Quantum Mechanics-I
Prof. Dr. S. Lakshmi Bala
Department of Physics
Indian Institute of Technology, Madras

Lecture - 29
The Wavefunction: Its Single-Valuedness and its Phase

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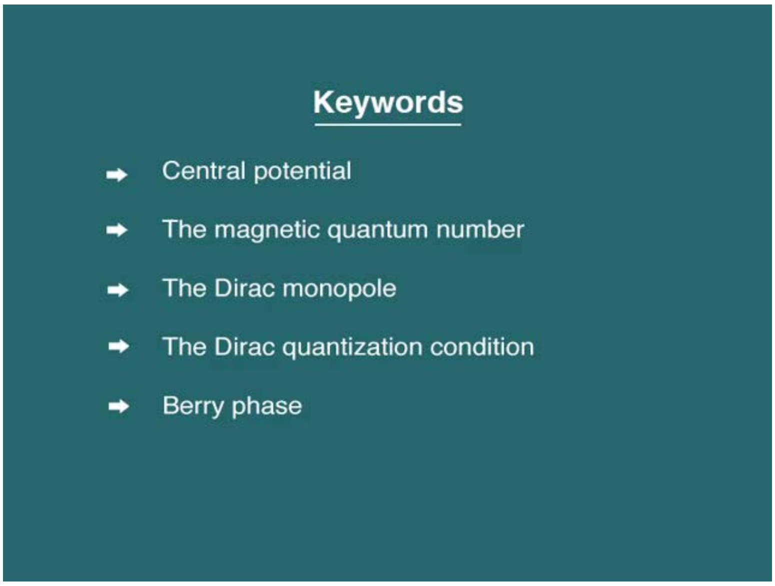

Till now, we have been looking at one dimensional problem. We looked at the linear harmonic oscillator, also at the square well potential. Now, in all those cases we could identify a length factor, an object of the dimensions of length which I could make with the parameters available in the problem.

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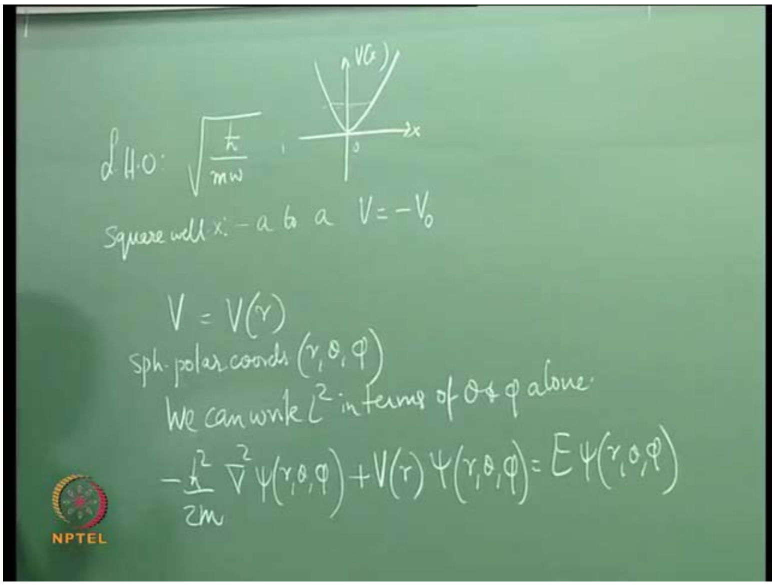

Now, you will recall that in the case of the linear harmonic oscillator we had root of h cross by m omega and that gave us the dimensions of length. Similarly, in the case of the square well potential, the potential itself was nonzero from minus a to a. So, x taking values minus a to a, V was minus V naught. Therefore, I had a length scale which was naturally there in the problem and that is needed in order to have a bound state. The surprising thing about the quantum mechanics as I have also reiterated in the past, is that there are classically forbidden regions which are accessible to the quantum state. For instance, in the case of the harmonic oscillator, if you had plotted of V of x verses x, that was a parabola going all the way up. And a classical particle would be confined to this region.

In fact, for a given value of energy the classical oscillator goes there. That is the turning point turns back, comes here, that is another turning point and that is the oscillation like this. But, as you know even in the ground state of the harmonic oscillator, the wavefunction is a Gaussian and therefore, the Gaussian dies down only at large x. So, the probability of seeing the object away or outside of the potential is also nonzero and therefore, classically forbidden regions are accessible. Similarly in the case of the square well, the particle which was confined to the square well could well have a nonzero probability of being outside the square well. So, this was one of the surprises from the quantum version of these problems.

