Quantum Mechanics - I
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Lecture - 7
The Uncertainty Principle

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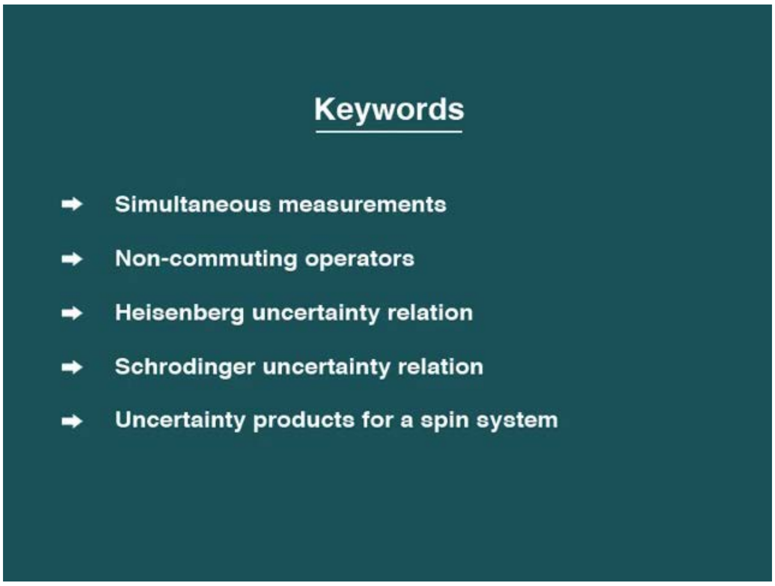

In the last lecture, I had spoken in some detail about the 3rd postulate of quantum mechanics which pertain to measurement, basically expectation values and Eigen values of operators which represented observables. We spoke about measurement outcomes, repeated measurements and how exactly expectation values are obtained and related to measurement outcomes in an experiment.

Now, in this context a very important question arises. How well can you make simultaneous measurements of two observables? You can always make simultaneous measurements of observables. Question is; can you measure both of them simultaneously, to arbitrarily high precision? And this is really the content of the uncertainty principle, primarily due to Heisenberg, and it is the uncertainty principle, that I will discuss in some detail today.

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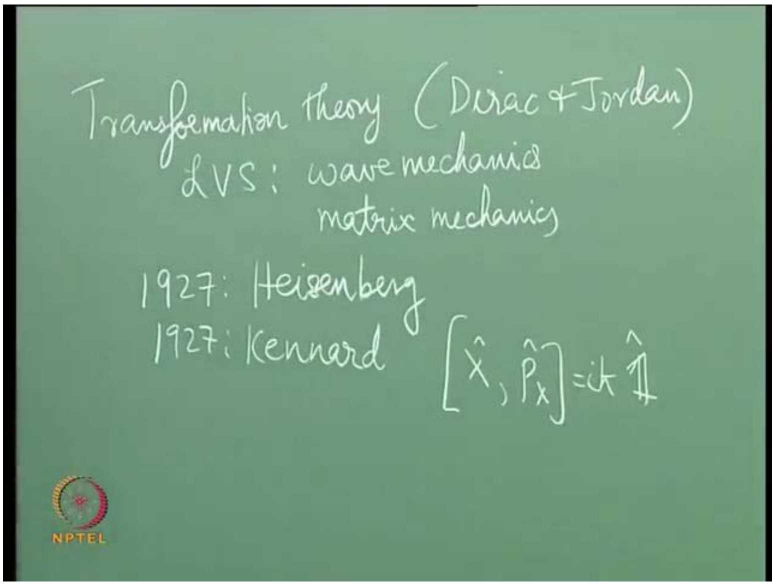

So, the topic today is the uncertainty principle. The history of the uncertainty principle is rather interesting. Dirac and Jordan had proposed, the transformation theory. The transformation theory by Dirac and Jordan, was really based on the theory of linear vector spaces. And the theory helps one understand both wave mechanics due to Schrodinger, primarily and matrix mechanics primarily due to Heisenberg, within the mathematical structure of liner vector spaces. This was due to Dirac and Jordan.

Now Heisenberg, in an attempt to understand transformation theory, had a glimpse of the uncertainty relation in the following sense. He realized that it was not possible to simultaneously measure, to arbitrarily high level of precision certain observables. For instance, the position of an object along with the corresponding linear momentum cannot be simultaneously measured to an arbitrarily high level of precision. So, he constructed a Gedanken experiment, a thought experiment to prove this point. Now, the thought experiment was all about a gamma ray, electron microscope, where the gamma rays interacted with the electrons. Except that the experiment as conceived by Heisenberg, assumed that the interaction was like collisions between mechanical objects, which is not quite the truth.

However, although the experiment itself as conceived in its original form by Heisenberg is not precise. The outcome of the experiment is certainly true, and it is really to the attributed to Heisenberg, the fact that simultaneous observation in quantum mechanics is very different from simultaneous observation of two variables in classical physics. So, it was in 1927 that Heisenberg suggested this, that there was this relation between position and linear momentum and so on. But, in 1927 later Kennard, actually produced a very precise mathematical expression for this uncertainty relation. The Kennard relations talks of from the fact, that x and P x position and the corresponding linear momentum, do not commute with each other.

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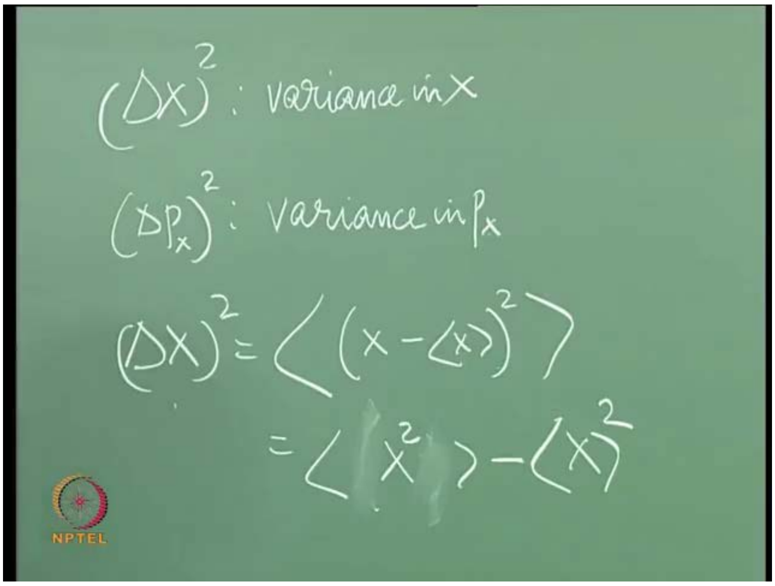

I had spoken about this towards the end of my last lecture and just using the fact that this is the commutation relation, between X and P x. Kennard showed, that the variance in x if the variance in x is delta x square, and the variance in P x, is delta P x square, where the variance of course, is defined as the expectation value, of X minus the average value of X the whole squared and so on. So, this can be easily checked to be expectation value of X squared that is the first term, minus expectation value of X the whole squared. So, this is the variance similarly, I can give a definition for P sub x.

