Quantum Mechanics - I
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Lecture - 30
The Central Potential

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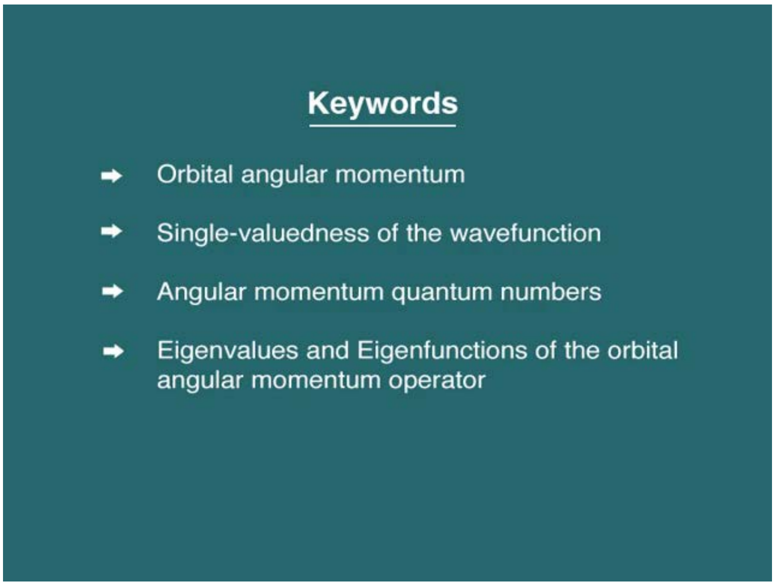

In the last lecture, we were doing a separation of variables, in spherical polar coordinates. This was in the context of working with del squared.

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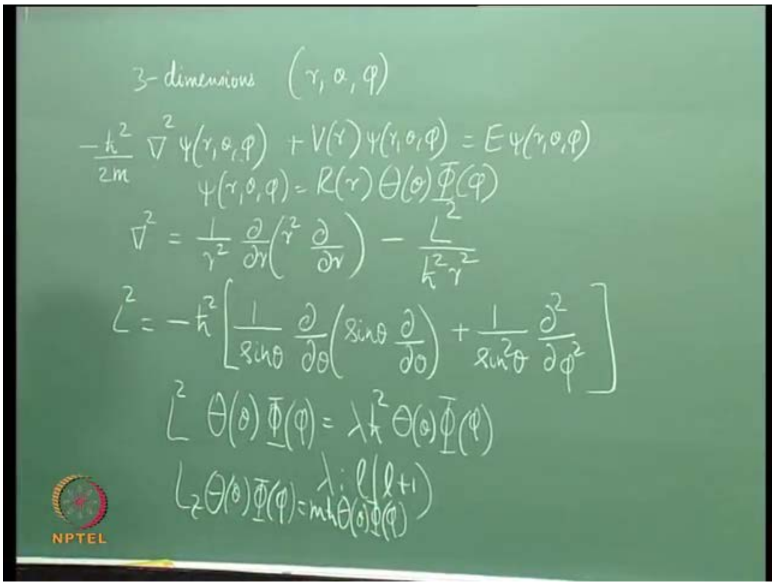

So basically, we have the equation in 3 dimensions, we use spherical polar coordinates, r theta phi and typically the Schrodinger equation. Would become minus h cross squared by 2 m, del squared psi of r theta phi. Plus V of r, because, we are considering central potentials, psi of r theta phi, is E psi of r theta phi. I will recapitulate the salient features of what we did pertaining to this equation, in the last lecture and then move on to new things. I will also elaborate on the separation of variables, done in this context. So, essentially, we wrote del squared, in terms of l squared. There is a radial component, which is 1 by r squared, delta by delta r of r squared delta by delta r.

Then there is a function of theta and phi that appear here, operator dependence on theta and phi. That can be written, as minus L squared by h cross squared r squared. We have seen this in the last lecture, we explicitly wrote out del squared in spherical polars and identified, L squared by h cross squared r squared, to be the rest of this, the angular part. So, that gave us L squared, as minus h cross squared times. 1 by sin theta, delta by delta theta of sin theta delta by delta theta, plus 1 by sin squared theta, delta 2 by delta phi squared. So, this is the operator representation of L squared, in polar coordinates, spherical polar coordinates.

Clearly, if you are looking at a term like this, it would involve the action of 1 by r squared delta by delta r, of r squared delta by delta r, on psi and the action of L squared on psi. So, we decided to first look at, L squared action on psi, but, since L squared only has dependence on theta and phi. I would like to write psi, as r of r, some function of theta phi and that was further reduced to this form. So, we wrote psi of r theta phi, as r of r some function, theta of theta. Some other function phi of phi and therefore, the L squared equation would simply be L squared. Acting on theta of theta phi of phi, is lambda h cross squared, theta of theta phi of phi.

Now, one thing is evident, this is an Eigen state of L squared and therefore, lambda should be of the form, L times L plus 1. This is the message learnt by looking at the spin system, which had the same algebra as orbital angular momentum. The values of L, however, could get restricted. Compared to what we saw earlier, in the case of the spin system. Because, in the case of the spin system, just from the algebra from the angular momentum algebra, we realised, that L could take values 0 half 1 3 by 2 and so on. But, here because, we work in physical space, there are some inputs that go in.

Inputs relating to the wave function. One of them being, that the wave function must be single valued, at any point and that would restrict L. Because, to begin with it restricts m, which is the Eigen value corresponding to L z. In fact, we saw that m could take values only 0, plus minus 1, plus minus 2, plus minus 3 and so on. And therefore, that is going to restrict L as well. Just to recapitulate what we did we therefore, went ahead and wrote out this equation, in an expanded form.

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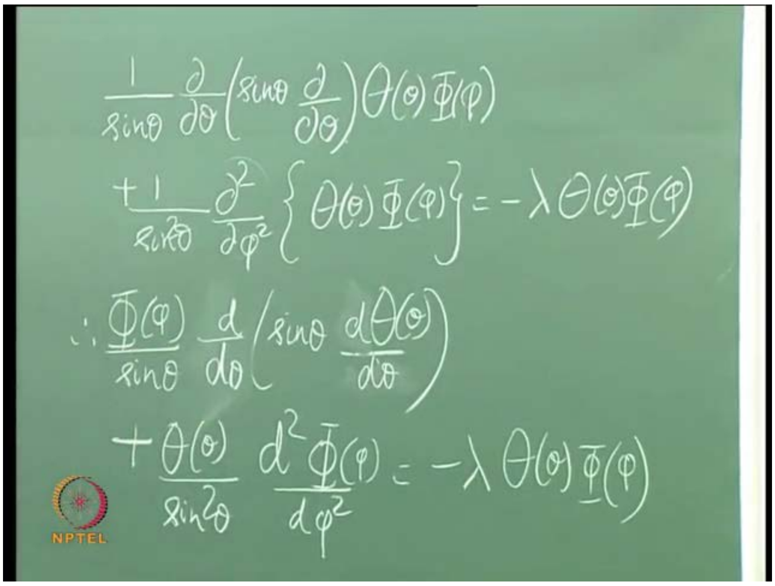

So, we have, 1 by sin theta delta by delta theta of sin theta, delta by delta theta, acting on theta of theta phi of phi. Plus 1 by sin squared theta, delta 2 by delta phi squared, acting on theta of theta phi of phi. (Refer Slide Time: 00:26) There is an h cross squared there, which cancels with the h cross squared here, when I pull the minus sign to that site.