Today I would like to look at three dimensional problems. The simple example that we can talk about is the case, where there is a central potential. So, if the potential V is simply a function of r and I have used spherical polar coordinates. An example could well be the coulomb problem, where the potential is minus z e square by r. So, we look at situations like this where we will be dealing with central potentials, a three dimensional problem. For instance, the hydrogen atom where there is proton in the nucleus and an electron outside of the nucleus. So, such problems would show a certain amount of symmetry, because if the potential is central then you know that orbital angular momentum is conserved.

L square, the angular momentum operator square, is the generator of space rotations and therefore, I would expect that if L square the operator were written in spherical polar coordinates it would merely be a function of theta and phi. We can write L square in terms of theta and phi and not r, as a differential operator involving delta by delta theta and delta by delta phi, and not the radial coordinate. The problem that I want to solve is of course, an Eigenvalue problem which is minus h cross square by 2 m generally, a particle of mass m, del square psi of r theta phi. This is the p square by 2 m part plus V of r, psi of r theta phi is e psi of r theta phi.

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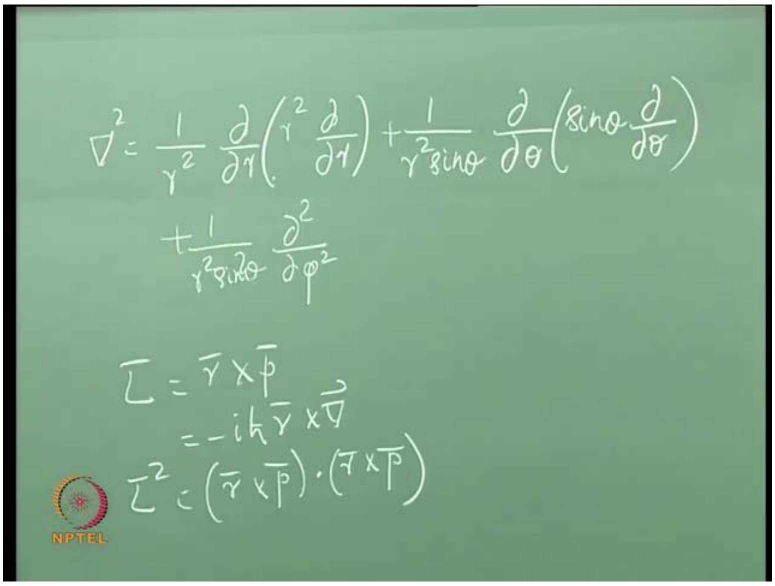

So, instead of writing psi as a function of x, y, z I write it in terms of r theta and phi. What I want to check is the following thing. del square of course, can be written in

Spherical polar coordinates in the following manner: sin theta delta by delta theta plus 1 by r square sin square theta delta 2 by delta phi square. You will recall that in spherical polar these scale factors are 1, r and r sin theta and that is why I have put them out here. That is 1 here and an r square there and r sin theta the whole square here. Those are the scale factors, when I go from Cartesian to spherical polar so this is del square and since L is r cross p which is minus i h cross r cross del, L square would be r cross p dotted with r cross p and will therefore, involve del square.

So, I would like to write del square. I would like to write a relation between del square and L square. Now, if I did that (Refer Slide Time: 00:36) I can reexpress del square in terms of the radial part 1 by r square delta by delta r of r square delta by delta r plus an angular part, which will depend only on L square. When L square acts on psi it is obviously going to change only the angular part of the wavefunction.