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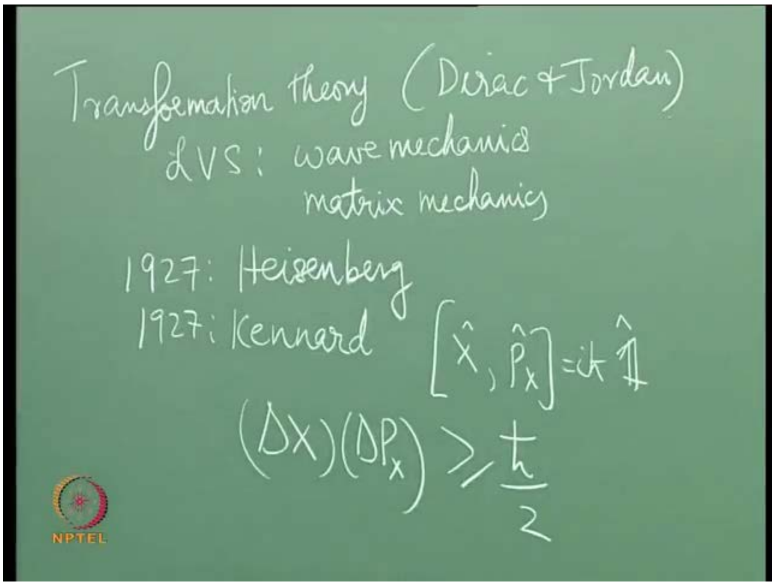

What Kennard showed was that delta X delta P x, was greater than or equal to h cross by 2. This is a very interesting statement. Because, however good the experimental measurements may be however good the instruments may be, there is always a minimum uncertainty delta X delta P x equal to h cross by 2, in some state, which is a minimum uncertainty state and for other states it is greater than this minimum value.

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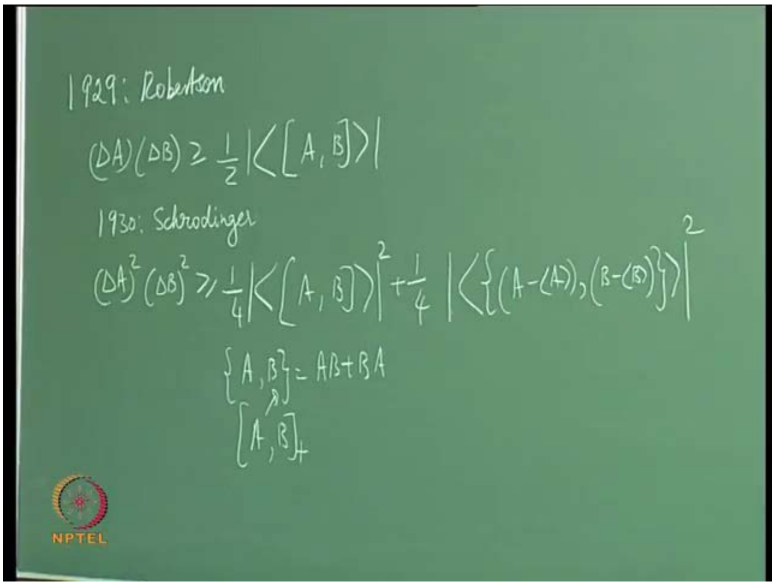

In 1929 Robertson, derived this uncertainty relation in a more general setting. Where two operators a and b were considered, Hermitian operators. Because, they represent observables and Robertson showed that delta A delta B, was greater than or equal to half. The modulus of the expectation value of the commutator of A with B It is pretty clear,.

that if A and B commuted with each other then this quantity became 0. And you have delta A delta B, greater than or equal to 0. But, in general this is true.

In 1930, Schrodinger generalized it to give his uncertainty principle, which can be stated in this fashion. Certainly the first term is present. And then there is an addition, which involves the anticommutator, of a minus expectation value of A, with B minus expectation value of B. The anticommutator is defined between two operators A and B, as A B plus B A. Sometimes one uses the notation, A B plus for the anticommutator and just A B for the commutator, the square bracket A B for the commutator and A B with the plus out there for the anticommutator. So, you see there are two terms that contributed to delta A square delta B square. These are the variances in A and B in this particular state.

So, it is clearly a state dependent statement. And in a given state the product of these variances is greater than or equal to this contribution one from the commutator and the other from this anticommutator. So, since these are all positive quantities, it is pretty clear that, delta A square delta B square is definitely greater than the first term and that is why you get the relation which Robertson gave. So, this is due to Schrodinger. This is a Schrodinger's uncertainty principle. In order to derive this uncertainty principle and study the various ramifications of the principle, one really starts with the Schwarz inequality. So, i will first derive the Schwarz inequality and then proceed to explain the uncertainty principle in this context.

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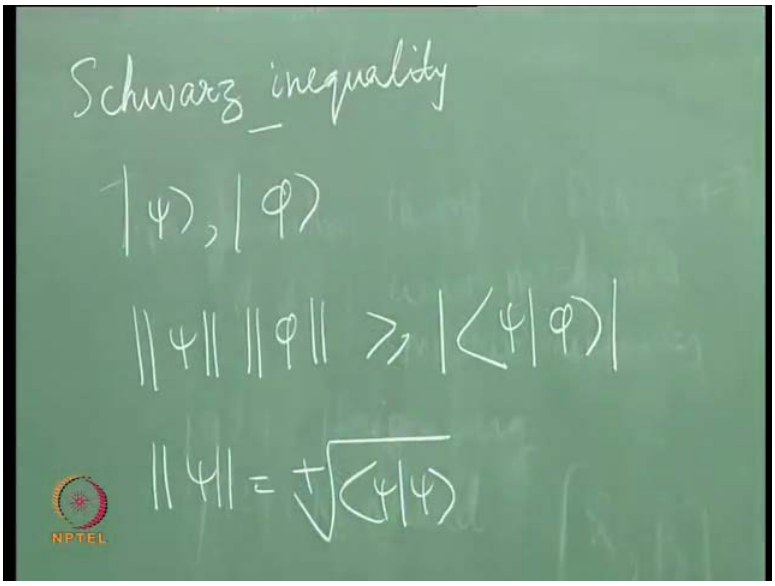

So, Schwarz inequality is merely this statement. Suppose, you have two states psi and phi, norm of psi times norm of phi is greater than or equal to the modulus of the inner product of psi with phi. It is clear that if phi with the null vector, or if psi with the null vector, then the equality holds, because this side is zero and so is this side. But, in general the nontrivial statement would pertain therefore, two vectors which are not null vectors. Let me recall that the norm is the positive square root of the inner product of psi with psi. It is a length of the vector. So, generalization of the modulus or the length of a three vector, a vector in usual three dimension space.