So, this was the Eigen value equation that we had. Of course, it is clear that this acts only on theta and not on phi and this operator acts only on phi and not on theta. So, I have phi of phi by sin theta, delta by delta theta of sin theta, delta theta by delta theta. But, remember that I can now replace a partial derivative with the total derivative, because, what I have here, is only a function of theta. The 2nd term, gives me plus theta of theta by sin squared theta, d 2 phi of phi by d phi squared, is minus lambda theta of theta phi of phi.

This was the Eigen value equation and therefore, I have this equation here. Clearly, in the separation of variables by now we know that you just divide by theta phi and then, let me write the equation on the other side.

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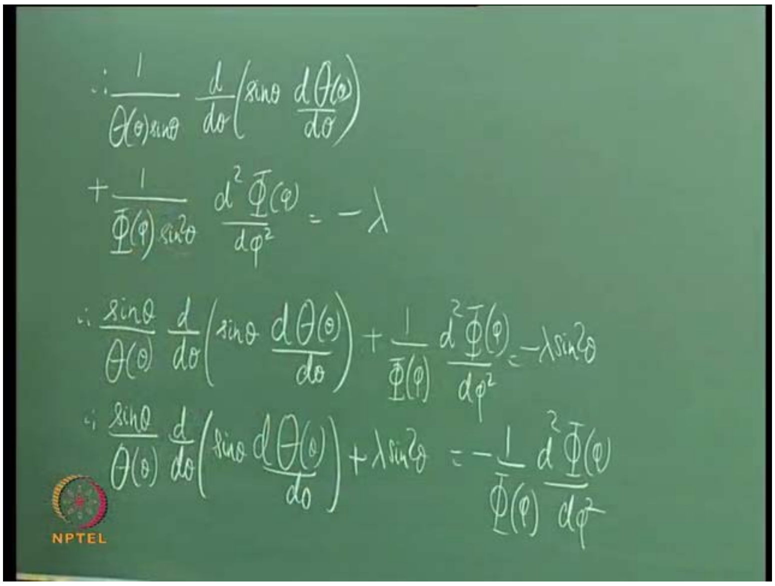

Therefore, I have 1 by sin theta, theta of theta, d by d theta, of sin theta, d by d theta of theta of theta. That is the first part. Plus 1 by phi of phi, sin squared theta; d 2 phi of phi by d phi squared, is minus lambda. So, that is what I have, after having divided by theta phi, we are in function space. These (Refer Slide Time: 06:16) are not kets, these are functions of phi and functions of theta. This is a function of theta and that is the function of phi. So, such a division is warranted.

I can make this better; suppose, I multiply by sin squared theta, I have the following equation. The sin squared theta disappears from here, leaving behind only a phi dependence, minus lambda sin squared theta. That is what I have. I can make this better, I can pull the lambda to this side and therefore, I will have, I have the following equation, plus lambda sin squared theta. Because, I have got it to this side, is minus 1 by phi of phi, d 2 phi of phi by d phi squared. It was here that, I pointed out to you, that this left hand sides depends only on theta. The right hand side depends only on phi and therefore, if these 2 have to be equal, each should be equal to a constant.

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And we decided therefore, that d 2 phi by d phi squared, would be written as minus m squared phi. This gave us a solution, phi phi is e to the i m phi, normalized with the root of 2 pi and also because, the wave function is single valued. If phi changes by 2 pi, the wave function should not change, this restricted m to integer values. You will recall that L z, is minus i h cross delta by delta phi, in spherical polars. Therefore, d 2 phi by d phi squared is minus m squared phi of phi implies that L z phi of phi, is m h cross, phi of phi.

So, delta 2 by delta phi squared, pull out the minus i h cross out here and you get this. Not surprising, that phi of phi is an Eigen state of L z as well. Because, L squared and L z commute with each other and therefore, this object phi of phi here is a common Eigen state, of L squared and L z. The Eigen value corresponding to L z would be m h cross. We have to find the Eigen value corresponding to L squared and, as I suggested (Refer Slide Time: 00:26) since, this is lambda and we know from earlier experience, that lambda would be of the form, L times L plus 1. We can now put 2 and 2 together, since m takes values, from the angular momentum algebra, I already know, that m takes 2 L plus 1, values for a given L, ranging from minus L to plus L.

And therefore, if m is allowed only integer values, this l here (Refer Slide Time: 00:26) should clearly take only integer values, unlike any respectable l, it has to take values 0 1 2 3 4 and so on, all positive values and all positive integers and 0. So, this is a difference between orbital angular momentum and what you got for other objects. That comes, with the angular momentum algebra, like spin, or isospin. Where, we realise that L could take positive values 0 half and so on. An m takes values minus L to plus L that is 2 L plus 1 values, in steps of 1, for a given L.

Now, this (Refer Slide Time: 00:26) restriction has come, because of a very crucial input. That the wave function should be single valued, in physical space. Should have the same value, whether we work with phi, a definite angle phi, or with phi plus 2 pi. So, that is one feature, which is special to orbital angular momentum and which is worth remembering. So, now we can just go back, to this equation (Refer Slide Time: 08:28) and substitute and then try to solve for the theta equation.

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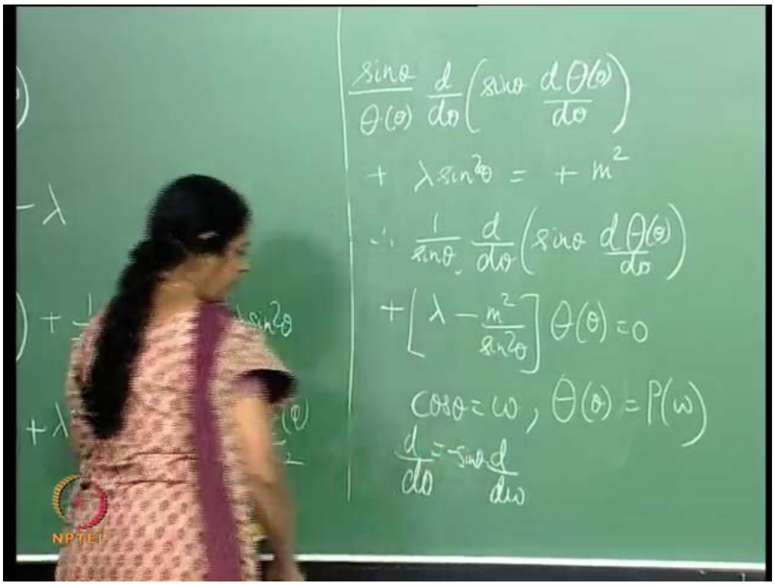

So, I have sin theta, by theta of theta, d by d theta of sin theta, d theta of theta by d theta, plus (Refer Slide Time: 08:28) lambda sin squared theta, is minus m squared. Because, we said d 2 phi by d phi squared as minus m squared phi. So, this would be plus m squared, multiply throughout by divide by sin squared theta first.