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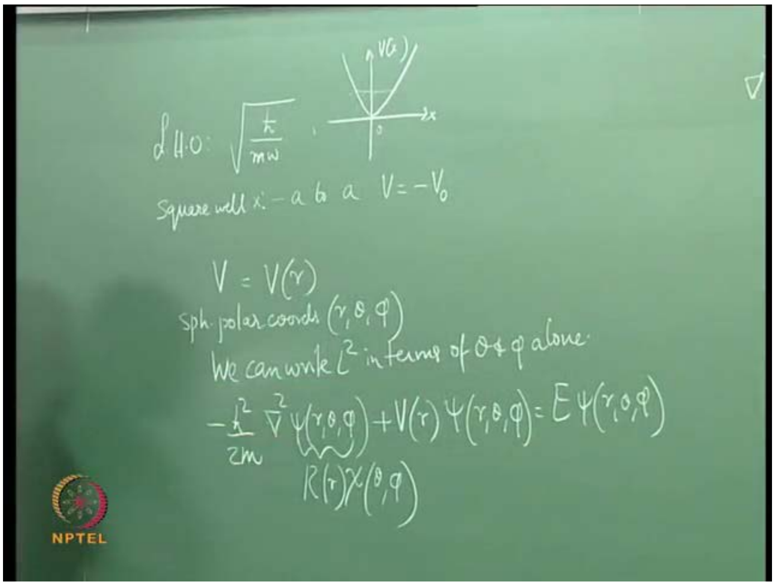

Because angular momentum is the generator of space rotations and that will only involve the angular coordinates and therefore, it would be possible to try to do a separation of variables here and write psi of r theta phi as R of r some chi of theta phi so that L square acts only on chi of theta phi. The equation would then separate into a redial equation, differential equation and an angular differential equation. We will be solving for the Eigenvalues and Eigenfunctions of L square. (Refer Slide Time: 05:35) We will use that as an input in this equation and we will solve for the Eigenvalues of del square. So, that would be the procedure that we will adopt in this problem.

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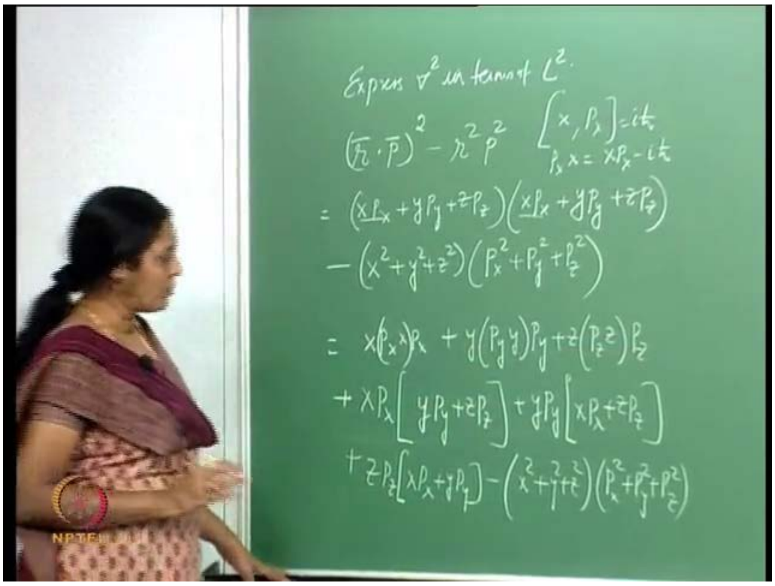

In order to write del square, in terms of L square I need to do little bit algebra. First of all, I consider r dot p the whole square minus r square p square. This is the straight forward exercise, but it is worth going through it so that we understand the things do not automatically commute with each other and we put in the appropriate factors which arise because of non-commutativity in quantum physics.

So, this object is x P x so when I expand this gives me x P x x P x that comes out of using terms like this and from this I subtract x square plus y square plus z square times P x square plus P y square plus P z square. Now, it is clear that if I commute x across P x using the fact that the commutator of x with P x is i h cross. I get a term x square P x square similarly, a y square P y square here and a z square P z square there. And that will cancel out with x square P x square plus y square P y square plus z square P z square here.

So, I will be left with terms from here, which involve i h cross. So use the commutator x P x is i h cross similarly, for y and P y and z and P z. So, P x x is x P x minus i h cross. Therefore, what I will be left with here is minus i h cross x P x, minus i h cross y P y minus, i h cross z P z, which is minus i h cross r dot P.

