Quantum Mechanics - I
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Lecture - 33
Illustrative Exercises-I

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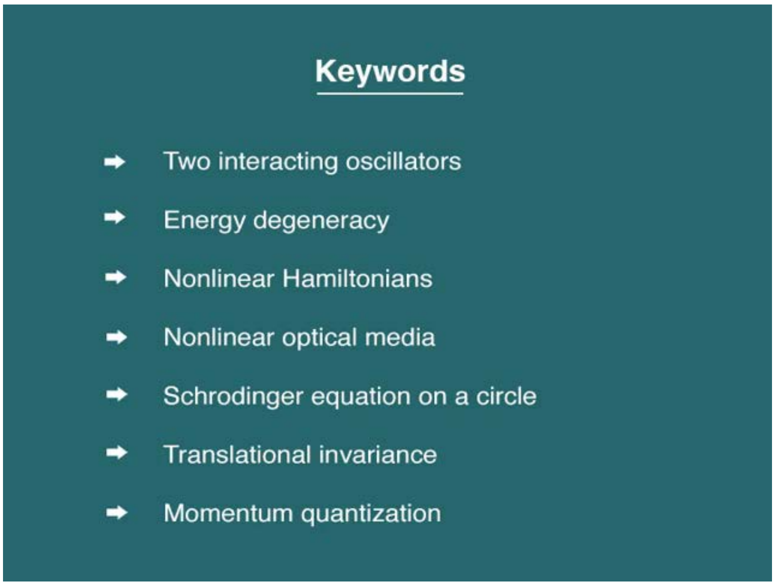

So, in the past weeks we have been looking at a variety of physical systems, largely one dimensional systems, but also we have looked at the central potential problem and over the set of lectures that I have given we have considered physical systems both in terms of abstract operator methods and using wave mechanics, where the wavefunction is a function of space and time. We have not yet concentrated on dynamics. We will do so in subsequent lectures, but today I would like to use the body of knowledge that we have acquired to look at more general systems.

General in the sense, for instance, two interacting oscillators or perhaps even a one dimensional oscillator but not quite a harmonic oscillator because you could have a Hamiltonian which has powers of a dagger a sitting in it. You could think of a situation where the particle of mass m is moving on a circle. So, you can generalize whatever we have learnt to a set of new physical situations and that is what I would like to consider today.

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So, I would like to call these applications of whatever we have learnt, maybe I should say some applications. So, the 1st problem that I would like to consider is the problem of two interacting oscillators. So, I have a Hamiltonian, both oscillators of mass m. So, the kinetic energies are p 1 squared by 2 m plus p 2 squared by 2 m and then, we have a potential term. Both oscillators have the same angular frequency. So, you have x 1 square plus x 2 square. The notation is evident, because corresponding to the coordinate x 1 there is linear momentum p 1 and corresponding to the coordinate x 2 there is linear momentum p 2. And then I put in an interaction of this form. It is clear that there is an interaction because x 1 and x 2 are coupled through some constant lambda.

Now, the 1st thing I know about lambda is lambda cannot be imaginary. It is a constant and it is a real constant, because if it were imaginary, it is clear that the Hamiltonian would not be Hermitian. If I take the dagger which means take the transpose complex conjugate and if this had an i in it, the Hamiltonian does not behave like a Hermitian operator. So, lambda is a real constant. Now, my aim is to make this problem appear simpler in the sense, that I would like to decouple this into two oscillators with coordinates say zeta and eta and corresponding momenta P and shall we say p. Of course since, the commutator x 1, p 1 is i h cross, x 2, p 2 is i h cross and x 1, p 2 is 0 and so on. I would expect the same kind of commutation relations to be obeyed here. In the since that zeta P commutator must be i h cross and eta p commutator p must be i h cross and so on.

So, the idea is to decouple this into two oscillators by going to new coordinates zeta and eta, zeta being a function of x 1 and x 2 and eta being another function of x 1 and x 2. Similarly, these two new momenta are functions of the old momenta. To many of you, this may appear to be a pretty simple problem indeed it is. But, it is worth going through this exercise because one learns a few things, good lessons which are worth remembering. So, let me look at this term.

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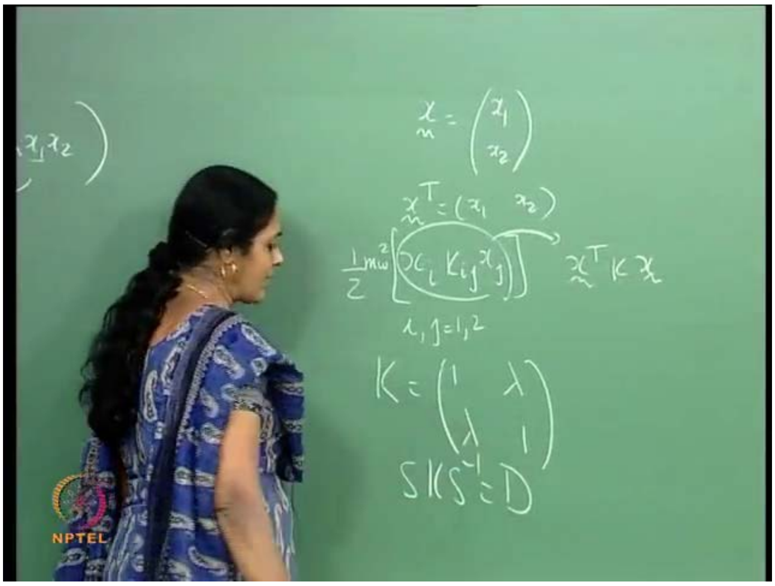

The simplest way to move to zeta and eta is to define an object x, a column vector which is given by x 1, x 2. So, that it is transpose is the rho x 1, x 2. So, basically I write this (Refer Slide Time: 01:49) whole thing, the potential term as half m omega squared x i k i j x j. Note that I have summed over repeated indices and i and j take values 1 and 2. So since, or I could well write this whole thing as x transpose K x. Since, this is 2 component row and that is a 2 component column, that is a 2 by 2 matrix. It is easy to see what the elements of the matrix K are. When i is equal to j, I get x 1 squared and x 2 squared. So, that is just going to give me 1 here and then otherwise there is a lambda x 1, x 2. So, k is simply this object.

Now, the idea is to make this diagonal and you can make K into a diagonal matrix by using a similarity transformation, which means I must find out that matrix with which I do the similarity transformation. In other words, I find a matrix S such that S K S inverse is a diagonal matrix. Of course, in order to do that I need to find the Eigenvalues and Eigenvectors of K and that is simple.

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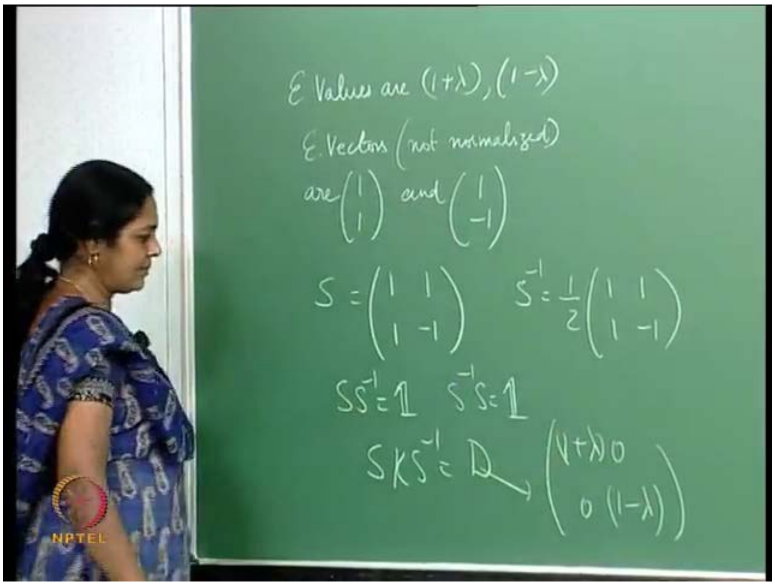

It is clear that the Eigenvalues of K are 1 plus lambda and 1 minus lambda and the Eigenvectors, not normalized are 1, 1 and 1, minus 1. Now, that tells me what S is because if I write S as the matrix made up of these Eigenvectors in this manner we can easily find S inverse. S inverse is half 1, 1, 1, minus 1 so that S S inverse is identity and because we are looking only a 2 by 2 matrices. It also implies that S inverse S is identity.