So, I have, 1 by theta of theta sin theta d by d theta of sin theta, d theta by d theta. That is the 1st term. Plus and bring this to this side, bring everything to the left hand side. So, lambda minus m squared, sin squared theta equals 0. So, let us multiply by theta of theta and that is what I have. So, I have 1 by sin theta, d by d theta of sin theta d theta by d theta, plus lambda minus m squared by sin squared theta theta equals 0. If you go back to the expression for L squared (Refer Slide Time: 00:26) that is simply retained.

I have a total derivative because; I have done a separation of variables and what I got from the phi equation, gives me this. I substituted the Eigen value, corresponding to the equation for the function of phi, phi of phi and there was a lambda which was already there. From here (Refer Slide Time: 00:26) that is the Eigen value equation for L squared. that gave me the lambda and therefore, this is the equation for theta of theta, which I have to solve. So, if you look at this, just for the sake of writing things in a compact fashion, let me define cos theta as w. Implies theta of theta is some P of w.

I need to find out, d by d theta so d by d theta would be d by d w, times d w by d theta. That gives me a minus sin theta .So, let us just recast this equation, in terms of w. This is analogous to what we did, in the harmonic oscillator problem. Where, I just wrote had an equation in as a function of x, was a 1 dimensional oscillator and the variable was x. Because, we were working with Cartesian coordinates and x went from minus infinity to infinity. For convenience, we defined rho which was alpha x, then wrote an equation for rho. It was a differential equation in rho that we solved; the variable was rho, rho is alpha x, in the context of the harmonic isolator.

Here, I am just calling it w and now rewriting the equation. This would give me a d by d w, the minus sin and the sin theta cancels out. So, I just have a minus d by d w, all this gets replaced, by minus d by d w. Now this, again picks up a minus sin theta, so that gives me a minus sin squared theta, which is a minus 1 minus w squared. But, there was already an overall minus sin. So, this whole thing just becomes d by d w, of 1 minus w squared d by d w of P of w.

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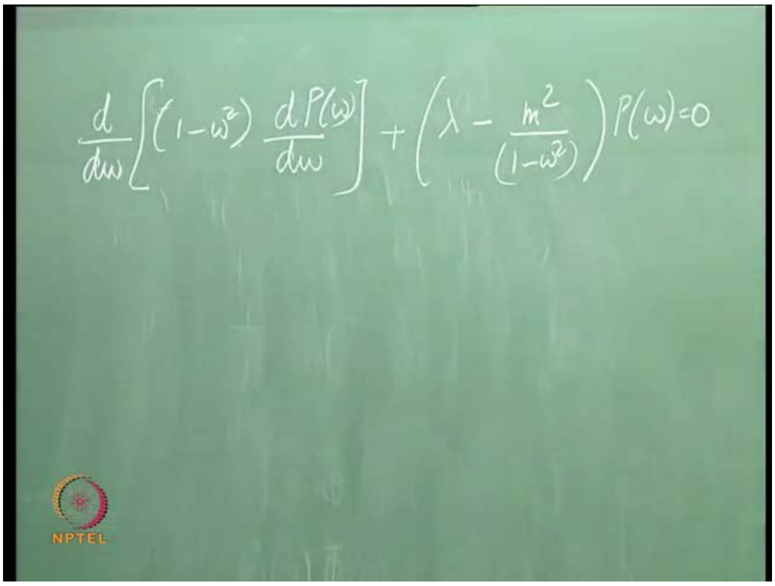

So, the 1st term, just becomes this, (Refer Slide Time: 15:13) plus lambda minus m squared by 1 minus w squared, p of w equals 0. So, this is the equation I have. As I said, this is analogous to the procedure we used, when we worked with the linear harmonic oscillator.

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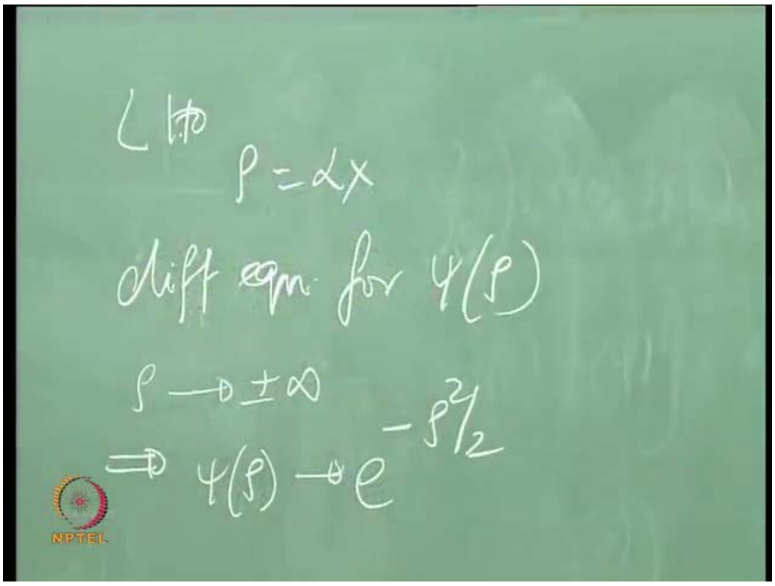

Let me quickly recapitulate, what we did in the case of the harmonic oscillator. There we had an equation, a differential equation, for psi of x. Where, I defined rho as alpha x and therefore, wrote it as psi of rho. Then we examined that equation, asymptotically. That means, what happens when x goes to plus infinity, or minus infinity, which are the limits of x. You want the equation and the wave function, to be well behaved; you want the wave function to be well behaved asymptotically.

So, the asymptotic form. So, rho going to plus minus infinity, implied that psi of rho went as e to the minus rho squared by 2. So, the Gaussian form was determined, by looking at what happens when x, or equivalently rho go to plus minus infinity. I do the same thing here, so this was the linear harmonic oscillator.