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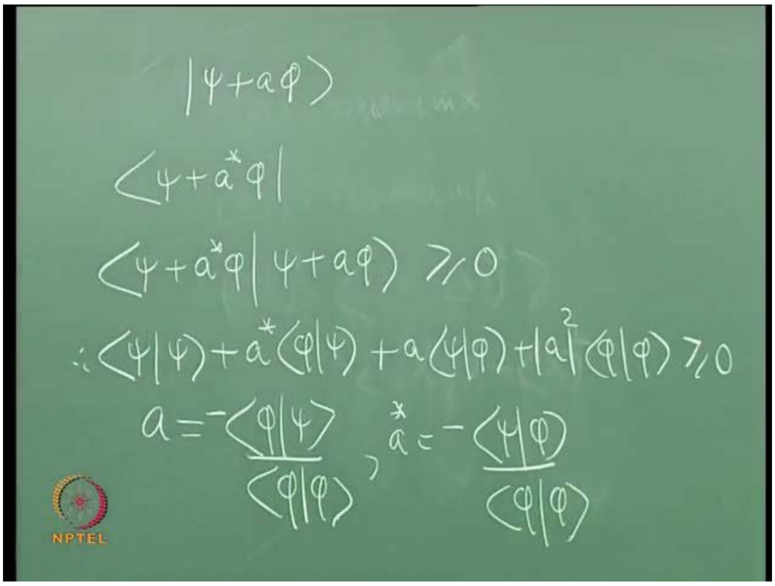

So, let me first consider, this object ket psi plus a phi, where a is some constant a complex constant Now clearly the bra is this. These are states in the linear vector space. I wish to find the inner product of psi plus a star phi with psi plus a phi. Now, this object is clearly the inner product of a state with itself.

And therefore, is greater than or equal to 0 and since these are not null vector psi and phi, it is greater than 0. In any case this object, in general is greater than or equal to zero or I can expand this and I therefore have the following, if i did a term by term expansion. This quantity is greater than or equal to 0, a is an arbitrary constant. Therefore, I choose a specific a. Let me choose a to be this object, minus phi psi by the inner product phi phi. So, all I have to do is substitute for a and a star, it is clear that a star is minus of psi phi, divided by phi phi.

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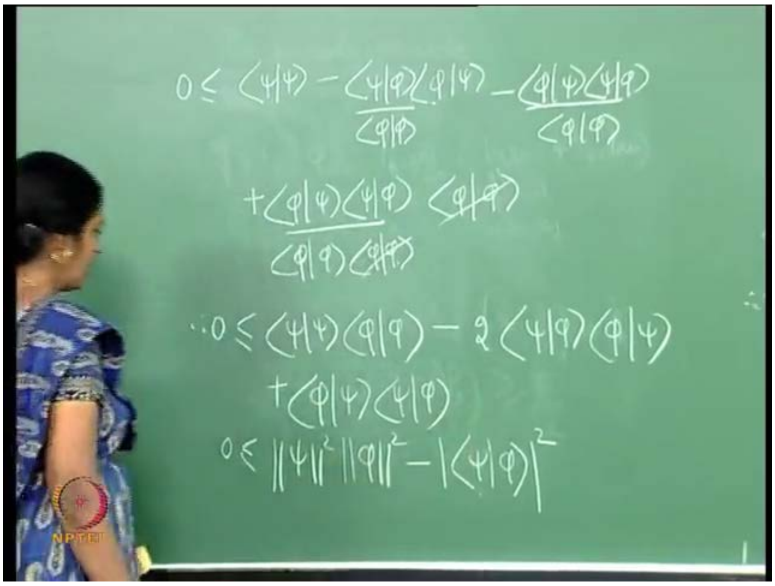

So once I substitute, I have psi plus a star phi or anyway that is greater than 0. 0 is less than or equal to, psi psi substitute for a. This object is in general a complex number and this is its complex conjugate so that is like z z star. The 3rd term is the same as the 2nd term because these are numbers and therefore, they become commuted across.

Then I have another term modulus of a squared which is a a star. That multiplies phi phi. So, I can score that out and now i multiply throughout, by this inner product. That is from the first term, I am just clubbing the 2nd and the 3rd terms and the 3rd term is simply this, the same as the 2nd term, half of the 2nd term. And therefore, I have 0 is less than or equal to this is, norm psi square, multiplied by norm phi square. Between these two I just have minus psi phi, with phi psi, and that is simply the modulus of psi with phi the whole square. I could have well written this as phi here and psi there, I should write this clearly that is a psi ok. Therefore, taking it to the other side, I have got Schwarz inequality.

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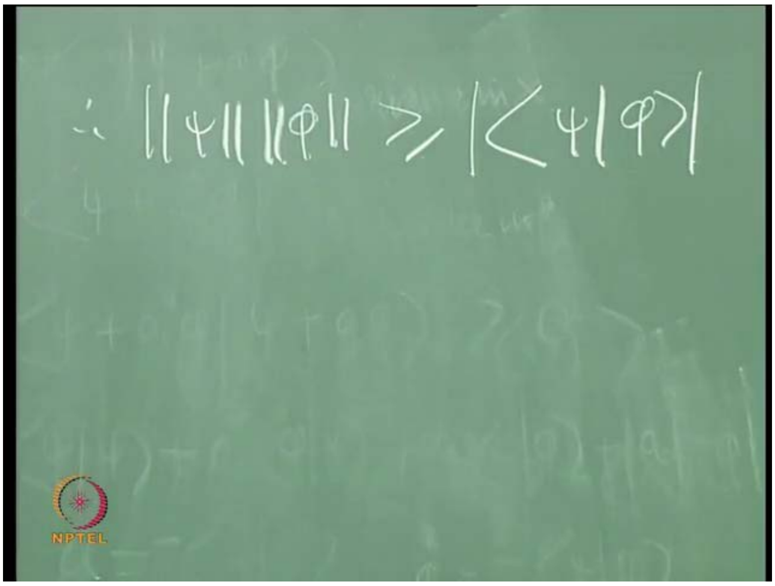

Therefore, norm psi norm phi, is greater than or equal to modulus of the inner product of psi phi, and that is Schwarz inequality. With this is something we use in order to derive, the uncertainty principle, which is what I let him to do now.

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For this purpose, I consider two operators A and B. Hermitian operators, representing observables. I look at this state A minus expectation value of A, acting on psi. If I want to find a variance, that is the expectation value of a minus expectation value of A, the whole square. So, I can well write this as psi and this is Hermitian so this is what it is. I could called this object psi 1, just for notation and therefore, this is psi one psi 1 inner product which is norm of psi 1, the whole square. Similarly, I can construct a state psi 2. So, delta B square is obtained in the same manner, as I did delta A square and that is psi 2 square. The idea is to find out (Refer Slide Time: 06:54) what is delta A square delta B square. In other words, norm psi 1 square norm psi 2 square, (Refer Slide Time: 16:10) so this is like psi 1 and that is like psi 2. And therefore, I use this Schwarz inequality right way.