So, this is the matrix with which I do the similarity transformation. So, basically S K S inverse is the diagonal matrix with entries 1 plus lambda and 1 minus lambda. So, how do I get K in terms of this diagonal matrix? I have to pre-multiply this by S inverse and post-multiply that by S. Do the same thing here.

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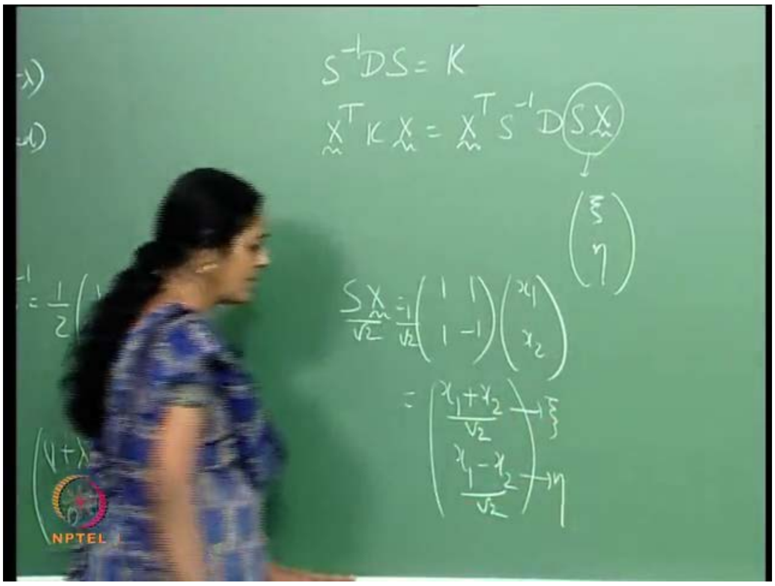

And therefore, I have S inverse D S is K. So, X transpose K x is the same as X transpose S inverse D S X. It is evident that this should be the new vectors. I will call them new coordinate zeta, eta like I wrote x 1 and x 2 I am now writing zeta and eta here. So, what is S X? S X is 1 1, 1 minus 1 acting on x 1, x 2 and that just gives me x 1 plus x 2, x 1 minus x 2.

As I said, the Eigenvectors when not normalized. So, if we take care of that I will have to put a root 2, this quantity by root 2 is what I want and therefore, I have this. Recall that I had a half here. (Refer Slide Time: 07:21) So, I am splitting it up as 1 by root 2 here and 1 by root 2 there and therefore, I do this. So, I have these objects: zeta and eta. Now, correspondingly it is clear that p (Refer Slide Time: 01:49) I can define the momenta P and as linear combinations of the old momenta.

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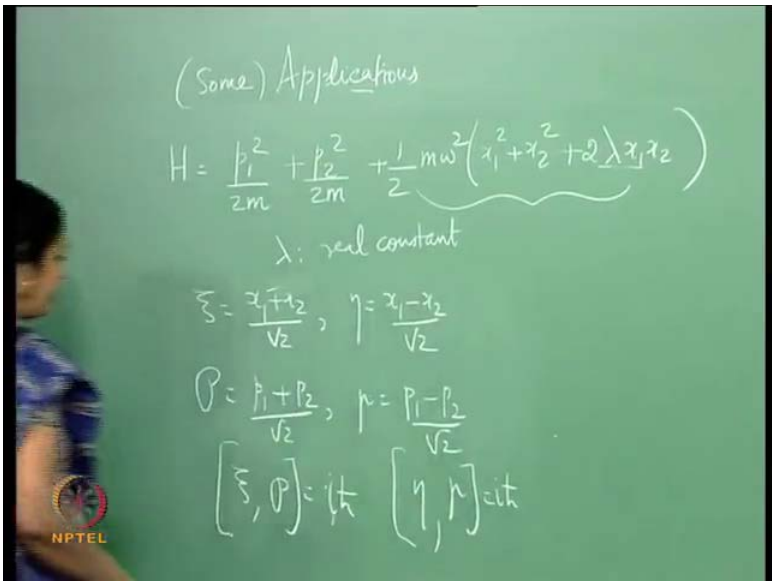

So, zeta is x 1 plus x 2 by root 2 and eta is x 1 minus x 2 by root 2. P is p 1 plus p 2 by root 2 and p is p 1 minus p 2 by root 2. It is not enough. We need to check that the commutation relations are obeyed. In other words, this should be equal to the same kind of commutation relation as x 1 and p 1 obeyed and x 2 and p 2 obeyed. So, I should get an i h cross here and similarly, eta with p must give me an i h cross. Indeed that is true. Because, if you look at the commutator of zeta with P it pulls out a half outside and x 1 with p 1 is i h cross and x 2 with p 2 in the commutator is i h cross. So, that gives me a 2 i h cross by 2 that is an i h cross.

Similarly, out here x 1 with p 1 and minus x 2 with minus p 2 and therefore. So, basically I have gone to a new pair of coordinates and their corresponding momenta such that I have the commutation relations obeyed. So now, I can write down my Hamiltonian in terms of zeta and eta. So, what do I get?

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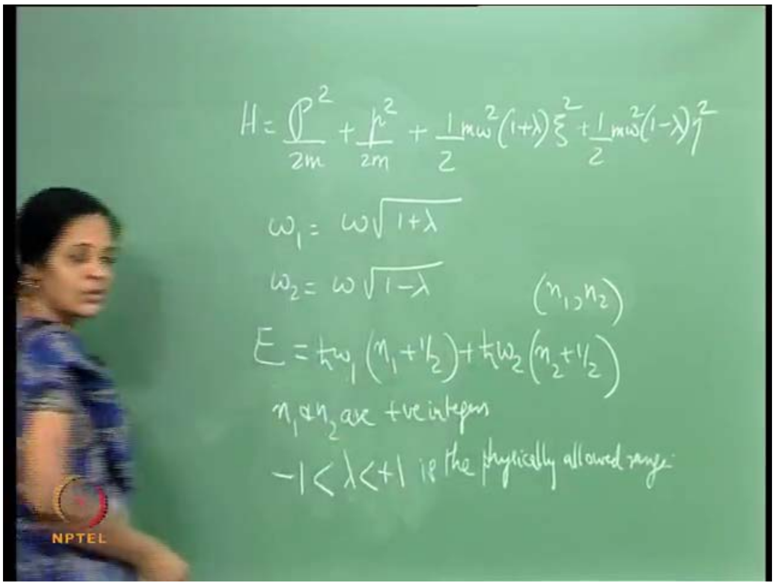

I ((Refer Time: 12:48)) I have H is equal to P squared by 2 m plus p squared by 2 m plus, (Refer Slide Time: 09:11) now look at what happens here. There is a zeta and an eta but then the diagonal elements are 1 plus lambda and 1 minus lambda. So, I have half m omega square times 1 plus lambda zeta squared plus half m omega square times 1 minus lambda eta squared. So, I really have two oscillators P squared by 2 m plus half m omega squared 1 plus lambda zeta squared and then p squared by 2 m plus half m omega squared 1 minus lambda eta squared. So, I had decoupled this into two oscillators.