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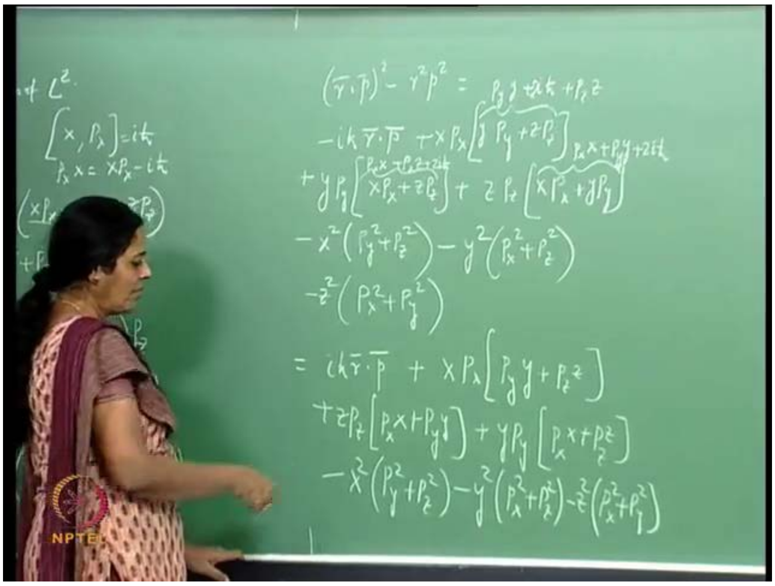

So, therefore I have r dot p the whole square minus r square p square is minus i h cross r dot p. (Refer Slide Time: 08:22) All these terms continue to survive up to this point. As I have already told you x square P x square cancels out from here after putting in the commutation relation. Therefore, I am left with minus x square times P y square plus P z square that is the first term. (Refer Slide Time: 08:22) Similarly, plus there is an overall negative sign so minus y square times P x square plus P z square minus z square times P x squares plus P y square. So, I am left with this.

Now, once more I can do the following. I can write here y P y as P y y plus i h cross and this as P z z plus i h cross. Similarly, here this would be P x x plus P z z plus 2 i h cross. Here too, P x x plus P y y plus 2 i h cross. Therefore, this simplifies to the following: look at this 2 i h cross x P x plus 2 i h cross y P y plus 2 i h cross z P z which is 2 i h cross r dot P minus i h cross r dot P so that just gives me i h cross r dot P. And then of course, I have the following things left behind: P y y plus P z z out there plus z P z times P x x plus P y y plus y P y times P x x plus P z z. And then of course, I have to subtract out these things: minus x square P y square plus P z square minus y square P x square plus P z square minus z square P x square plus P y square. So, this is what I have. Look at these terms and compare this with an object like L square.

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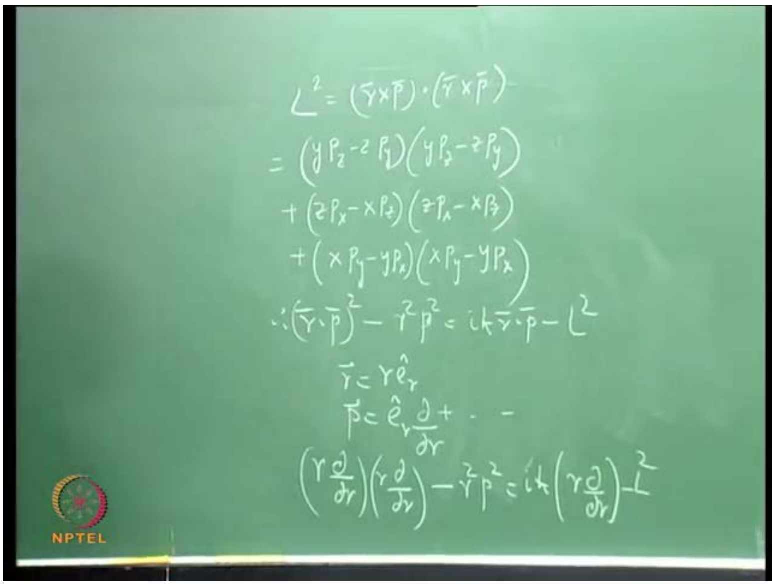

So, let us look at l square now. L square is r cross p dotted with r cross p. Of course, if you now compare term by term, you will find for instance (Refer Slide Time: 10:49) look at this term. There is an x P x P y y and out here you would see the same kind of term that appears there is an x P x P y y, the order is right because P x can be commuted across y and P y. So, this is an minus x P x P y y (Refer Slide Time: 10:49) and here I would have plus x P x P y y. You can compare it term by term and you find that what occurs here, apart from i h cross r dot p is minus L square. Therefore, this object r dot p the whole square minus r square p square is i h cross r dot p minus L square.

Now, I move on to Spherical polar coordinates. First of all I write r as r e r and p as e r delta by delta r plus so on, apart from minus i h cross which we should remember to put in. But, what matters in r dot p is simply r delta by delta r and therefore, I have r delta by delta r times r delta by delta r which is the first term. Minus r square p square is i h cross r delta by delta r minus L square so this is the kind of thing that I have, but p square itself is minus i h cross del the whole square.