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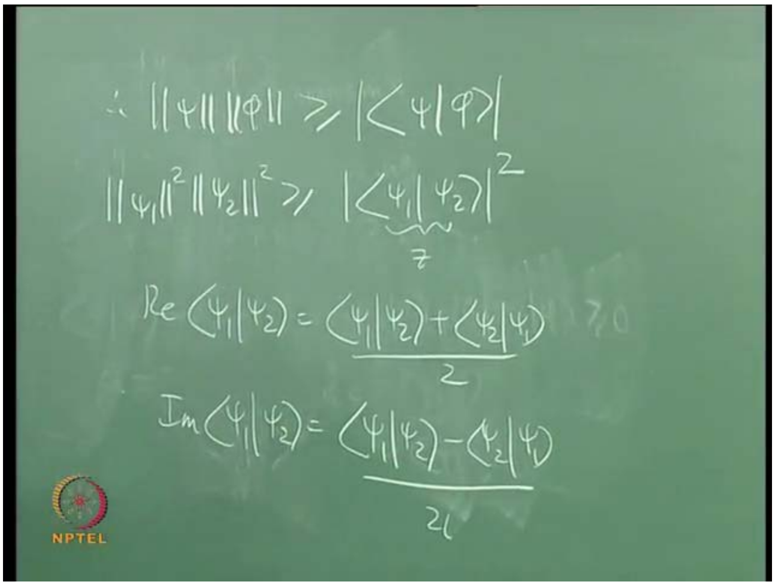

And because of this, I know that norm psi 1 square norm psi 2 square, it is greater than or equal to the modulus, of the inner products of psi 1 with psi 2 the whole square. This is a complex number. It has a real part and an imaginary part. So, the real part of this complex number is this. And the imaginary part, so this complex number can be written in terms of its real part and its imaginary part. We wanted modulus squared, the modulus squared is clearly going to be the real part squared plus a imaginary part squared. So, let me consider that.

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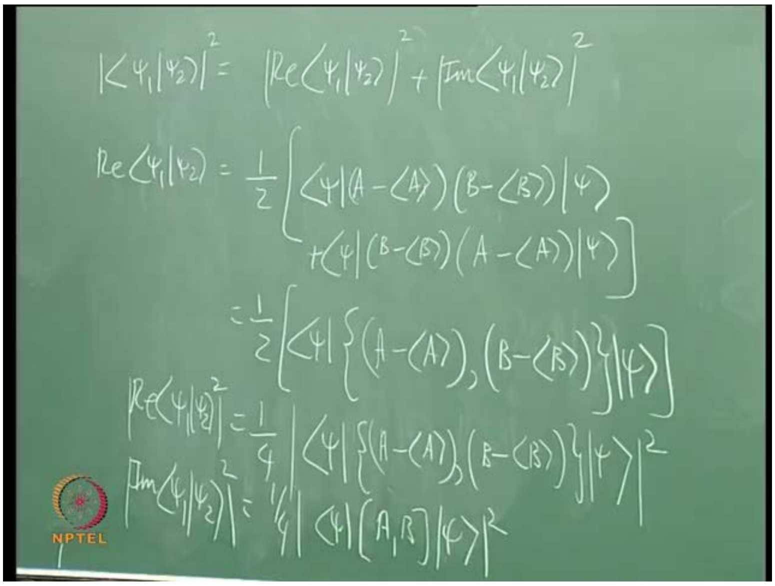

So, if i want the modulus square that is as good as saying, has two parts. So, if we consider first the real part. What i expand it, it is simply half the inner products psi 1 psi 2 plus psi 2 psi 1, plus B minus expectation value of B, A minus expectation value of A. You could put hats on all these objects to show that they are operators. But, in this notation it should be pretty clear, what the operators are. But, this is simply half of anticommutator of A minus expectation value of A, with B minus expectation value of B.

We are interested in the square of this and therefore this is quarter. Modulus of the expectation value of the anticommutator of A minus expectation value of A, with B minus expectation value of B, squared. (Refer Slide Time: 06:54) Which is the structure that we see here, this term from the Schrodinger uncertainty relationship has already appeared, we have accounted for the anticommutator. So, next thing is to look at the commutator which is the imaginary part. In a similar manner, we can show this, commutator of A with B, the expectation value of that mod square.

So, this really comes from the imaginary part of an appropriate inner product and this comes from the real part of the inner product. So putting the two together I get dealt A square delta B square is greater than or equal to this term plus that term. And that is the proof of the Schrodinger uncertainty principle. Now it is good to discuss certain cases in this setting. First of all, let us go back to the two level atom model. And since the uncertainty relation involves commutators, it would be good to look at objects which do not commute.

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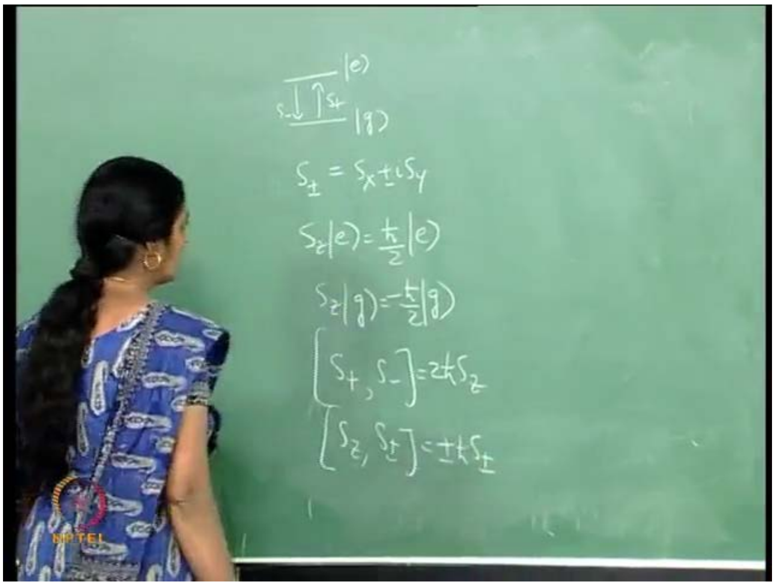

So, I first example would be the two level atom, and just to remind you S plus takes us from ket g to ket e, and S minus takes us from the excited state down to the ground state, and S plus is S x plus i S y, and s minus is S x minus i S y. I want to look at an Eigen state of S z. So we know that these two are the bases states which are Eigen states of S z. And just to refresh your memory, these are the corresponding Eigen value equations. We also have a commutator algebra, in terms of S plus S minus and S z, they look like this.