Look at the frequencies: Oscillator one has frequency omega 1 which is omega root of 1 plus lambda and oscillator two has frequency omega 2 given by omega root of 1 minus lambda. So, while the decoupling has been accomplished, originally I had two interacting oscillators which had the same frequency. Now, there are two non interacting oscillators, but with different frequencies. So, what is the total energy? Total energy, the 1st oscillator will give me an n 1 plus half h cross omega 1. The 2nd one gives me an n 2 plus half h cross omega 2, where n 1 and n 2 are positive integers taking values 0, 1, 2, 3 and so on.

What is the physically allowed range for lambda? If indeed I want it to be two separate oscillators I know that lambda has to be less than 1 and greater than minus 1 so minus 1 less than lambda less than plus 1. This is the physically allowed range, because I am thinking of it as two independent oscillators. Do I have degeneracy in energy? Clearly this carries a number n which is n 1 plus n 2. If you wish I can write it as an ordered pair. Let me just call it E n and n is n 1 plus n 2 or n 1, n 2 that is the ordered pair.

The point is the following: if there is degeneracy, it means that I can have another set n 1 prime, n 2 prime satisfying omega 1 n 1 prime plus omega 2 n 2 prime is equal to omega 1 n 1 plus omega 2 n 2. So, let us see what that gives us.

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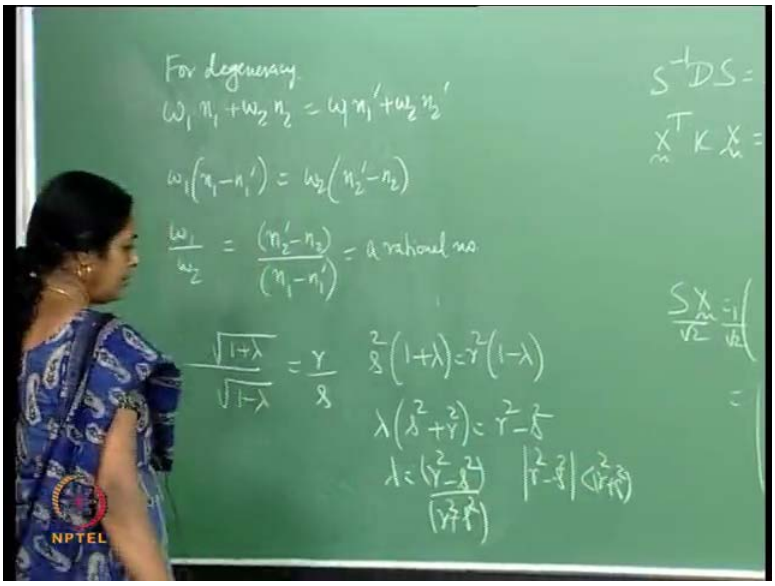

So, I work in the physically allowed range for lambda and I have omega 1 n 1 plus omega 2 n 2 is equal to omega 1 n 1 prime plus omega 2 n 2 prime, if there is a degeneracy. So, what does that mean? It tells me that omega 1 by omega 2 is n 2 prime minus n 2 by n 1 minus n 1 prime. But, remember these are integers so the statement is that omega 1 by omega 2, the two frequencies and the ratio must be a rational number in order that there is degeneracy.

So, what does that mean for lambda? Omega 1 is omega root of 1 plus lambda and omega 2 is omega root of 1 minus lambda and this is some rational number r by s. So, what is that imply for lambda? s squared times 1 plus lambda is r squared times 1 minus lambda or lambda times s squared plus r squared is equal to r squared minus s squared. So, lambda is r squared minus s squared by r squared plus s squared. So, if there is degeneracy and you can see that the degeneracy is not generally ((Refer Time: 18:25)). It is there only if omega 1 by omega 2 is a rational number. So, those are very very specific ratios that we are talking about.

In fact, in general if you had the original problem and (Refer Slide Time: 11:02) there was no interaction. There would have been degeneracy; because you are looking at a two dimensional isotropic oscillator and you know that there is degeneracy in that problem. This interaction actually lifts the degeneracy. In other words, there is a degeneracy only for specific values. That is, when omega 1 by omega 2, it is a rational number. Now, because lambda is allowed only the range minus 1 to 1 it is clear that modulus of r squared minus s square is less than r square plus s square.

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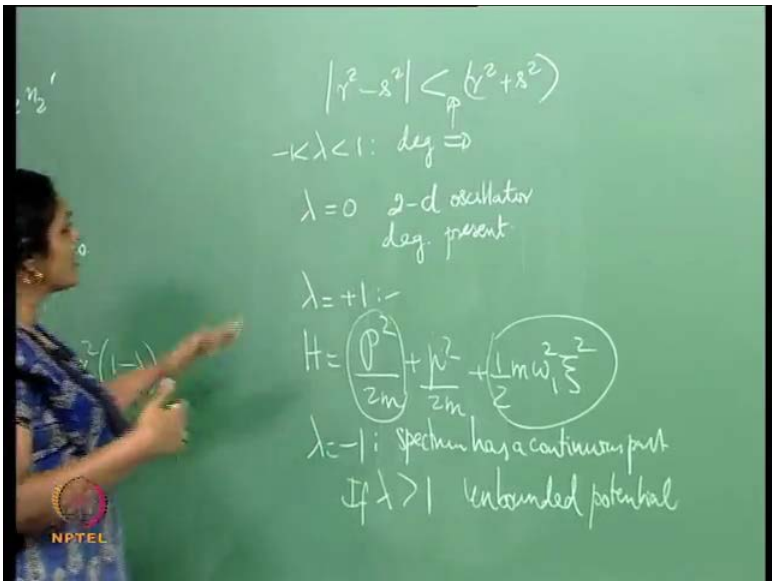

Let us write that properly. In order for the problem to be the problem of two uncoupled oscillators so this is what you have. So, in this problem the degeneracy is not a generic feature. It happens only sometimes. What are the important lessons that one learns from this? First of all, of course, minus 1 less than lambda less than 1 is the allowed range and degeneracy implies this. Second thing: What happens if lambda is equal to 0? Go back to the original problem. (Refer Slide Time: 11:02) If lambda is equal to 0, you have the isotropic two dimensional oscillator. So, oscillator and degeneracy present. Now, what happens if lambda is plus 1? Let us look at the frequencies. (Refer Slide Time: 12:47) If lambda is plus 1 there is no omega 2 and you just have an omega 1. So, what does the Hamiltonian look like? The Hamiltonian is P squared by 2 m plus p squared by 2 m plus half m omega 1 squared zeta square.

What kind of spectrum does this Hamiltonian have? Now, this P squared by 2 m plus half m omega 1 squared zeta squared is just an oscillator. So, that is a discrete spectrum n plus half h cross omega 1. But then, any value of p is allowed and therefore, the energy has a continuous part and a discrete part, very similar to the problem of the charged particle in a homogenous magnetic field which we have discussed earlier. So, the same thing happens when lambda is equal to minus 1. You will have the other oscillator. You have p squared by 2 m plus half m omega 2 squared eta squared plus P squared by 2. So, the spectrum really has a part which is continuous energy spectrum.