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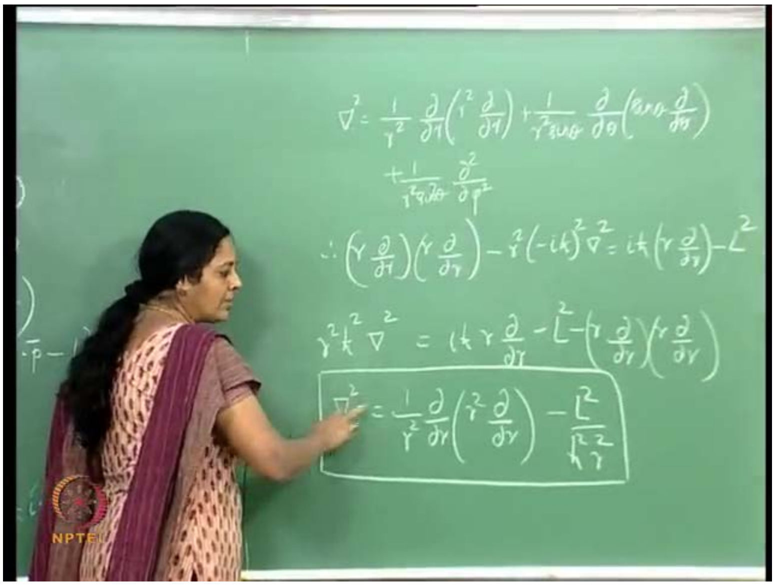

So, let me put that down here. Therefore, I have r delta by delta r, times r delta by delta r minus r square times minus i h cross the whole square del square is equal to i h cross r delta by delta r minus L square. This can be easily done now to say that del square can be written in terms of L square. So, I have r square h cross square del square from here is i h cross r delta by delta r minus L square minus r delta by delta r of r delta by delta r. But I can now write del square in spherical polar and that expression is given here.

I substitute for del square from here and then if I check by substituting for del square here I can show that del square is 1 by r square delta by delta r of r square delta by delta r the radial parts survives minus L square by h cross square r square. The h cross square because we know that angular momentum occurs in units of h cross and therefore, this should carry dimensions h cross square.

So, I scale that out and 1 by r square, there is a dimension matching del square has dimensions of 1 by r square and that is what happens here. The minus sign is simply kept from here I have merely substituted for del square from there. And this becomes an important relation between del square and L square. So, this is one way of seeing it. I could have done it by moving onto spherical polar coordinates right at the beginning, could have done it in different ways. It is straight forward algebra, but the result is worth demonstrating. (Refer Slide Time: 08:22) Because, it is matter of exercise it teaches you that right at the very first step you cannot simply interchange P x and x, any way you wish and have to use the commutator relation and maintain order of things. So, this is what you get for del square.

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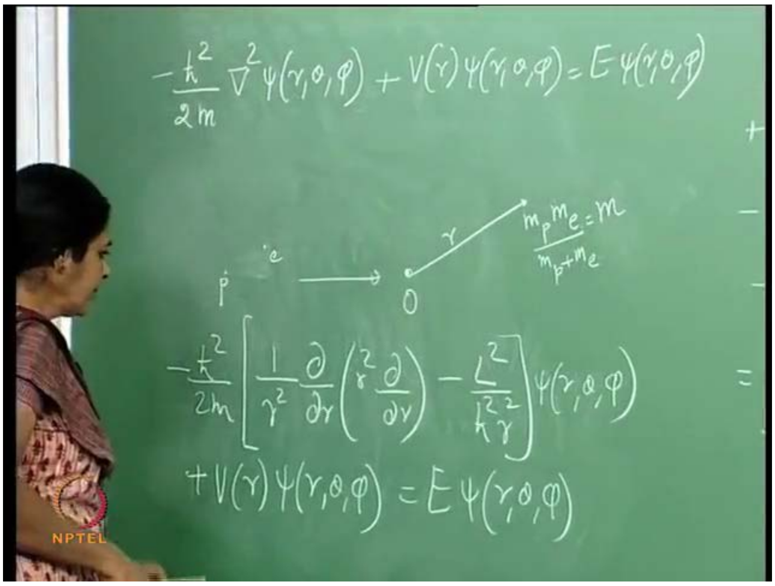