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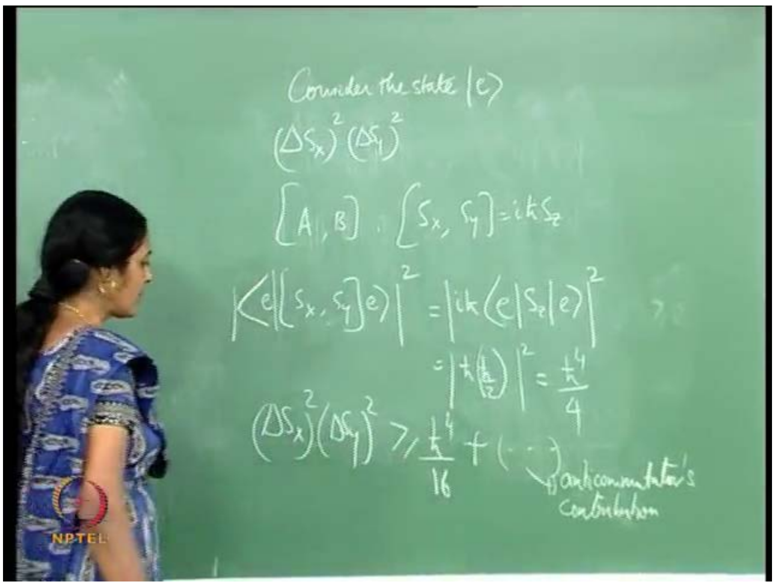

So, I am interested in finding out delta S x squared delta S y squared. The product of the variances in S x and S y and the state that I consider, is an Eigen state of S z, say ket e. So, I use the uncertainty relationship. Now, the first term is the commutator, so look at the left hand side of the uncertainty relationship, first of all the commutator of A with B.

In this case it is the commutator of S x with S y, which is i h cross S z. I want to find the expectation value of the commutator of S x with S y, in the state e and take the mod square of that expectation value. That is the same as modulus I pulled out the i h cross, S z e gives me, I can forget the i because I am taking the modulus. So, that gives me an h cross the S z gives me another h cross by 2 and this is to be squared. So, I basically have h cross to the power 4, out there and there is an h cross by 2 here and therefore, I have h cross to the 4 by 4. So this is what I have.

So this is the contribution from the first term. (Refer Slide Time: 06:54) So delta S x square delta S y square, is greater than or equal to quarter of this, plus h cross. Now, this comes from the anticommutator. This is a contribution from the anticommutator, of S x with S y. So, let us find out what that is?

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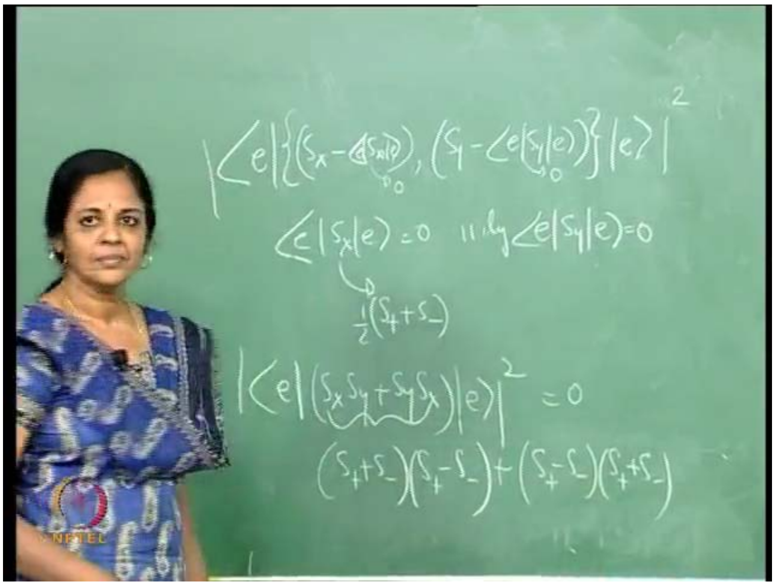

I wish to find out the expectation value, of S x minus expectation value S x, in this same state e. The anticommutator of that with S y minus expectation value of S y in that state, and then of course the mod square and so on. First of all in the state e, the mean value of S x is zero. The simplest way of seeing this is to write S x as S plus plus S minus. And since S plus acts on e to destroy it, there is no contribution from that term and S minus acts on e to give me g, ket g and ket e and ket g are orthogonal to each other. Therefore, there is no contribution.

Similarly, this average value is also 0. So these terms drop out and all I have, is to find out the modulus of the anticommutator, its average value and square it. I need to estimate what is S x, S y plus S y, S x. Now, apart from factors, numbers which I can pull out. This quantity is S plus plus S minus, S plus minus S minus, plus S plus minus S minus, with S plus plus S minus. So, the first term is an S plus squared which doubles up. So, I have a 2 S plus square but when S plus acts on e it destroys it and therefore, there is no contribution from there.

Similarly, there is a minus S minus squared and that comes twice over but again there is no contribution from that, because s minus acting on e twice. First time gets it down to g and the next time destroys it. Then there are cross terms, there is an S minus S plus, plus S plus S minus and that cancels out. Then there is an S plus, S minus with a negative sign and that cancels out with the term here and therefore, this contribution is 0. Does it turns out that the anticommutator does not contribute at all and any contribution that comes, to the right hand side of the uncertainty relation is from the commutator.

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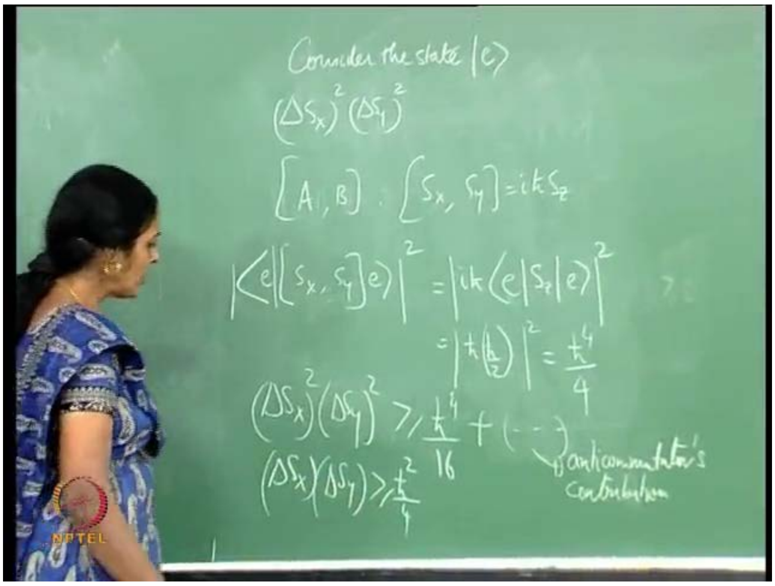

And so I have delta S x, delta S y it is greater than or equal to h cross squared, by 4. So, here is an example, where I have objects that do not commute with each other and therefore, I have a minimum uncertainty value of h cross square by 4. This happens in the state e, which is an Eigen state of S z. You can construct states where it is greater, where this uncertainty product is greater than h cross square by 4. It would be a good exercise to see, what is the value of this product delta S x delta S y? In the other base state ket g, as also in super positions of ket e and ket g. Now, let me look at another example. My next example is to find out again in this two level model.