Now, what happens when lambda is outside the range? Let us imagine that lambda is greater than 1. If lambda is greater than 1, so lambda equals minus 1 again spectrum has a continuous part, same here. Now, if lambda is greater than 1, let us look at this. (Refer Slide Time: 12:47) A better still here. (Refer Slide Time: 11:02) Suppose, lambda is greater than 1 and x 1 and x 2 take large values with x 1 and x 2 being of opposite signs. Then surly this potential term, this whole thing becomes negative and the potential gets unbounded. So, it is an unbounded spectrum in general and that is the reason why we do not want to discuss it. It is not as if there is a lower bound for the potential term. So, these are the very many features that come out of this (Refer Slide Time: 11:02) problem.

You see a situation which is equivalent to the two dimensional oscillator and therefore, there is a degeneracy. You see a situation where the degeneracy gets lifted and you have degeneracy only for very specific values of the ratio of frequencies. Then you see a situation where there is a discrete part contribution to the energy and there is a continuous part. Then the other case is the one where the potential is unbounded and therefore, you do not want to consider that.

So, these are the very many aspects that you can see from this problem. This (Refer Slide Time: 11:02) is an example of two oscillators which are interacting with each other. The next example I want to talk about, the next application is to that of a single one dimensional case. But, where the Hamiltonian is not merely of the form a dagger a, but has higher powers of a dagger and a plus present in it. So, let us look at the next problem.

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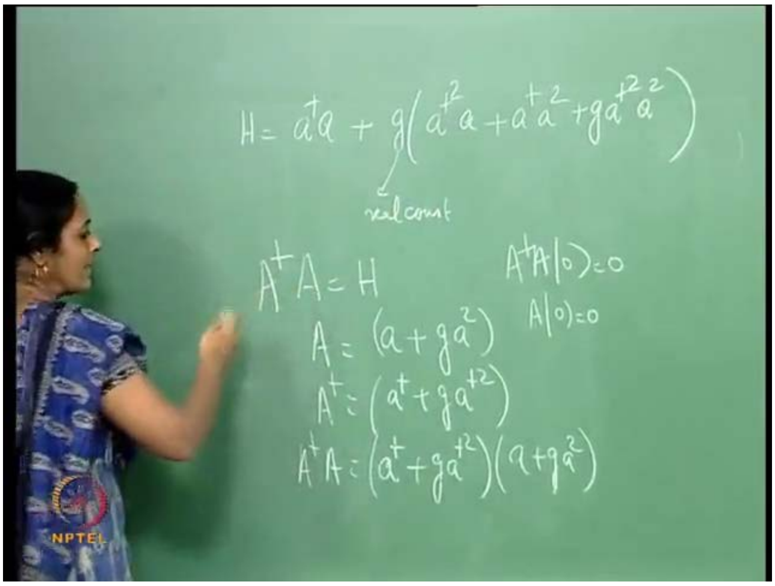

So I give you a Hamiltonian. This is my next problem. The Hamiltonian is a dagger a plus some constant, real constant g and, there is an a dagger squared a plus a a dagger squared, it is Hermitian conjugate plus a dagger a squared. It is Hermitian conjugate, plus g a dagger squared a squared then this is Hermitian in itself and this guy is the Hermitian conjugate of that. Now, you look at this Hamiltonian. First of all this Hamiltonian has a lot of significance when we deal with nonlinear optical media. As you know, I have used the harmonic oscillator Hamiltonian extensively in explaining aspects of quantum optics. Normally, I would have worked with just a dagger a. Now, if I put in an a dagger squared a squared term, that effectively models a nonlinear optical medium. Nonlinear because it has more than one power of a dagger and a and the same thing happens here.

So, there are physically realizable optical media, nonlinear media which can be modeled by this Hamiltonian. Is it possible? Let me ask the same question that I asked in the earlier problem. Is it possible to define an A and an A dagger such that A dagger A equals this Hamiltonian? Now, I can do that just by inspection. Because, if I write A as a plus g a squared then A dagger is a dagger plus g a dagger squared and A dagger A is a dagger plus g a dagger squared times a plus g a squared. Keep the order the same because A and A dagger do not commute with each other. So, the first term is a dagger a and then you have a g a dagger squared a as g a dagger a squared plus g squared that is the g squared a dagger square a squared.

So indeed, it is true that this is an example of a Hamiltonian which can be written in this diagonal form A dagger A except that, A and A dagger themselves have quadratic terms in a squared and a dagger squared respectively. What about the spectrum while the spectrum itself is little bit messy to get? First of all, is the ground state of the harmonic oscillator also a ground state of this Hamiltonian? That is true because A dagger A acting on ket 0 is 0 because a on ket 0 is 0 and a square on ket 0 is 0.

So, it really does not matter what happens here. A on ket 0 is 0 and therefore, it is true that both this Hamiltonian and the harmonic oscillator Hamiltonian have ket 0 as the lowest energy state.

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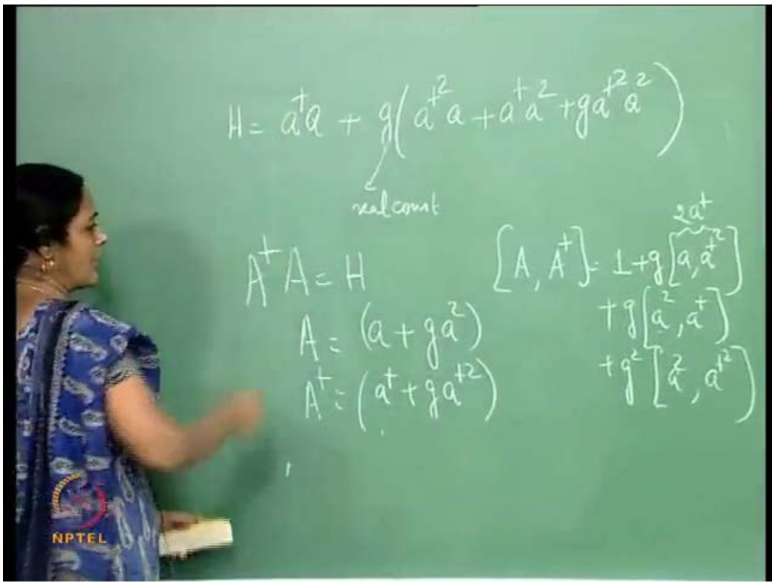

Having said that, in contrast to the earlier problem this transformation from a and a dagger to A and A dagger does not preserve the commutation relation, because if you now find commutator of A with A dagger. Well, the 1st term is A with A dagger which is 1 and then there is an a with a dagger squared. So, there is a plus g commutator of a with a dagger squared plus there is a commutator of a squared with a dagger and of course, there is a commutator of a squared with a dagger squared. Now, none of these vanish. Use the a b c rule and this just gives me an a dagger plus a dagger that is a 2 a dagger and this is 2 a and then of course, there is this object. So, this transformation does not preserve the commutation relation, in contrast to the earlier problem that I spoke about.

Now, suppose we look at the harmonic oscillator itself and that is by way of as an ((Refer Time: 29:47)). Suppose, I forget all this and I just looked at a standard oscillator Hamiltonian a dagger a. But instead now I want to go to the position representation.

