psi, there is an n n square where n n is a normalisation factor this object must be 1. This immediately tells me that the normalised state is ket 11. I do not have to do any work for this because I know that the 11 photon state acts such as a normalised state.

So, what is the expectation value of a dagger in the new state? Expectation value is 11, which is believable because I took the 10 photon state. I added a photon to it by applying a dagger to it and therefore, the mean value of a dagger a has moved up to 11 I have added a photon.

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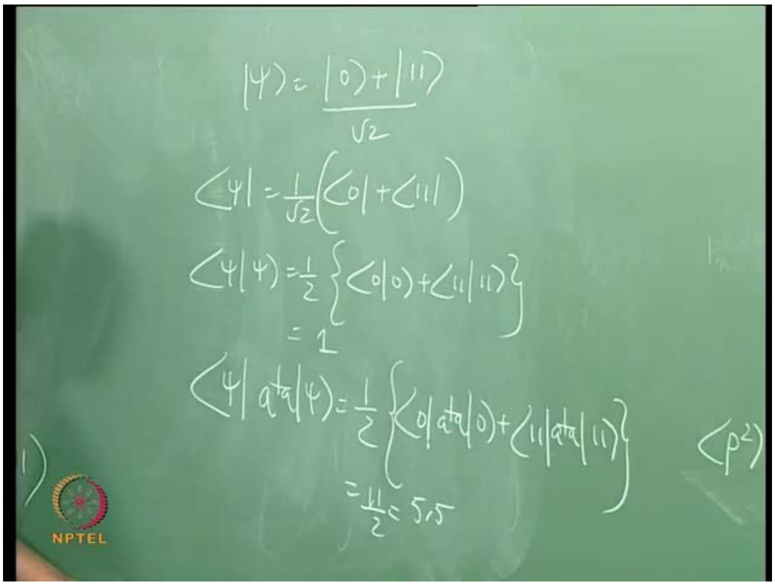

Now instead of this, let me consider this state psi which is the 0 photon state plus 11 photon state. It has to be normalised so let me consider this state. This is normalised I can check it out because bra psi is 1 by root 2 and therefore, I have the inner product given by this. This is normalised to 1 and that is normalised to 1 and therefore, I have the inner product of psi with itself to be 1. So, this is a normalised state. The total number of photons in the state is 11, because this is a 0 photon state and that is a 11 photon state. So, I have a total of 11 photons.

Let me find the mean number of photons in this state. So, I can calculate the mean number of photons in this state and I have a certain mean number of photons. First of all this gives me a 0, but that gives me 11. (Refer Slide Time: 41:34) That is the mean number of photons. I have brought down the mean number of photons from the initial value by doing a quantum superposition of the 11 photon state with the 0 photon state. So, naively one may imagine that since a 0 photon state has no photons it cannot make a contribution to anything, but it has brought down the average from 11 to 5.5, this is what I have for this state. Now, let me pump in photons into the state.

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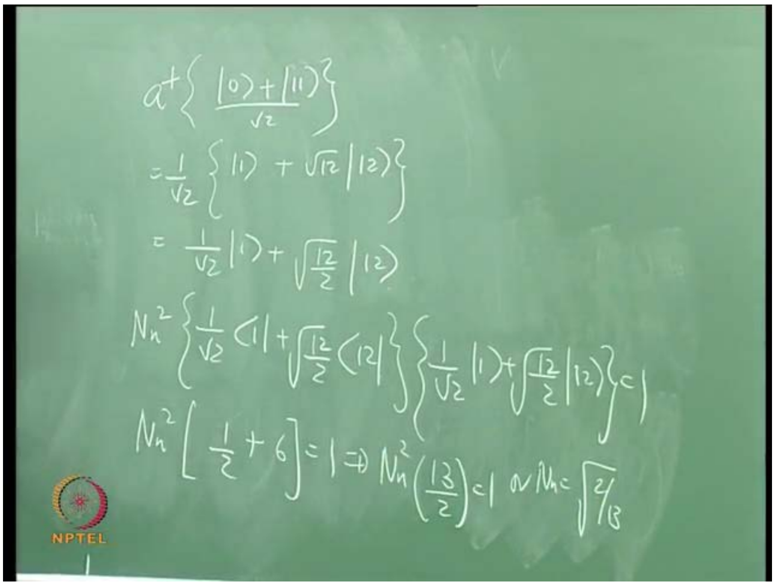