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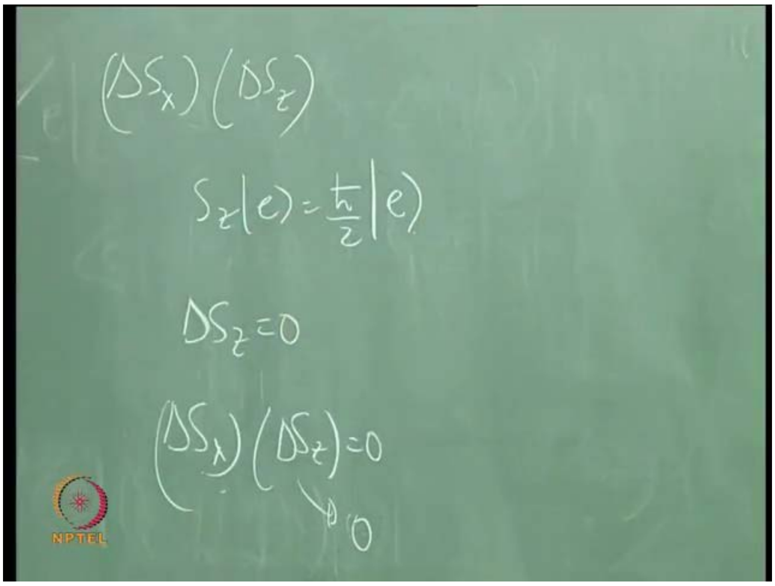

The uncertainty product delta S x, delta S z, in the Eigen state of S z, that is ket e. So, what is it that I have here? This is an Eigen state of S z. Therefore, delta S z is equal to 0. And therefore, delta S x, delta S z, is equal to 0. But, that does not been that delta S x is 0, in this state ket e. This is in fact finite and we can check this out. So, what is that variance of S x in this state?

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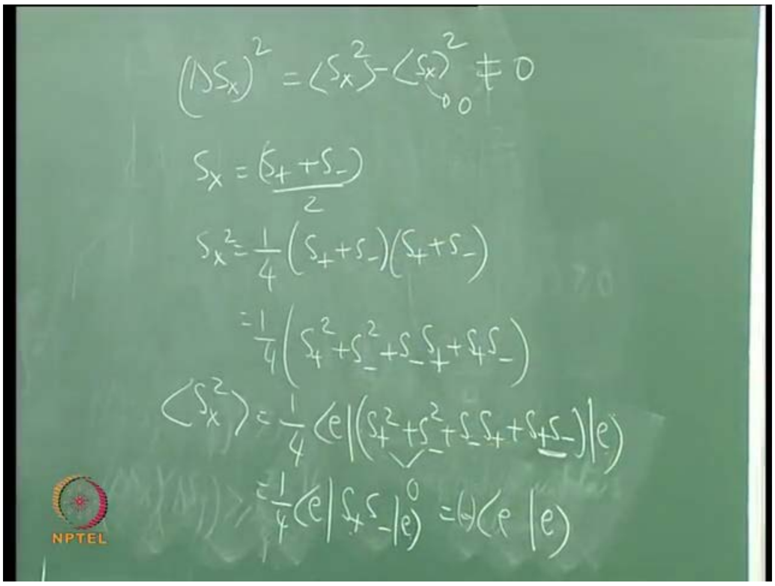

So, delta S x squared, in the state ket e, is expectation value of S x squared, minus expectation S x the whole square. This is 0 as we have seen earlier but this object is not 0. There is a spread in the measurement of S x. We can compute that right away. Therefore, S x square this quarter, S plus, plus S minus times S plus, plus S minus. Now, it is clear that when these operators are sandwiched, in the following manner, between the states ket e and bra e. This does not contribute and this does not contribute. So the contributions are 0 from here. But, this task because S plus, S minus out here S minus acting on e, gives me g and S plus acting on g, takes it back to e and there is an e on this side.

Similarly, here I have an S plus acting on e which will annihilate and therefore, not make a contribution. But, I do have this term, which states. So, essentially this is quarter expectation e S plus S minus e and S minus e takes it to g, but S pus takes it back to e. So, apart from a number which is non zero, there is a non zero value for expectation S x square. And therefore, delta S x the whole square, is not equal to 0, although, the mean value of S x in the state ket e is equal to 0. The following points emerge. We have an uncertainty product of this kind. (Refer Slide Time: 06:54) That if you make a measurement of observables A and B. Look at their average value their variances and so on. Then the product of the variances satisfies this relation, there is an inequality here, could become equal for some states.

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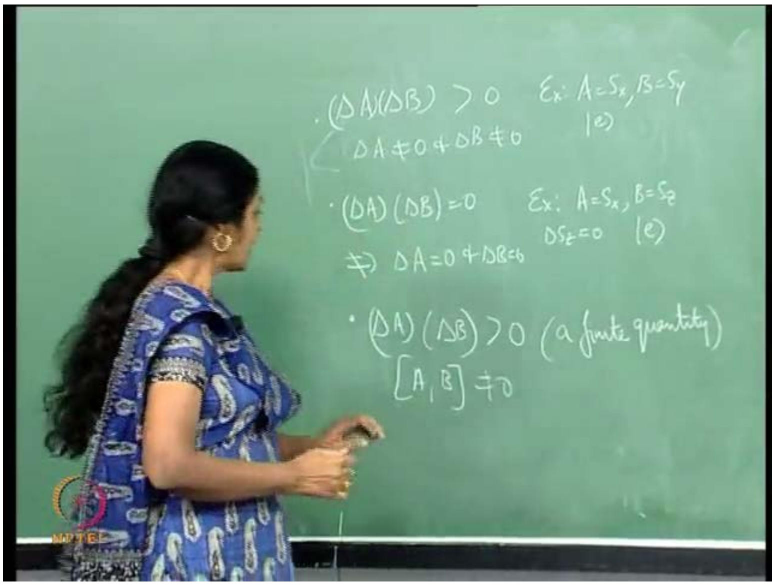

We have seen the following cases. If delta A delta B product, is greater than zero, then clearly the contribution has come because this or these two are non zero. It is clear therefore, that delta A is not equal to 0, and delta B is not equal to 0. So, this was the first example we had. Example: A was S x and B was S y and the state considered was ket e, which was an Eigen state of S z. Suppose delta A delta B is equal to 0, which is what we had in the second example, A was S x and B was S z. So, delta S z was equal to 0. The state considered was ket e, it does not mean it does not imply that delta A is 0 and delta B is zero, it does not imply this at all. It simply means that one of them 0 in general and that is what we have seen in this example.

Now, it is possible that delta A, delta B greater than 0, equal to a finite quantity and A, B commutator is not equal to 0. And that is how the contribution came. It is possible to make delta B 0, which means you measure B with infinite precision. In that case the spread in A is going to be very large, such that the product of 0 and infinity gives me a finite quantity. So, it is possible to be in an Eigen state of B, such that there is infinite precision in the measurement of the B. But, then you have really very little idea about the actual value of A, when you measure it in that state. So, I look at another example right now and that has to do with the position momentum, uncertainty relationship. Relation which is what Heisenberg gave us and Robertson formulated precisely.