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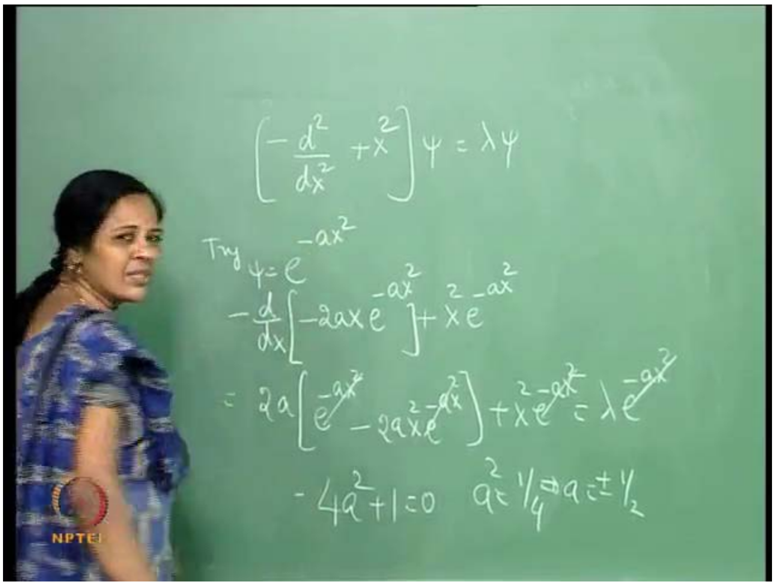

So, I start with an operator minus d 2 by d x squared plus x squared and you will recognize this as essentially the harmonic oscillator Hamiltonian. This is the del squared part of p squared and there is an x squared and clearly I have scaled out things set 2 m equals 1, h cross equals 1 and things like that. But then, you see if I now want to look, let me just do this problem. I want to look out for an Eigenstate psi of this Hamiltonian and I started with e to the minus a x squared. I am not normalizing it but suppose, I started with this because that is a twice differential that is going to come out.

So, you can see that this is minus d by d x d by d x of psi tri psi equals this, not normalized, plus x squared e to the minus a x squared. Now that is a 2 a. So, the 1st term is just e to the minus a x squared and the 2nd term is minus 2 a x squared e to the minus a x squared plus x squared e to the minus a x squared. Now, it is clear that once I cancel these things out I need a number. So, basically all that should survive is the 1st term 2 a. If at all this should be an Eigenvalue problem I should get lambda e to the minus a x squared which of course, I have cancelled out. I cancelled out e to the minus a x squared all over.

So, this 2 a should be equal to lambda which means that the coefficients of the x squared term should go to 0. So, I basically have minus 4 a squared plus 1 equals 0 or a squared is quarter or a is plus or minus half. But if I use a equals plus half here that is ok because things are all right.

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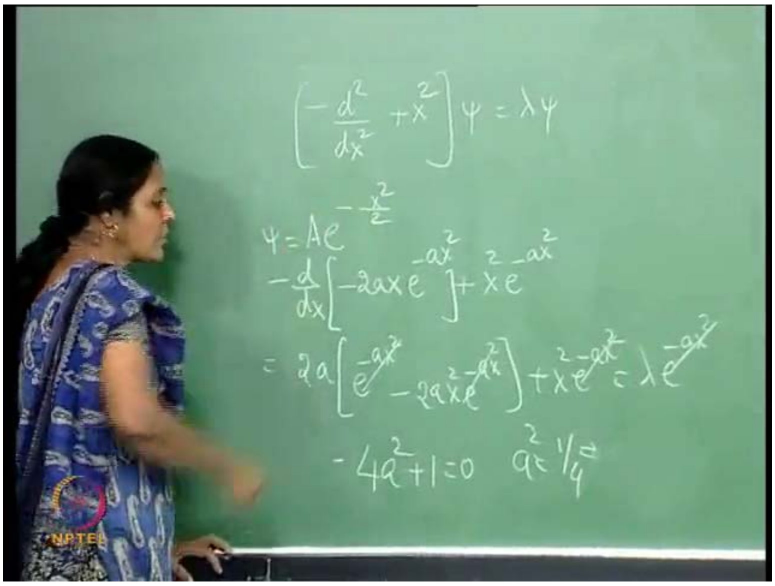

When a is minus half I have an e to the plus x squared and that blows up and no longer am I in a position to normalize it. So therefore, I get a is equal to plus half and my wavefunction would be minus x squared by 2, apart from a normalization. Now, you also know that this ground state of the harmonic oscillator. Not only this state but all the ground state, the 1st excited state, 2nd excited state and so on. You can find out expectation value of x.

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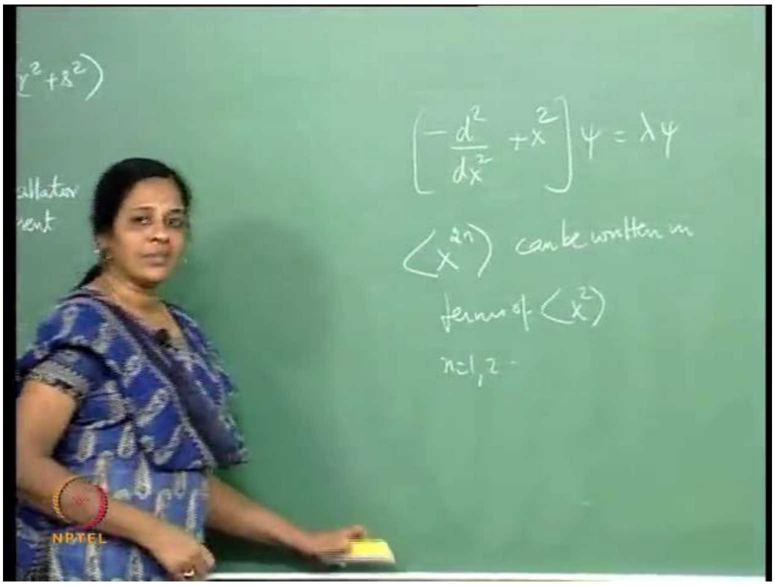

You can also find out expectation x squared and as you know expectation x is 0, you can find out expectation x squared in these states and since it is a Gaussian what survives is expectation x to the 2 n can be written in terms of expectation x squared, n equals: 1, 2, 3 and so on. Because it is a Gaussian all these higher even moments can be written in terms of expectation x square. I leave it to you as an exercise to do this problem and find out how exactly expectation x to the 2 n can be written in terms of expectation x squared. That is because it is a Gaussian.

Now, let us move on to the next application, so much for the harmonic oscillator problem and it is cousins and ramification. Let us do another problem which is working out the Schrodinger equation solution, the wavefunction on a circle. That again teaches us several lessons.