Now, let us go back to the Eigenvalue problem. The problem that I have is minus h cross squared by 2 m del square psi of r theta phi plus V of r psi of r theta phi is E psi of r theta phi. So, this is what I have. I substitute for del square therefore, minus h cross square by 2 m I should say what those m is all about. Suppose, we were looking at the hydrogen atom problem for instance, it is a two body problem, there is a proton and there is a electron. But, instead of this two body problem I could go to an effective one body problem where there is an origin of coordinates and there is an object here with an effective mass m p m e by m p plus m e which I can call mu or m at a distance r, where r is the radial coordinate. So, this is the equivalent one body problem as you know.

You reduce two body problem to an equivalent one body problem, where this radial coordinate is from the origin of coordinates and there is a reduced mass which is m p m e by m p plus m e, if we are talking about the hydrogen atom. Otherwise of course, any two objects whose mass is m 1 and m 2 would be an m 1 m 2 by m 1 plus m 2. So, the m that I write here is essentially this reduced mass perhaps, I can call it m. But, you should remember that by m I mean the reduced mass in the equivalent one body problem.

So, that is what we have here. And in that picture I can now substitute for del square as 1 by r square delta by delta r of r square delta by delta r minus L square by h cross square r square. This acts on psi of r theta phi plus V of r psi of r theta phi is equal to e psi of r theta phi. This is the Eigenvalue problem that I am ought to solve. It is now clear what del square is, what L square is. Just by identification from here (Refer Slide Time: 16:45) I can write L square.

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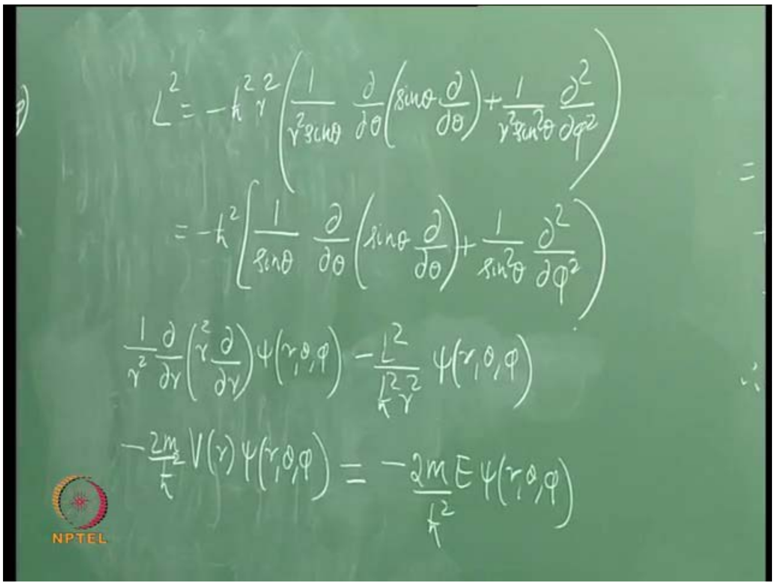

And therefore, L square is minus h cross square r square times 1 by r square sin theta, delta by delta theta of sin theta delta by delta theta plus 1 by r square sin square theta delta 2 by delta phi square. I can cancel r square and therefore, I only have an angular dependence in L square as I would expect. So, this is the expression for L square in spherical polar. I can see that L square depends only upon the angular variables and it is essentially the angular part of del square (Refer Slide Time: 16:45) that is what this equation tells me.

So, when del square acts on a wavefunction which is a function of r theta and phi, this part of del square acts on the radial wavefunction and this on the angular part of the wavefunction. In other words, I can try a separation of variables and write the total wavefunction as r of r some chi of theta phi.