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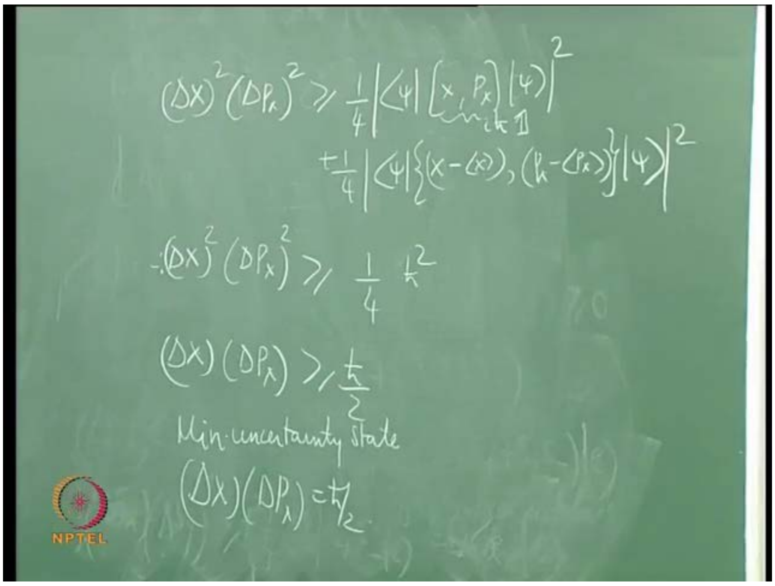

In that case, looking at the general uncertainty relationship I have variance, delta X square delta P x square, is greater than or equal to quarter of the modulus of the expectation value in the state considered of the commutator of x with P x, whole square plus quarter, of the anticommutator expectation value, the anticommutator being X minus expectation value of X, with P x minus expectation value of P x, in that state. Now, if you get this, X p x is i h cross identity and therefore, delta X square delta P x square, is greater than or equal to quarter h cross squared, telling us definitely greater than this because this could also contribute perhaps.

So delta X, delta P x is greater than or equal to h cross by two. The minimum uncertainty state is 1, corresponding to which delta X delta P x, is equal to h cross by 2. And we will see that the ground state of the oscillator for instance, is in minimum uncertainty state. In optics zero photon state is a minimum uncertainty state and so on, minimum uncertainty in these variables.

In order to actually make this estimation and given example a concrete example. I have to consider a specific system and it will be good right now to move on to a slightly more complicated system, the system which does not have two discrete levels, as its Eigen spectrum for the Hamiltonian, but an infinite number of discrete levels for the Eigen spectrum. In other words, it would be good to take an example; which pertains to an infinite dimensional linear vector space. A discrete infinity of levels, the simplest thing that one can consider is a multilevel system, where the Eigen vectors are equally spaced and indeed this is a situation with the harmonic oscillator. So, it is a good thing now to consider the next level of complexity, which is an infinite dimensional linear vector space but with a discrete equally spaced energy Eigen spectrum.

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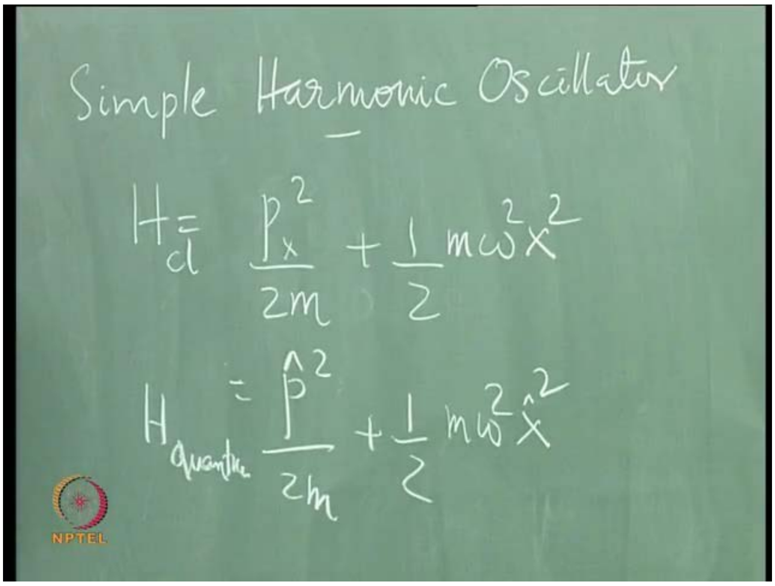

So, if you consider this simple harmonic oscillator and quantize it, before we actually proceed to do the problems, solve the problem explicitly, we take cognizance of the following. A classical Hamiltonian assuming that the oscillator is in along the X axis. This P x square by 2 m, oscillator of mass m, plus half m omega square X square. I am going to drop the suffix x, simply because there is no other momentum in consideration. In terms of operators, that would be p square by 2 m, in quantum physics this is h classical, h quantum is an operator, plus half m omega square, X square. We will see that there are an infinite number of energy levels for this Hamiltonian. When you quantize a system, the ground state of this system happens to be a minimum uncertainty state in X and P x, and if you wish to give a position representation to the ground state. It will turn out to be a Gaussian, a Gaussian in X, as also a Gaussian in P. Simply, because I will establish shortly in the course of the following lectures.

That the position space and the momentum space are Fourier transforms of each other. And therefore Fourier transform of a Gaussian being a Gaussian, when you go to the momentum space the structure of the wave function the form of the function as a function of momentum does not change. So, the Gaussian turns out to be a minimum uncertainty state. Now, X and P happen to be infinite dimensional matrices. I could construct operators out of these which would take me from one energy Eigen state of the oscillator to the next energy Eigen state of the oscillator. And these operators are going to be analogues of S plus and S minus in the spin system. With of course, the important difference, that there is an infinite set of energy levels in the case of the oscillator, as we will set out to prove shortly.

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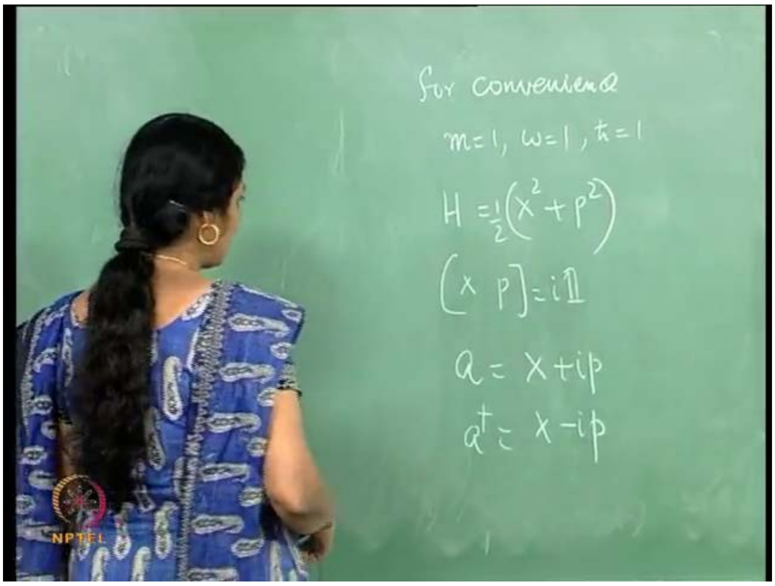

Suppose we forget all the constants here and I consider a generic system. Where, I have set for convenience I set m equal to 1, omega equal to 1 h cross equal to 1 and so on. If I did that well there are only these three in this context, then my Hamiltonian the quantum Hamiltonian, is simply X square plus P square by 2. I put back the m and the omega and the h cross later, with the commutation relation that X P is equal to i times the identity operator. I could find combinations, I could find a combination a which is X plus i p and a dagger which is X, minus i p, and obtain the commutator of a with a dagger. I could represent this algebra in terms of the commutator between a and a dagger. Now, first of all let us find out what is a dagger a? It is clear that a and a dagger are individually not Hermitian operators.