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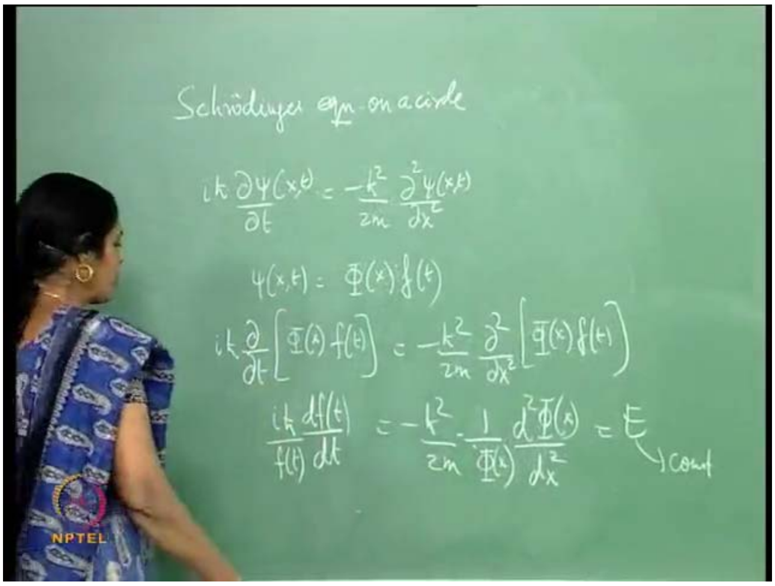

So basically, I want to now look at the next application, Schrodinger equation on a circle. Before that, let us consider a one dimensional Schrodinger equation with time dependence and you know that the equation is this. In fact, when you looked for stationary state solutions you wrote psi of x t as some phi of x T of t, a chi of t is perhaps a notation that I used out it would not quite recall, may be f of t. And then, when you did this, when you substituted it here you got two equations. Let us just go through that. Delta by delta t of phi of x f of t is minus h cross squared by 2 m delta 2 by delta x squared phi of x f of t.

So, this just pulled out a phi of x and I had an i h cross delta f of t. Now, it is d f of t because partial derivatives have become total derivatives here. It is minus h cross squared by 2 m f of t d 2 phi of x by d x square. Then of course, the standard trick, done it many times, divide by phi of x f of t and this will give me 1 by phi of x and we said that since this object is only a function of t and that is only a function of x, each of this should be equal to some constant E.

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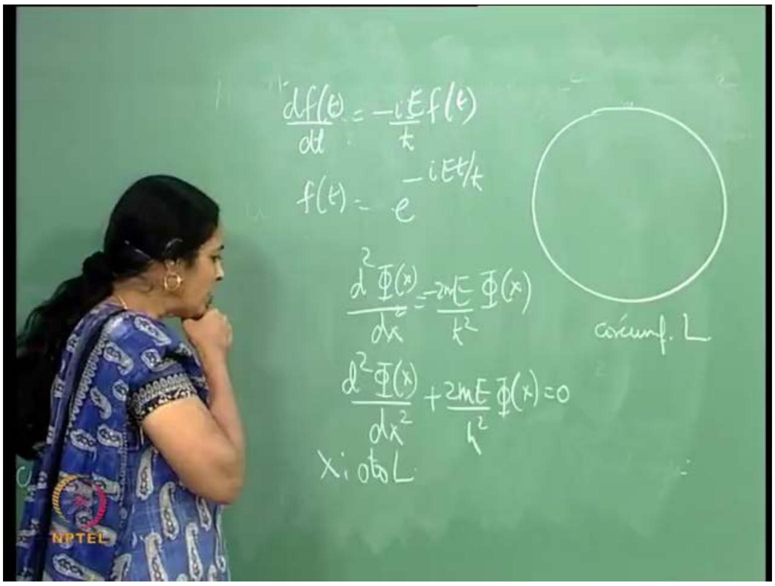

That gave me the equation for d f of t by d t 1 by i h cross E f of t. And the solution of course, was e to the minus i E t by h cross. But, that is as far as f of t is concerned. Look at the equation (Refer Slide Time: 34:45) for phi of x. I have minus h cross squared by two m d 2 phi of x by d x squared equals E phi of x. So, take this to that side. This is equal to minus 2 m E by h cross squared and therefore, d 2 phi of x by d x squared plus 2 m E by h cross squared phi of x equal to 0.

So essentially, this is the kind of equation that we have been looking at. Now, suppose I also had a potential of course, there would be an E minus v there. The important thing to note is the following: that if I were working with the particle in a box of size 0 to L. Suppose, x went from 0 to L, what kind of boundary conditions can I impose on this? See, first of all I have to be careful. It depends upon the barriers. There is no translation in variance in this case. Because clearly, if there is an infinite barrier at x is equal to L then the boundary condition there is very different from the boundary condition at x is equal to 0, if that is a finite well on that side.

Now, you look at this particle on a circle. So, this is the equation. I am going to think of a circle whose circumference is L. I want to keep track of the length dimension therefore, I call it L could have chosen 1, but then I decided to call it L because I want to keep track of the length scale in the problem. Now, on this circle I have this particle of mass m which is moving around. Got the temporal solution but now look at the space part. What kind of boundary conditions should I impose? Now, this is some k squared.

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Therefore, the general solution is simply going to be phi of x some A sin k x plus B cos k x.

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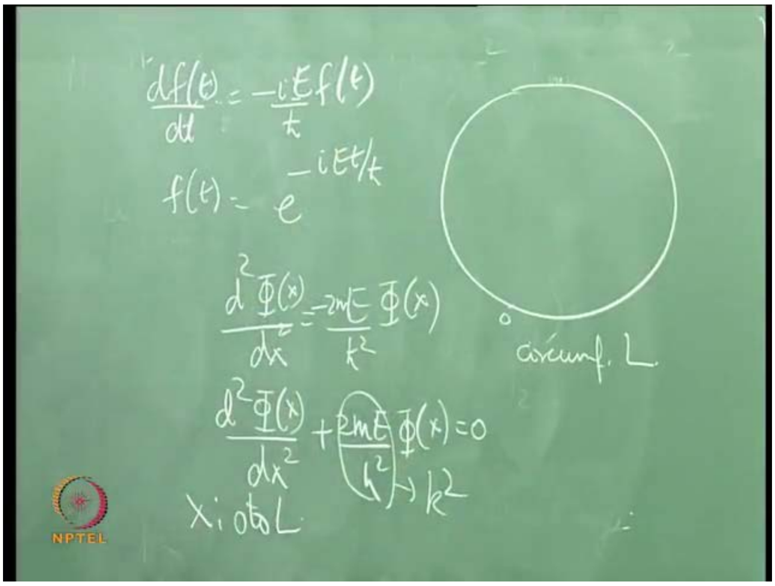

I could start off by saying that on this circle (Refer Slide Time: 37:07) if I start at some point which I call 0 then when I comeback that means, I have travelled a length L phi at 0, phi of 0 must be equal to phi of L. But I have also realized that there is nothing special about this point. I could have started from any point on the circle and if I go around it once I would expect periodicity. That is, I would expect that the wavefunction comes back to itself. Now, this is not new to you. You did this when, in a sense, when you did separation of variables in the problem of finding Eigen values for the arbitrarily angular momentum and then you wrote. You wrote the wavefunction. It is a function of r, theta and phi spherical polar coordinates and then the equation for the phi dependent part crucially, fixed the values for the magnetic quantum number to be integers and not half integers because you demanded single valuedness of the wavefunction.