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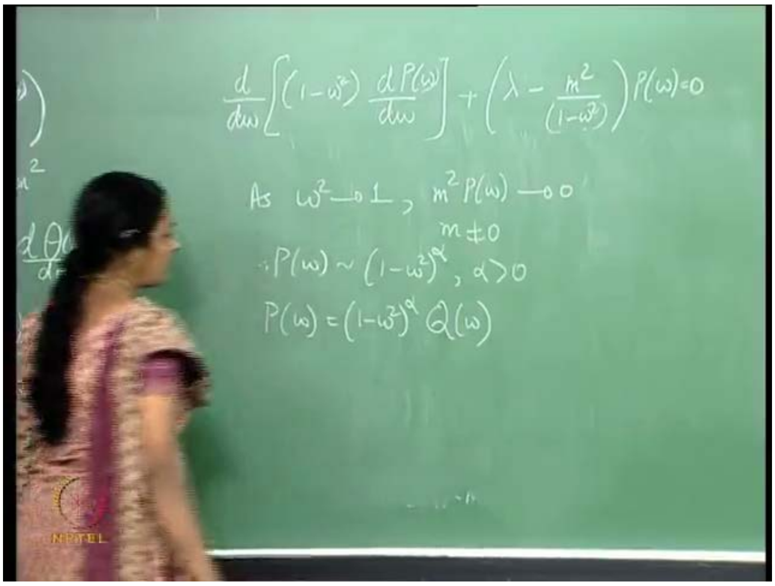

Now look at this equation, (Refer Slide Time: 19:44) w is cos theta. So, trouble comes when w squared goes to 1. This term is going to create trouble, because, a denominator has 1 minus w squared. So, the analogue of looking at, what happens when x goes to plus minus infinity? In this case, is to look at what happens when w squared goes to 1. So, if things have to be well behaved, as w squared goes to 1, m squared P of w should go to 0. For m not equal to 0. Recall, that m can take values 0 plus minus 1 plus minus 2 and so on. So, for a non negative m, I require m squared p of w to go to 0, as w squared goes to 1.

So, this means, that P of w, should go as 1 minus w squared to the alpha, where alpha is greater than 0. This statement is the analogue (Refer Slide Time: 20:44) of the statement psi of rho, goes as e to the minus rho squared by 2 for large rho. So, that is the 1st step, in all these problems where, you solve for 1, when you solve the differential equation an

Eigen value problem, which involves one argument, in this case w; therefore, I would like to write P of w, as 1 minus w squared to the alpha some Q of w.

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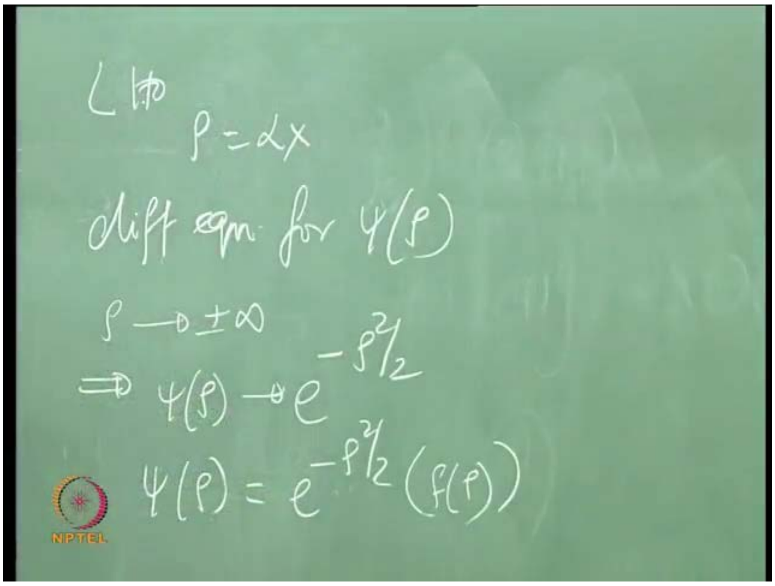

This is analogous to what we did there. We wrote psi of rho as e to the minus rho squared by 2, times some function of rho. And the behaviour of this function should not affect the asymptotic form. In fact you might recall that the differential equation for psi was then rewritten and we had a differential equation for f of rho in that case.

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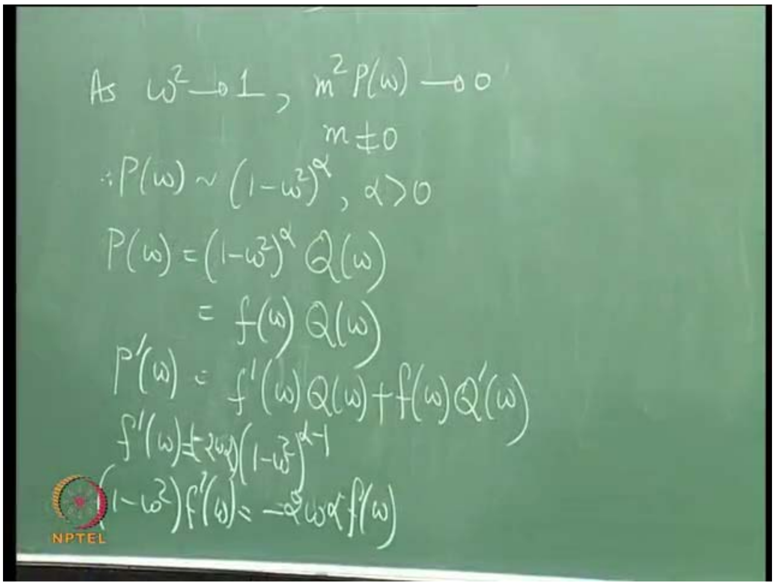

So, we will have to do the same kind of thing here and try to solve for Q of w. Now, so my aim is to get an equation a differential equation, for Q of w and that can be simply done. I am just simply going to call 1 minus w squared to the power of alpha as f of w and therefore, P of w is f of w Q of w. I need p prime; so p prime of w, by prime I mean the differential with respect to w, is f prime of w, Q of w plus f of w Q prime of w. So, f prime of w is alpha 1 minus w squared to the alpha minus 1, times minus 2 alpha, minus 2 w, sorry.

So, this is minus 2 w alpha, 1 minus w squared to the alpha minus 1. This comes from P prime of w. So, when I multiply with 1 minus w squared, as is expected here, I get a term 1 minus w squared times, f prime of w. So, let us see what that gives us. 1 minus w squared f prime of w is minus 2 w alpha 1 minus w squared to the alpha, which is f of w. So, that is what I have.

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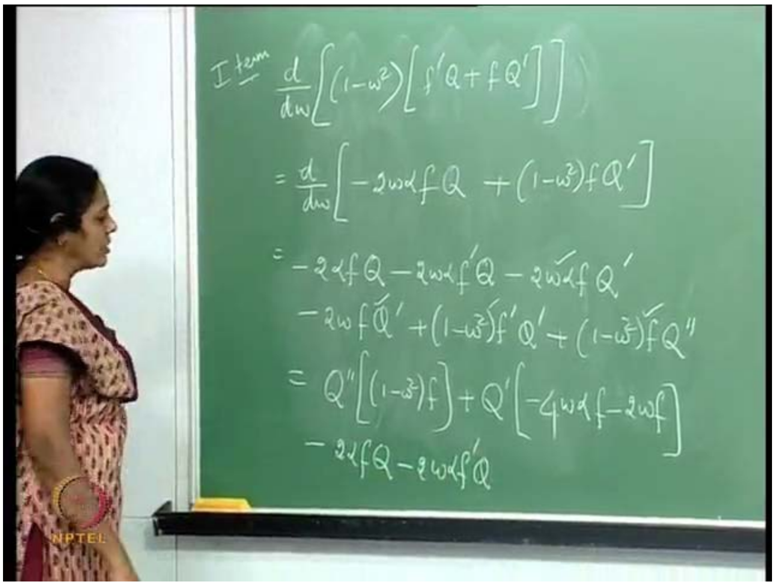

Now, I use all that, and I get the following. I have d by d w, of 1 minus w squared, p prime of w, which we have got there, f prime Q, just dropping the arguments for the moment, plus f Q prime that is the 1st term. So, let me consider that. That is d by d w, 1 minus w squared f prime, is what we have got here, that is minus 2 w alpha f. We have already got that there, time a Q plus 1 minus w squared f Q prime. So, this is the 1st term and when expanded this just gives me, minus 2 alpha f Q, minus 2 w alpha f prime Q, minus 2 w alpha f Q prime. Minus 2 w f Q prime from there, plus 1 minus w squared f prime Q prime, plus 1 minus w squared f Q double prime.