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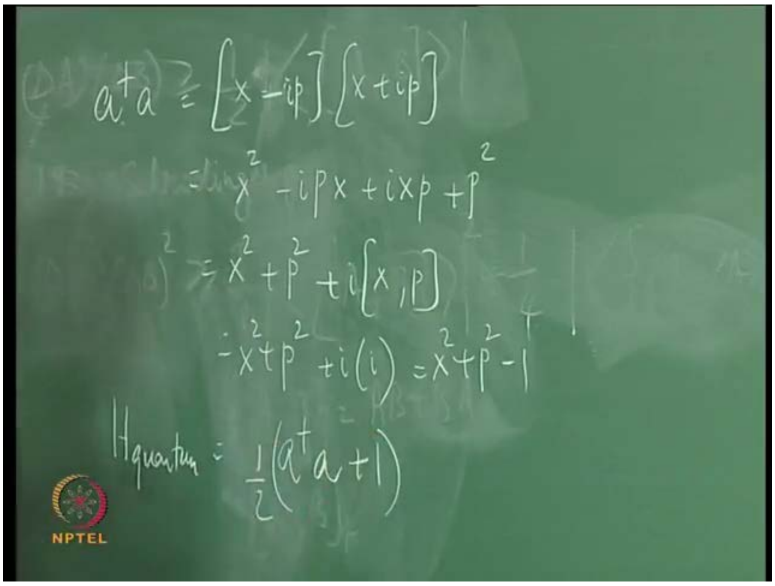

So, a dagger a is x plus i p, x minus i p, times x plus i p, that is x square minus p x, plus minus i p x, plus i x p, plus p square, which is a same as x square plus p square, plus i commutator of x with p which is x square plus p square and x p is i h cross. I have set h cross equal to 1. So, that is x square plus p square minus one. Therefore, the quantum Hamiltonian, can be written as a dagger a, plus 1 by 2.

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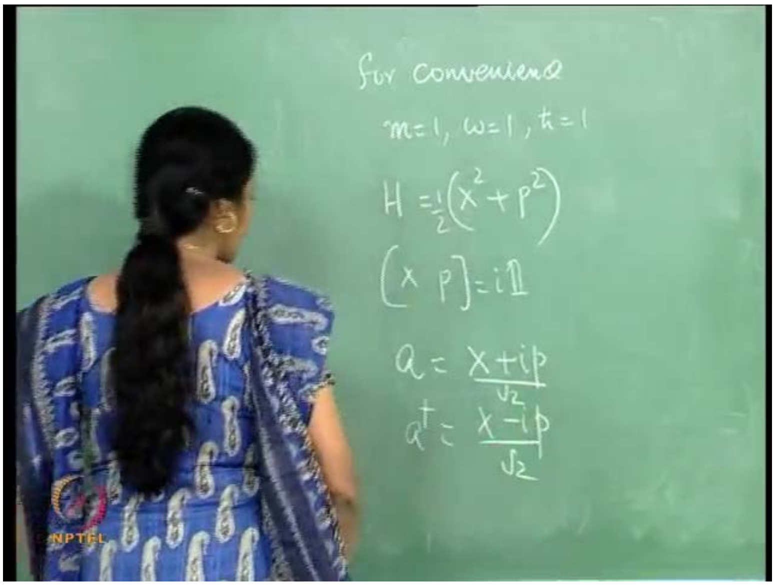

If you wish, it is easy to absorb, a root two in a and a dagger. So, that a dagger a already has a half here. So, in standard notation this is what we do, multiply the whole thing by a half, so that the quantum Hamiltonian can be written as a dagger a plus half. But, it is a Hamiltonian and if you put in the h cross the m and the omega properly. You actually land up with the oscillator Hamiltonian, to be a dagger a plus half, h cross omega. That has got the dimensions of energy in any case. One could study the quantum oscillator using the Hamiltonian a dagger a plus half h cross omega. I would formally define it properly in terms of X and p putting in the h cross the m and the omega, in the suitable places.

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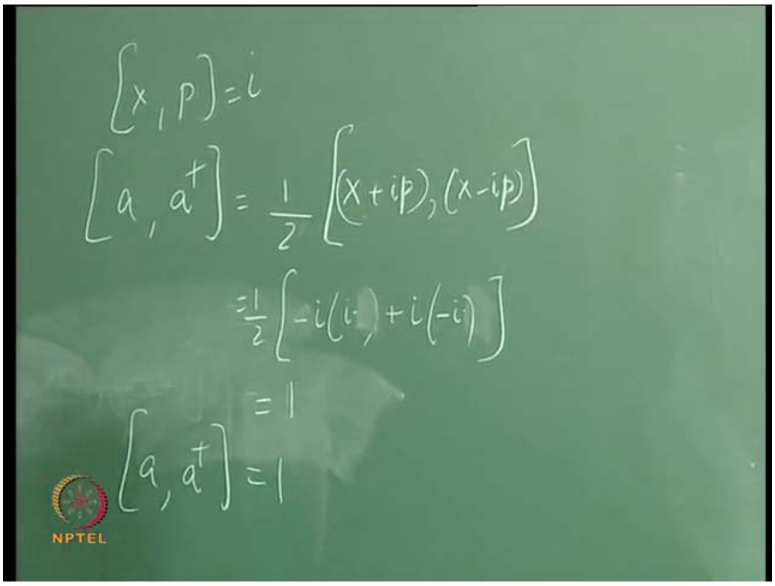

But, the algebra x p is equal to i h cross, (Refer Slide Time: 45:52) is equivalent to the statement. That a a dagger commutator, given that x p is equal to i h cross. In this case I am writing i because h cross is equal to 1. The commutator of a with a dagger, is the commutator of X plus i p, with x minus i p. I have the commutator of X with minus i p and that gives me an i h cross, out there. Then I have the commutator of p with x, which is minus i h cross. But, I have said h cross equal to one therefore, i am going to get rid of that. That is giving me 1 here.

Instead of writing x p is equal to i. I could have well written the commutator of a with a dagger is equal to 1. So, these are equivalent ways of writing the algebra of the commutators. Even as we wrote in the spin system, the commutation relation S x, S y commutator is i h cross S z cyclic or equivalently in terms of S plus S minus commutator. I could work with X p is equal to i or with a a dagger is equal to 1.

My next lecture I will use this Hamiltonian, a dagger a plus half h cross omega for the quantum oscillator Hamiltonian, and show what exactly is the minimum uncertainty state; in the case of the oscillator the equality, the Heisenberg relation equality, delta X delta p is equal to h cross by 2, would be established for the ground state of the oscillator.