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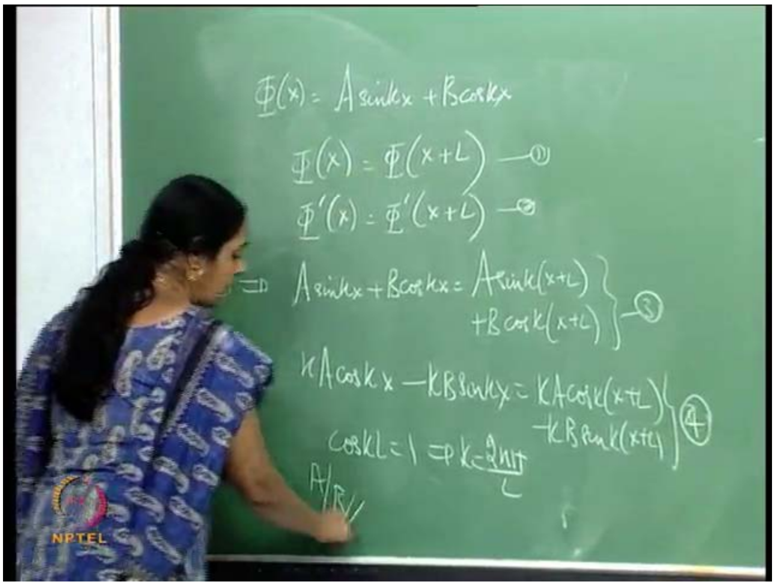

So, it is precisely that kind of thing that I would do here except that I could have started with any x of course. And, I should come back to the same point and nothing should change. So, I have phi of x equals phi of x plus L. That is fine as it goes, but what about the derivative? Should I also impose a condition? After all, this (Refer Slide Time: 40:27) is a 2nd order equation. Should I also impose a condition: phi prime of x equals phi prime of x plus L? x being any point on the circle.

Now, suppose I did that. Remember, I have two unknowns here A and B. So, the 1st condition implies that A sin k x plus B cos k x equals A sin k x plus L plus B cos k x plus L. That is the 1st condition. Now, look at the 2nd condition k A cos k x minus k B sin k x equals k A cos k x plus L minus k B sin k x plus L, that is a 2nd condition. Call that 4. But you see if I want to have, I want to solve for A and B. As I want to do that, the determinant should be 0. Now, if you solve this equation and find out what the determinant implies, it simply tells you that cos k L is 1 and you can solve for that. You want to solve for A and B and you have these two conditions and that tells me that cos k L is equal to 1 which implies that k is 2 n pi by L, one can take values 0, 1, 2, 3 and so on.

Now, if I substitute that back there I will find that I can address get A by B. I will not be in a position to determine A and B separately. So, what exactly is the problem? It is evident that the problem has arisen because I have put in these two conditions and I am not able to consistently solve, do not seem to have enough information to solve for A and B. The manner in which we resolve this problem is to say the following: first of all we know that there is translation invariance on the circle. (Refer Slide Time: 40:27) It is a free particle in contrast to the square well potential and so on where the particle was subject to a potential V of x a nonzero V of x. In this case, your Hamiltonian is simply P squared by 2 m, where P is the linear momentum of the particle.

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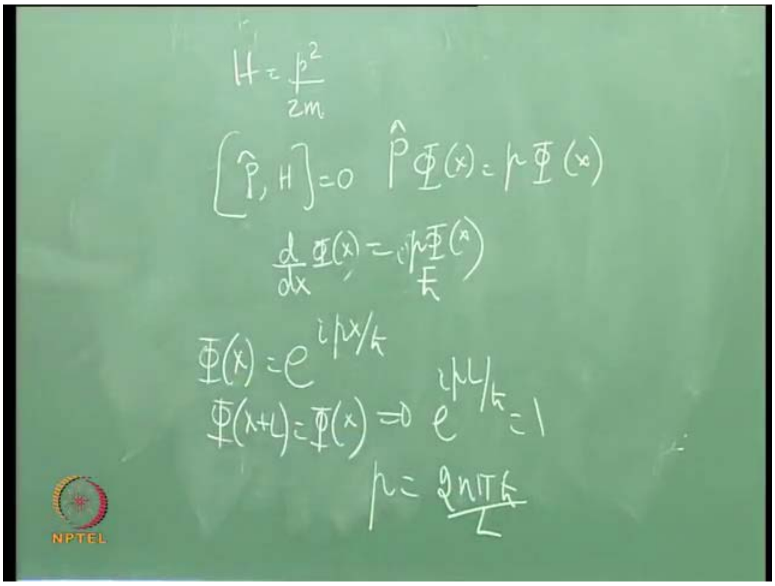

So, it is clear that because there is translation invariance, momentum is a good quantum number. I have already discussed in a different context, what you mean by a good quantum number? The context of the conservation laws in general. I told you that the total angular momentum is a good quantum number. Not necessarily, the orbital angular momentum or the spin angular or the spin because j which is L plus s, vector addition is what is conserved in any process. Be it a strong, weak, electromagnetic any process and therefore, I said the j was a good quantum number and that had ramifications, because j square commutes with the Hamiltonian. The outcome of the whole thing was the usage, the importance of the spin-orbit coupling for instance, in discussing the shell model of the nucleus.

So similarly, here P is a good quantum number. P commutes with the Hamiltonian. So, it should be possible to find Eigen states of P and the Hamiltonian. In the position representation, I have i h cross minus i h cross d by d x phi of x is equal to some E phi of x. Can well find the solution to this phi of x. But then, you see if I wrote P phi of x is some P phi of x, let me just call it p so that I avoid confusion. Then I just have the solution phi of x as e to the i p x by h cross.

You see this P is a number. It is a Eigenvalue here of the momentum operator because phi of x is a simultaneous Eigenstate of both the momentum operator and the Hamiltonian. Now, if I say phi of x plus L is equal to phi of x that tells me that e to the i p L by h cross is 1. For this momentum Eigenvalue is quantized. This becomes 2 n pi h cross by L, n being an integer 0, 1, 2, 3.

So, you see the whole business of demanding single valuedness of the wavefunction on the circle tells me that the momentum is quantized and it is of the form 2 n pi h cross by L. Needless to say, that now we can find the energy.

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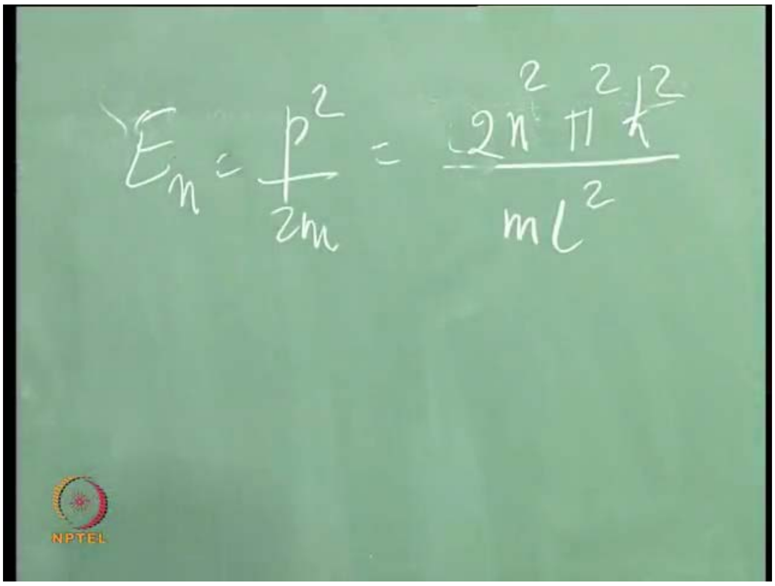

So, the energy itself for this particle moving on a circle is obviously quantized and E sub n which is the energy is P squared by 2 m and since P is quantized you just have 4 n squared pi squared h cross squared by 2 m L squared or let us just put a 2 here and that gives me a 2 n squared pi squared h cross squared by m L squared, n taking integer values, non negative values. So, this is the energy quantization. As to the solution itself, the solution happens to be of the form e to the i p x by h cross. (Refer Slide Time: 45:07) This is a plane wave kind of solution and that is because on the circle there is no block anywhere, we are considering a particle moving freely on the circle.