Now, since the 2nd term, there are only 2 terms here and since the 2nd term just has p of w, no derivatives. This just has an f Q here; there is neither Q prime nor Q double prime. It is clear that, the Q prime terms and the Q double prime terms come from here. So, this can be just written as, Q double prime times, 1 minus w squared f and that is all for Q double prime. Let me group the Q prime terms, remember prime is differential with respect to w. So, I have a minus 2 w alpha f from here, minus 2 w f from there, plus 1 minus w squared f prime Q prime, substitute back.

So, minus 2 w alpha f, Q prime, so the Q prime has been pulled out and, this can be just written, as minus 4, w alpha f, minus 2 w f. Then, I have terms which involve Q, well there is one term here, there is one term there. So, that is a minus 2 alpha f Q, minus 2 w alpha f prime Q that is the contribution from the 1st term.

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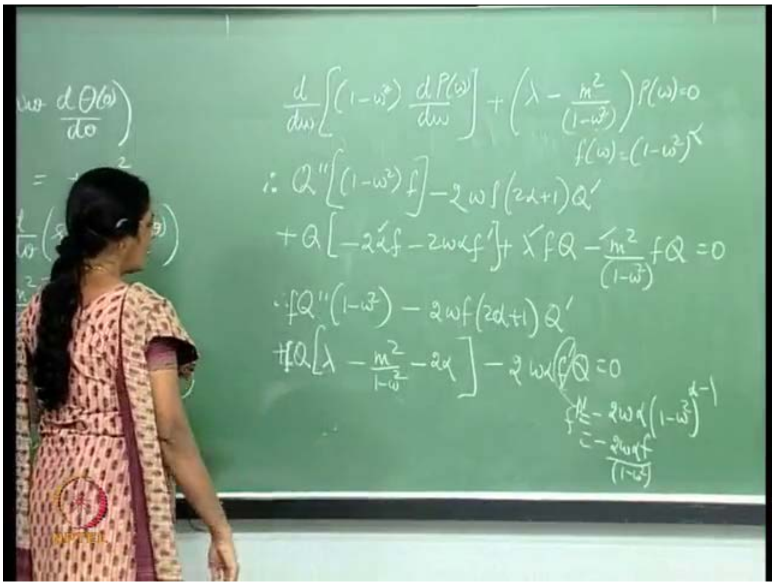

So, now I can just add it all up. So, this just becomes Q double prime time’s 1 minus w squared f. Recall that f of w, was 1 minus w squared to the alpha. Where alpha was greater than 0, I can write this as minus (Refer Slide Time: 26:18) 2 w f 2 alpha plus 1 times Q prime. And then, I have the Q dependent terms, from there I have, minus 2 alpha f minus 2 w alpha f prime and from here, I have plus lambda f Q. Remember that P was f times Q minus m squared by 1 minus w squared, f Q this is equal to 0. So, what it that mean? Q double prime time's 1 minus w squared, f minus 2 w f 2 alpha plus 1 Q prime and now group the Q terms. There is a lambda, so f Q if I pull that f out that gives me a lambda, minus m squared by 1 minus w squared. So, that is already there and then I have a minus 2 alpha, f Q and a minus 2 w alpha f prime Q equals 0.

But you see, once I substitute for f prime, as minus 2 w alpha 1 minus w squared to the power of alpha minus 1. I can rewrite this equation, cancelling f on all sides. This can be written as, this object f prime, which is this. Can also be written as minus 2 w alpha, f by 1 minus w squared. And therefore, the equation takes on a very nice form, you can cancel f everywhere and the equation simply becomes simplified form.

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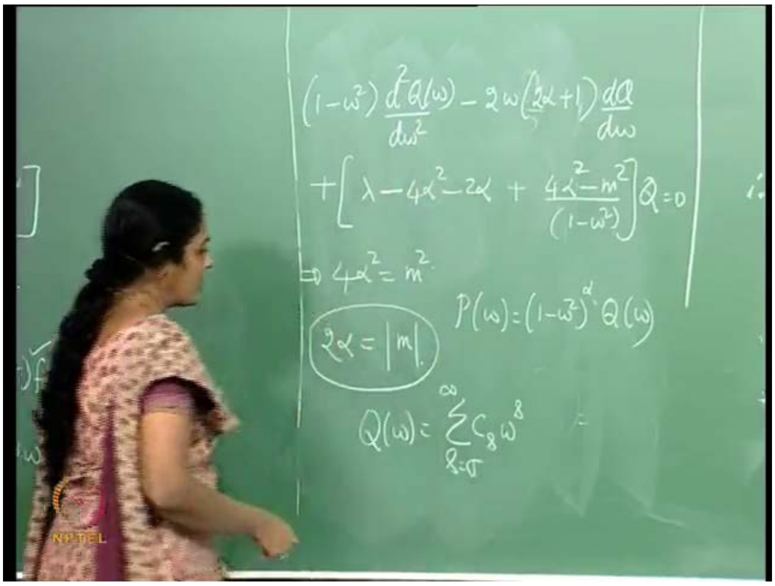

1 minus w squared, d 2 Q of w by d w squared. That is the 1st term (Refer Slide Time: 29:38) minus 2 w times 2 alpha plus 1 d Q by d w. That is the 2nd term, plus lambda minus 4 alpha squared minus 2 alpha, plus 4 alpha squared minus m squared, by 1 minus w squared Q, equals 0. Just added a 4 alpha squared and subtracted (Refer Slide Time: 29:38) out an extra p's, in order to get this term.

So, that is what I have and as you can see, trouble comes when w squared is equal to 1. As w squared approaches 1, this term is going to give trouble. Unless 4 alpha square equals m squared. You will recall, that p was written as, 1 minus w squared to the power of alpha Q of w. So, now you see, you are fixing the value of alpha. How does a value of alpha get fixed? Because, you do not want a problem, as w squared goes to 1. So, it is always the fact that the wave function is single valued at a point, or that the wave function should be well behaved.