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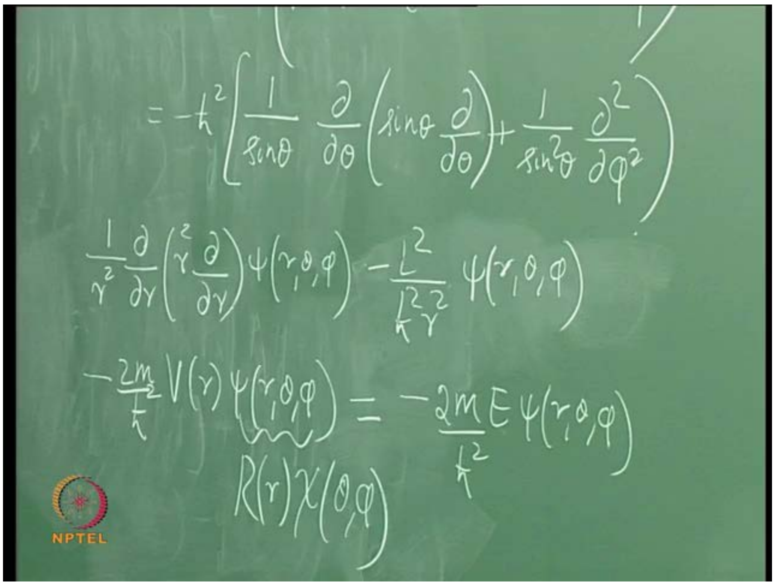

Now, if I did that I have the following equation so this is what I have in my equation and if I write psi of r theta phi as R of r chi of theta phi. So, I can do a separation of variables. I will demonstrate that later perhaps in my next lecture, but right now it is clear that chi of theta phi is an Eigenstate of L square after the separation of variables is done.

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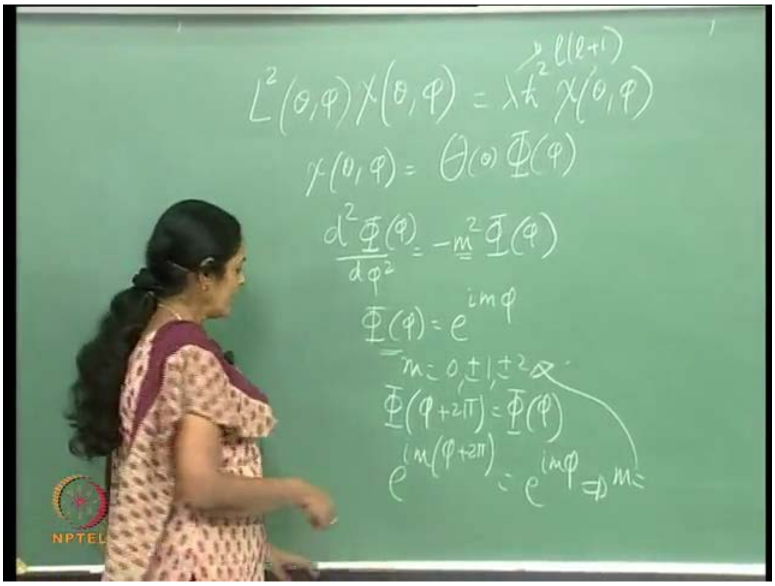

So, let me solve for that Eigenvalue equation. I can write L square which is an operator function of theta and phi, acting on chi of theta phi is lambda h cross square chi of theta phi. In fact, if you look at the structure of L square you can see that a further separation of variables can be done. In fact, I can write chi of theta phi is sum theta of theta phi of phi. I will put this into L square into the Eigenvalue equation for L square. Substitute for L square in terms of theta and phi explicitly. I can do a separation of variables. I will demonstrate that in detail in the next lecture. But right now I want to point out that after that is done, the equation for phi of phi would turn out to be this.

Going from there to this step would be shown in a subsequent lecture as I said, but this would be the phi equation and I want to focus attention on this for a considerable part of this lecture. What appears here is the separation constant that you would have when a function of theta is equated to a function of phi after you do a separation of variables. So, this is the separation constant. The solution for phi of phi is therefore, e to the i m phi, m taking value 0, plus minus 1, plus minus 2 and so on. I want to make an observation here. The facts that m takes these values. The fact that lambda should be of the form l times l plus 1 simply because this is the orbital angular momentum operator and it is orbital angular momentum operator square and therefore, in pattern it should be like the Eigenvalues for the spin square operator, s square.

So, the fact that lambda can be written as l times l plus 1 and m takes values like this would restrict l and I will comment about this in the next lecture. But, having said that let me focus on this equation. The wavefunction should be single valued at any physical point and therefore, phi of phi plus 2 pi must be equal to phi of pi. What does that mean? e to the i m phi plus 2 pi is e to the i m phi, which implies that m takes these values. So, that is how I get quantization of m, m h cross actually is the Eigenvalue for l z. But, the important thing is that the quantization comes because of single valuedness of the wavefunction at a physical point. This is like a phase and therefore, a phase change of 2 pi m is allowed. There is certain arbitrariness in the phase of the wavefunction up to 2 pi m, where m is an integer, but that is all the arbitrariness that is allowed.