So, it is not as if it can hit a barrier somewhere on the circle, bounce back. So, the question of forming a standing wave does not arise. So, the only solutions that are allowed are plane wave solutions and momentum itself is quantized on the circle. And therefore, the energy also is quantized on the circle. So, this is the difference that you see when you look at the Schrodinger equation on a circle compared to working with the Schrodinger equation in a one dimensional along the x axis without using periodic boundary conditions. So, the only boundary condition I use is this. (Refer Slide Time: 45:07) The single valuedness, phi of x is phi of x plus L and that should give me momentum qunatization and the energy quantization.

So, we will look at the next application which has a lot of significance, because now that we have started doing dynamics and we are invoking the time dependence Schrodinger equation. What is the following question?

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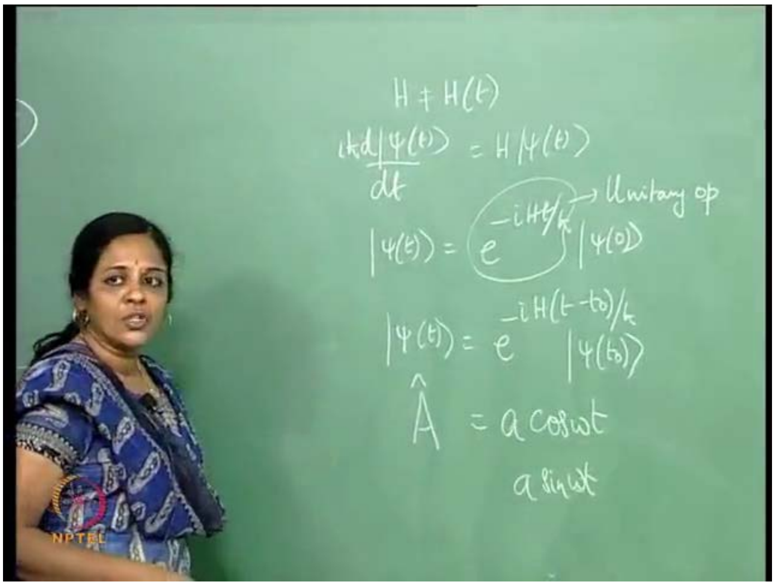

Till now, we have only looked at a situation where i h cross d psi of t by d t is H psi and I have used a Hamiltonian which is conservative. So, H is not a function of time. Then the formal solution for psi of t is e to the minus i H t by h cross psi of 0 or if my initial time were some t naught psi of t in terms of t naught would be e to the minus i H t minus t naught by h cross psi of t naught. It is convenient to set t naught equal 0 and that is what I have done here.

So, it is the state vector that evolves in time and the time evolution, the dynamics of the state vector is given in terms of the Schrodinger equation. This is a unitary operator by it is very structure because the Hamiltonian is Hermitian and you have e to the minus i times the Hermitian operator. Now, even if H were a function of time, we will consider that a little later, still the evolution is a unitary evolution and psi of t would be related by some unitary opertaor to psi of 0. It is a unitary operator acting on psi of 0 though it does not have such a simple form.

So, given psi of t what about the operators? Consider an operator A. The operator itself of course, it is a linear operator, the operator itself does not evolve in time in the following sense. There is no evolution equation which tells you how exactly this operator evolves. On the other hand, nothing prevents this operator from having an explicit time dependence. If this is the harmonic oscillator lowering operator you can have a cos omega t, you could have a sin omega t, anything. You can have an explicit time dependence. So, any time evolution of this operator would only be because of explicit time dependence and in the absence of such time dependence there is really nothing to say about the evolution of this operator.

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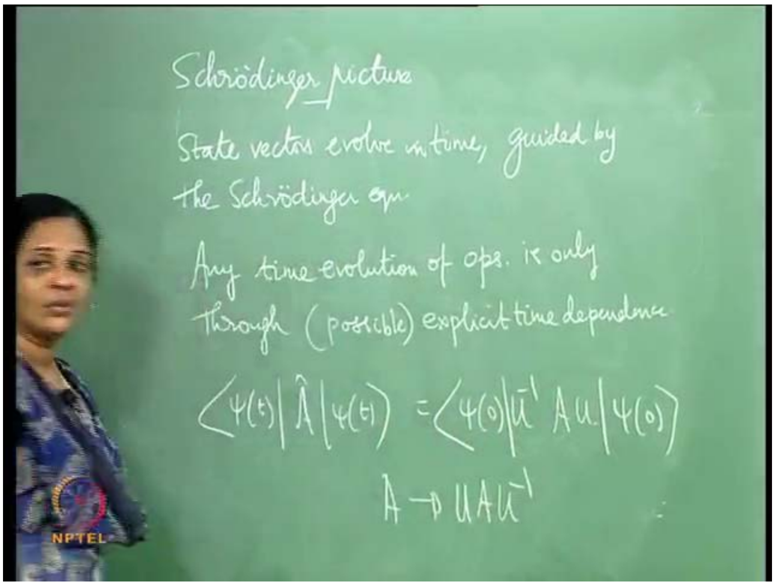

Now, this I call the Schrodinger picture. So, in the Schrodinger picture state vectors evolve in time, guided by the Schrodinger equation. Any time evolution of operators is only through possible explicit time dependence. So, it is not as if there should not be a time dependence in an operator, it has to be an explicit time dependence that is all. So, if you look at an expectation value like psi of t A psi of t. This is the same as psi of 0 u inverse A u sin of 0.

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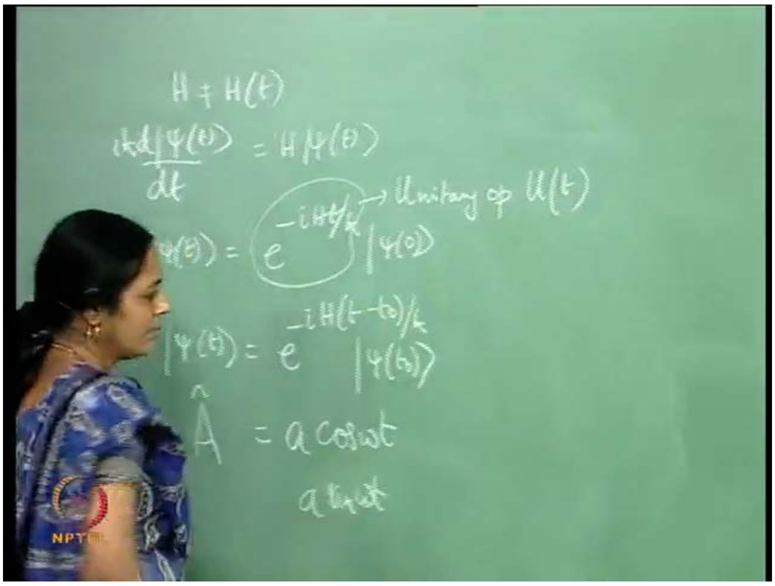

Remember that psi of t is u psi of 0. I call this operator u, u would normally be a function of t, t naught. I have set t naught to be 0. So, this would be a u of t of t u of t, t naught that is u inverse of t, t naught so that you have.

Now, if A itself, in the process, when to u A u inverse then clearly expectation values do not change because psi of t A is psi of t would simply be equal to, u inverse, u cancels and you have psi of 0 A psi of 0 and the expectation value does not change in time. This is what we know in the Schrodinger picture. On the other hand, I can think of a formalism which is more close to matters in classical physics where dynamical variables evolve in time and that is called the Heisenberg picture, that the operators have a time dependence and the state vectors do not have time dependence. So, the evolution equation will be an equation for operators. That is called the Heisenberg picture and I will take that up in the next lecture.