These are the properties of the wave function, very important properties of the wave function, as you can see. Because, they help you fix, the behaviour of the Eigen function. But, alpha is greater than 0, it has to be greater than 0. Otherwise, you would have had trouble. Therefore, alpha 2 alpha, is equal to mod m, because, m can take values, integer values positive negative and of course 0. So, I get this very important result, that alpha is mod m by 2. So, look at P of w, it is 1 minus w squared to the mod m by 2, times Q of w. So, this is the analogue (Refer Slide Time: 23:53) of this problem. I have already recast the equation in terms of Q of w. That is the analogue of what you did in the case of the linear harmonic oscillator. Where, you recast the differential equation, to a differential equation in terms of f of rho.

And then what did you do, you solved for f of rho. How did you solve for f of rho? You start with a series solution. So, here to we will now assume a series solution, sum sigma to infinity, C s w to the s. Where C sub s are the coefficients, as in the case of the harmonic oscillator, we have to find out what are the coefficients that survive. To begin with of course, I would imagine that this is an infinite sequence. I will substitute for Q of w, in this differential equation and see what it gives me, same way that we did for the oscillator problem. Except that we are doing it now, for an angular wave function, but the procedure is the same.

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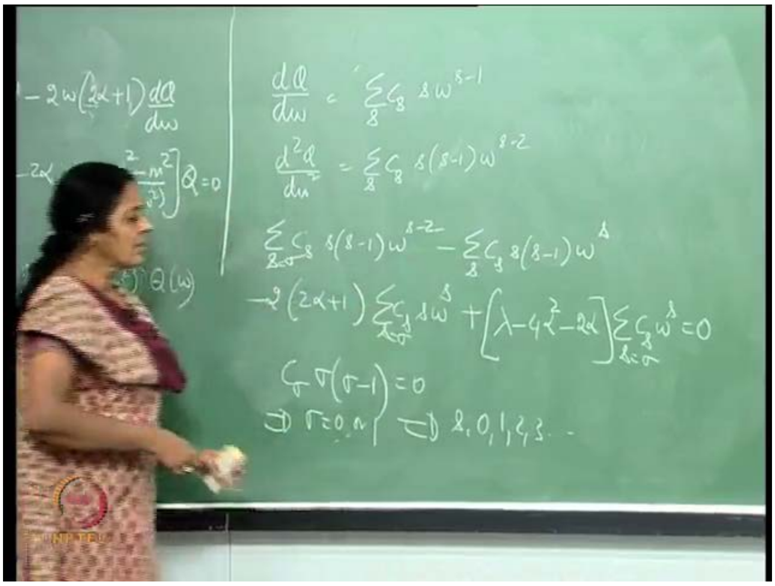

So, when I substitute there, d Q by d w, is summation over s C s, s time’s s minus 1, w to the s minus 1 and d 2 Q by d w squared. Summation over s C s, sorry, s times w to the s minus 1 and here it is s times, s minus 1 w to the s minus 2. Now, when I substitute back there, I have a term, 1 minus w squared, times d 2 Q by d w squared. So, the 1st term, 1 times this, gives me the following and then, (Refer Slide Time: 32:34) minus w squared times this, will give me a power w to the s. This would give me a power w to the s, because, d Q by d w comes with the power w to the s, minus 1 and then of course, I have all these terms.

But, this term drops out, because 4 alpha squared is equal to m squared and therefore, I just have, this time summation over s C s w to the s equals 0. So, let us look at the lowest order, w to the s minus 2 and s went all the way from some value sigma to infinity. We are supposed to determine sigma. In a series (Refer Slide Time: 32:34) expansion of this kind, you know that these behave like basis functions and that is precisely, why we are expanding it in this fashion. So, you have to equate each power of w to 0 and since this is the only contribution to w to s minus 2, let us first start with s equals sigma. So, that tells me C sigma time’s, sigma time’s sigma minus 1, equals 0.

But, C sigma is not 0. Because, you have written this series, as summation s is equal to sigma to infinity. C s w to the s and the lowest non-zero contribution comes from sigma. Since, C sigma is not 0, sigma is 0 over 1. So, this (Refer Slide Time: 32:34) series, starts from s is equal to 0 and s takes only positive integer values. This becomes an important thing, implies s takes values 0, 1, 2, 3 and so on. This is the equation, which this is this for small s, now for all other values of s. Because, coefficient is w to the s, for all other values of s, we can again equate the coefficient to be 0. And as in the case of the oscillator, we can get a recursion relation.

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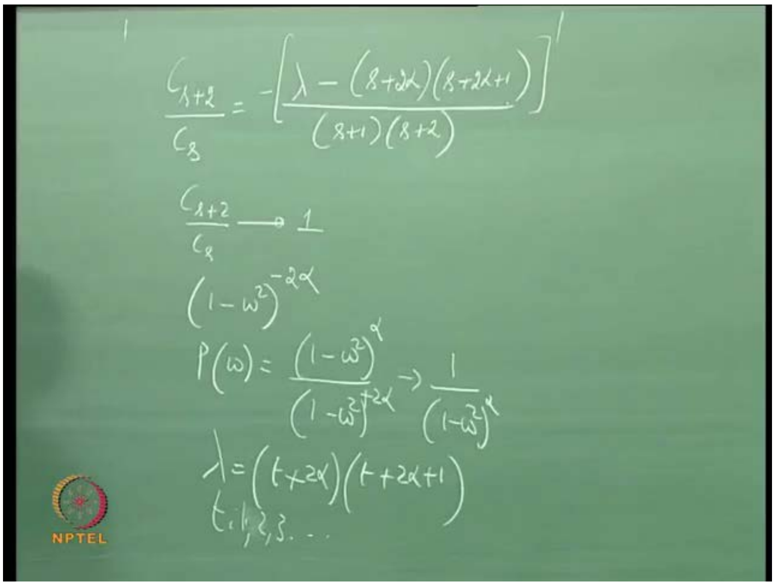

I leave it to you as an exercise, to show now that C s plus 2 by C s, is minus of lambda minus s plus 2 alpha, times s plus 2 alpha plus 1. All of it divided by s plus 1 times, s plus 2. You see now, let us find out what happens, for large s. For large s, this simply goes as 1. Because, you have an s squared over there and an s squared out here and therefore, it goes as 1. Now, that is bad news, because, it means that all contributions, are equally important and you cannot truncate the series, at some C s.

In fact, this is the kind of ratio of coefficients that you get, in the series expansion for 1 minus w squared to the minus 2 alpha. Because, there you will find, that if you expanded the series and took a ratio like this, it would go as 1. And that is bad news, because then P of w, is 1 minus w squared to the power of alpha, by 1 minus w squared to the minus 2 alpha, to 2 alpha. I have brought the minus 2 alpha down and that goes, as 1 minus w squared to the alpha. That is bad news, because, when w squared is equal to 1. The term is going to blow up. Now, the only way to avoid a situation like this is to do precisely the kind of thing, we did in an analogous situation in the case of the harmonic oscillator. You have to truncate the series.