I wish to do a dagger on ket 0 plus ket 11 by root 2. In other words I am going to add photons to this state a dagger on ket 0 is ket 1 and a dagger on ket 11 is root 12 ket 12. And therefore, this is just 1 by root 2 ket 1 plus root of 12 by 2 ket 12. This is not a normalised state. I need to normalise this state. So, I need to consider n n square which is the square of the normalisation with itself and that should tell me what the normalisation factor is. Half from here and 6 from there should be 1. Now, this normalisation is simply root of 2 by 13 and this is what I have for the normalisation.

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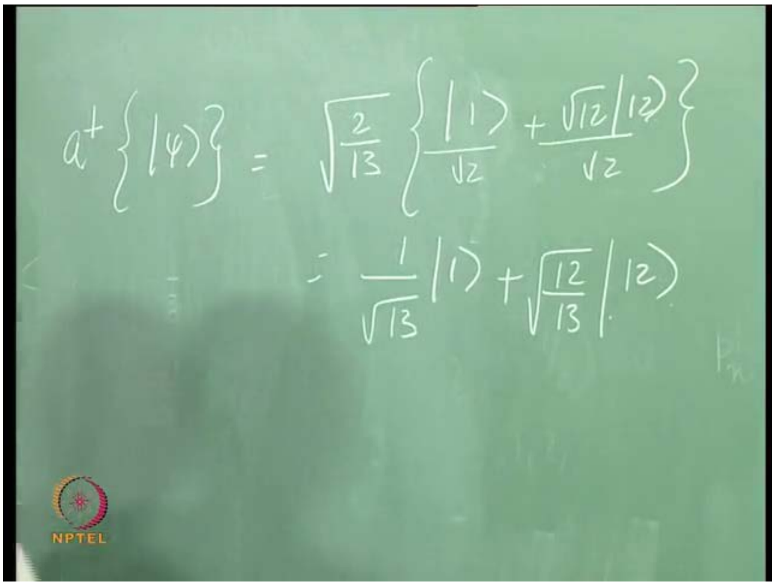

So, I have the new normalised state a dagger acted on my state psi and I normalised it. And the new state that I have got is root of 2 by 13 times ket 1 by root 2 plus root 12 ket 12 by root 2. So this is just 1 by root of 13 ket 1 plus root of 12 by 13 ket 12. This is what I have when I added a photon to this state. (Refer Slide Time: 46:02) So, to begin with since I have added a photon I would normally expect it to add here or add there, but a dagger acts on both states and adds a 1 here and makes it 12 out there. So, this again is the power of quantum superposition of course, this is a normalised state, because when I square it I just get a 1 by 13 plus 12 by 13 and I am through.

(Refer Slide Time: 46:02) So when you do a quantum superposition the photon sits here and the photon sits there. It is not as if the photon gets added to this state or to that state. So that is another aspect of quantum superposition which I want to bring about here. There are 2 aspects to quantum superposition, 3 in fact which I have demonstrated in this lecture. The 1st is suitable superpositions could produce squeezing either in the X quadrature or in the P quadrature. (Refer Slide Time: 43:43)

The 2nd is this that in a superpose state the average number of photons can come down drastically although a photon has been added that tells us the importance of the zero photon state, and thirdly when you add a photon it does not add to just this state or that state, but adds on the whole. So, these are aspects of quantum physics which do not see analogues in classical physics. I will stop here and take on more interesting quantum superpositions in the next lecture.