Now, truncating the series would mean the following, for some s, C s plus 2 by c s should go to 0. In other words, suppose I fix lambda to be, t plus 2 alpha time's t plus 2 alpha plus 1. Where, t is an integer. So, let us start with the 1, 2, 3 and so on. When s is equal to t, it is clear that the numerator vanishes and therefore, the series will truncate, because, beyond that, there is no ratio of coefficients worth talking about. This is precisely the kind of thing we did, with the recursion relation for in the case of the linear harmonic oscillator and it was this truncation of series, which gave us a differential equation, which had the hermit polynomial as a solution.

So, that is how, that special function the hermit polynomial came in into the picture. It also brought in a quantum number. Because, in the case of the oscillator it brought in the quantum number n and we had h n of rho, or h n of x. We would expect something like that to happen in this case as well. Set lambda equal to this value, the series will truncate. Check to see, if the differential equation now, once the series truncates, is a familiar differential equation in the sense, it has some special function as a solution. And indeed that is what we will get. We will also have a quantum number, coming up in the process. So, it is a standard ploy that I employ, in the case of a linear harmonic oscillator, in angular momentum algebra and so on, when I am looking out for series solutions.

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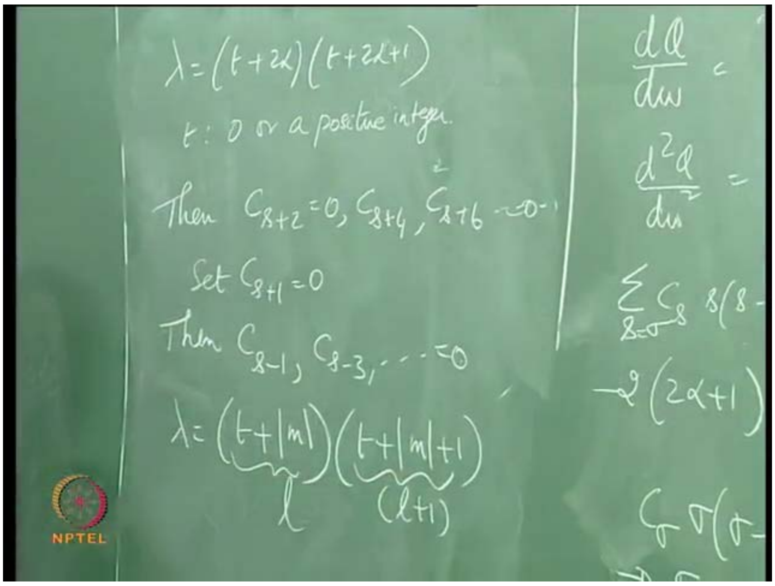

Choose, lambda to be t plus 2 alpha, times t plus 2 alpha plus 1. Where, t is 0, or a positive integer. Then, the series terminates, when s takes the value t, C s plus 2 become 0 and because, C s plus 2 is 0, C s plus 4 C s plus 6 etcetera are 0, (Refer Slide Time: 40:04) because, the same recursion relation holds. In the next stage, you will have C s plus 4 is something times C s plus 2 and since C s plus 2 is 0, all of them are 0. Set C s minus 1 also C s plus 1 also to be 0, because, you want to truncate the series. Then, by the recursion relation, C s minus 1, C s minus 3, they are all 0. So, depending on whether a series terminates with C s. Then s takes the value t and the important thing is this, it is not just a finite series. If s is even, is an even integer, only even objects contribute like C naught C 2, C 4, C 6 and so on, as you can see.

If s is odd, C 1, C 3, C 5 and so on contributes. So, that is the structure and you have terminated the series. (Refer Slide Time: 40:04) But this, gives you the value of lambda, because 2 alpha, is mod m. So, lambda is t plus mod m, times t plus mod m plus 1 and I would call this l and I would therefore, call this l plus 1. And t takes values 0, 1, 2, 3 etcetera; mod m takes values 0, 1, 2, three etcetera. Where, l takes values 0, 1, 2, 3 etcetera. So, the fact that the asymptotic behaviour, in the case of the oscillator. That means x going to plus minus infinity and here omega squared w squared going to 1.

That means theta taking values 0, or pi. Because, cos theta was w; The fact that the wave function must be well behaved; when these limits are reached, by the argument that is what has quantized the Eigen value. Quantization of m k m, because, you expected the wave function to be single valued, quantization of l has come, because, you truncated the series because the asymptotic behaviour, should not be jeopardized. So, it is really the well behaved nature of the wave function. A single valued nature of the wave function that gives you quantization. Returning now, to this problem, so I have lambda equals l time’s l plus 1.

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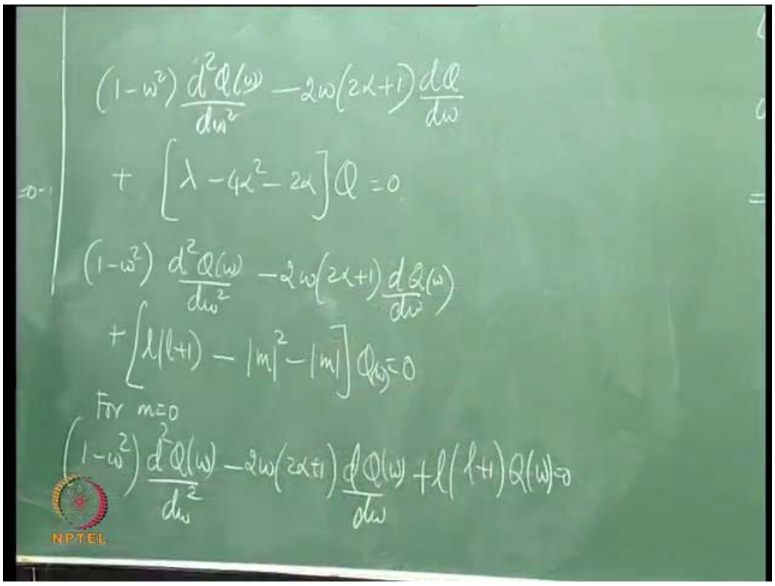

So, I can go back to the equation for Q and see what I have. The equation for Q simply said this, minus 2 w times 2 alpha plus 1 d Q by d w. Plus Q times lambda minus 4 alpha squared, minus 2 alpha plus 4 alpha squared minus m squared, by 1 minus w squared. Then that term dropped out, is equal to 0. So, in terms of m, which is what I will write now, I have d 2 Q of w by d w squared, minus 2 w 2 alpha plus 1, d Q by d w. Plus lambda is l times l plus 1, minus 4 alpha squared is mod m squared, 2 alpha is mod m Q is equal to 0, Q of w.

Let us see what Q can be. It is clear, that since w is cos theta. We are now looking for an expansion, for Q of w in terms of an orthonormal basis set of functions, which are functions of theta. Ok, in that term with 2 alpha plus 1, there too set 2 alpha equals mod m. Everything has to be written in terms of mod m. Further, if m is 0, for m equals 0 we have, 1 minus w squared, d 2 Q of w by d w squared, minus 2 w times 2 alpha plus 1, d Q of w by d w. Plus l times l plus 1, Q of w equals 0. But, this is precisely the equation corresponding to the Legendre polynomial.

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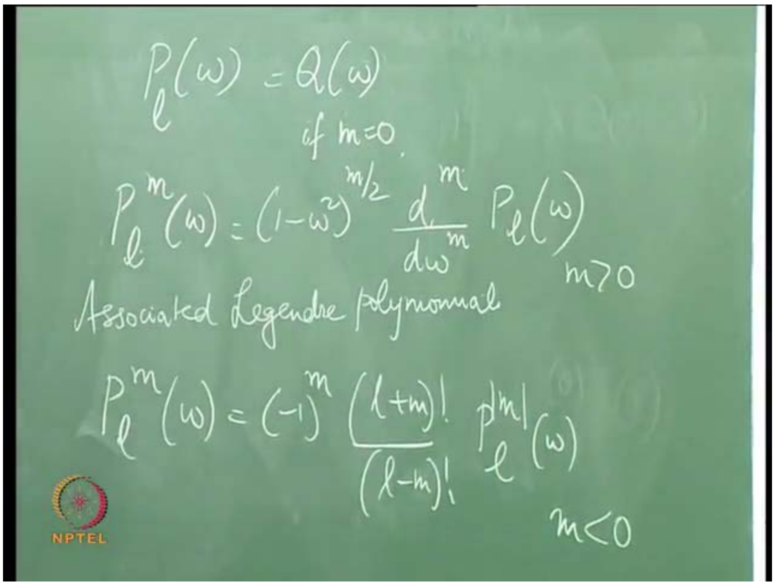

The Legendre polynomial, usually called referred to as P of w. Where, w is cos theta P l of w, that is a Legendre polynomial. That satisfies (Refer Slide Time: 48:022) precisely this equation. So P l of w is the same as Q of w, if m equals 0. Now, if m is non zero, I would therefore, suspect, that the equation for Q of w would be that, of the associated Legendre polynomials, because, they come, with another number P l m, another quantum number. So, I have another quantum number m, so I have 2 numbers associated with this. This is the associated Legendre polynomial and they are defined in terms of P l of w, as 1 minus w squared to the m by 2, d m by d w the m-th differential, of P l of w. Of course, I can define it also for this is for m greater than 0 and I can define it for m less than 0, simply. If m is less than 0 it is minus 1 to the m, l plus m factorial by l minus m factorial, P l m mod m of w, this is for m less than 0.

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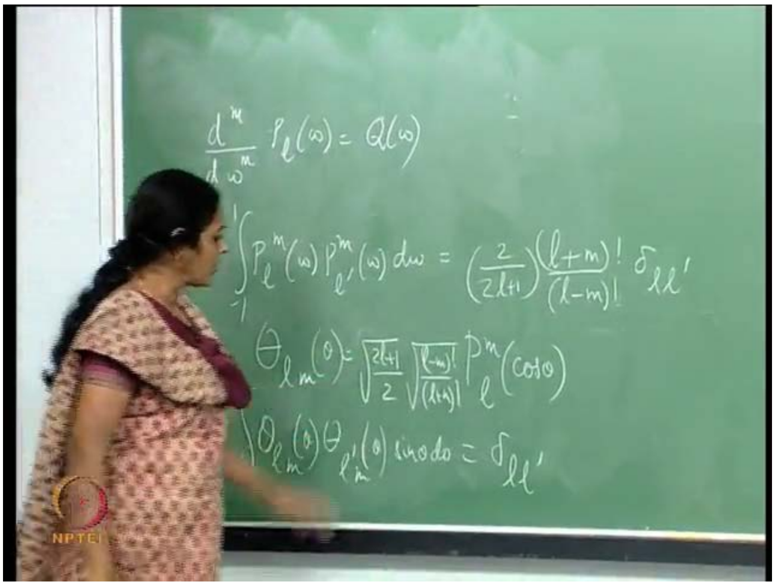

Now if you identify, Q of w in the following fashion. You identify Q of w as the n-th differential, d w to the m, of p l of w as Q of w. Then you can check for yourself and I leave it as an exercise. Now, the equation for Q of w, for m naught 0, is precisely the equation satisfied, by the associated Legendre polynomials. But, these polynomials have a normalization property, which says P l m of w, P l prime m of w d w. Remember the w goes from minus 1 to 1. It is not delta l l prime, it is 2 by 2 l plus 1, l plus m factorial by l minus m factorial, delta l l prime. And therefore, what I need, are orthogonal polynomials, I define a theta l m of theta, which is of course, P l m of cos theta.

But, in order to have the normalization straight, this would be root of 2 l plus 1 by 2, root of l minus m factorial, by l plus m factorial, p l m of cos theta. Then, it is clear that integral theta l m of theta, theta l prime m of theta, sin theta d theta, because, that is d cos theta. Theta itself going from 0 to pi, is delta l l prime.

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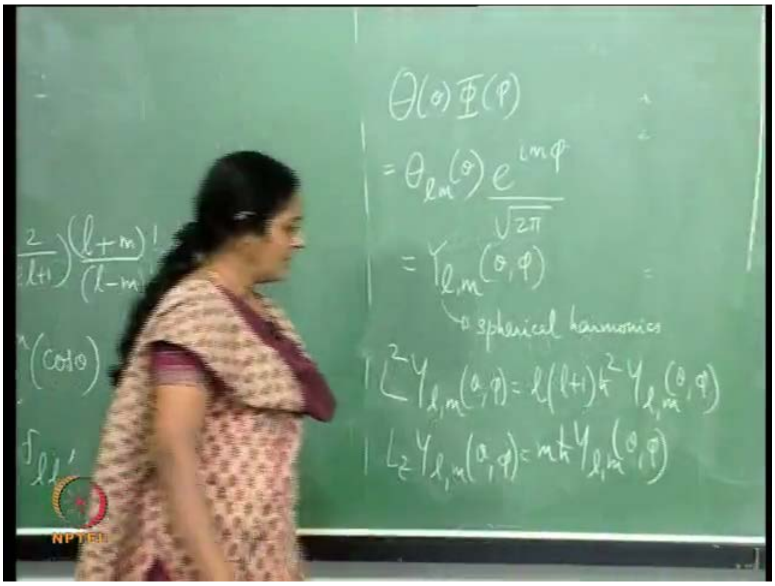

So, finally, I have the wave function. The wave function is given by theta of theta, phi of phi and that is simply theta l m, of theta. Defined here, (Refer Slide Time: 52:32) so that it is normalized to 1, e to the i m phi, by root 2 pi. So, that phi of phi is also normalized to 1. And this object, theta l m of theta, e to the i m phi by root 2 pi, is referred to, as the spherical harmonics Y l m of theta phi. So, these are in facts the angular wave functions, L squared Y l m of theta phi, is l times l plus 1 h cross squared, Y l m of theta phi. Instead of lambda h cross squared I have this and L z Y l m of theta phi, is m h cross Y l m of theta phi. Y l m of theta phi have, some very interesting properties, particularly with reference to parity and I will elaborate on that, in my next lecture.