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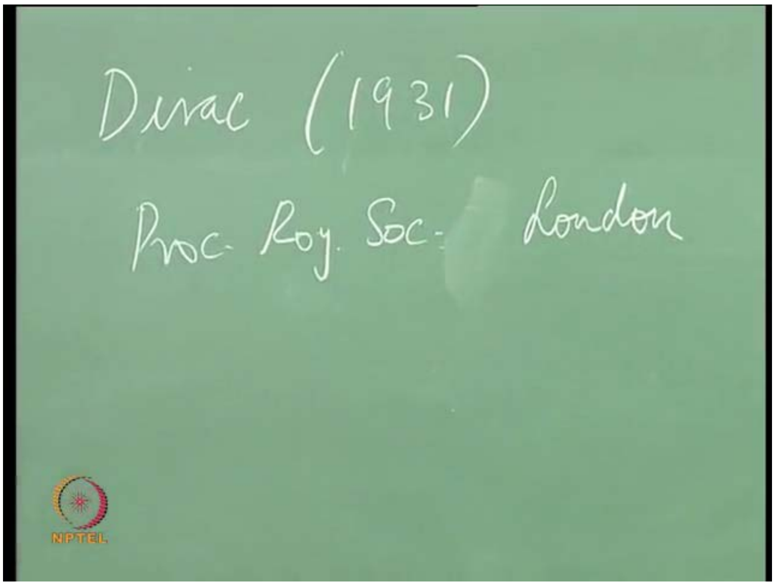

I now want to expand on the phase of the wavefunction particularly, in the context of a very important, enlightening and certainly a very extraordinarily educative paper and this was by Professor Dirac in 1931, in the proceeding of the royal society of London. What Dirac showed in this paper was that quantum mechanics does not preclude the existence of isolated magnetic chargers like you have electric charges, which are integer multiples of e or multiples of e. Dirac showed that in quantum mechanics you can also have magnetic charges and they will be quantized in units of some g which is the magnetic monopole, a monopole which is an isolated magnetic charge with magnetic charge g.

Now, in order to get through that result what Professor Dirac did was to talk about (Refer Slide Time: 24:43) the change in phase of a wavefunction, the fact that things should be single valued at a given space time point and so on and so forth. With very little formal mathematical equations, but with extraordinary logic, deductions have been made in this paper on the isolated magnetic charge and the quantization condition referred to as a Dirac quantization condition.

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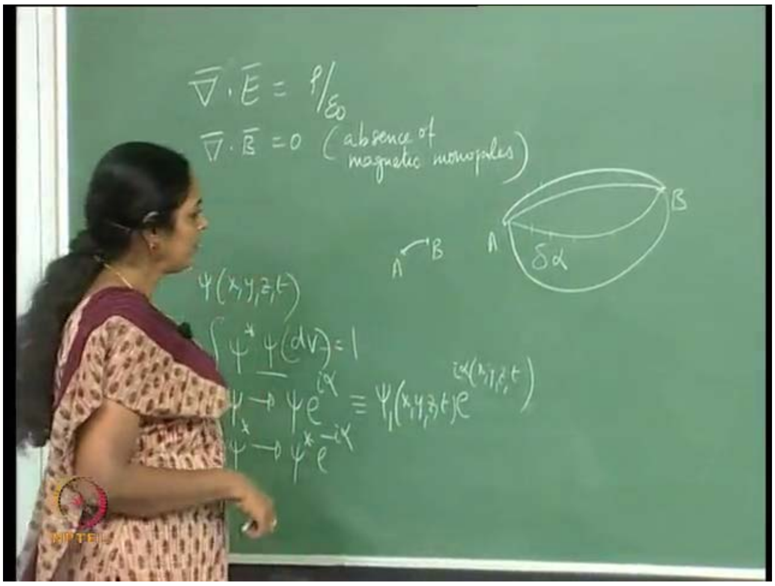

So, let me now talk about some of the highlights of that paper and sketch for you how exactly that result was obtained. The Maxwell equation in the presence of electric charges, reads del dot E is rho by epsilon 0. In the absence of magnetic charges or in the absence of magnetic monopoles, we would have an equation like this. Of course, if you had magnetic charges you should put a rho sub m here, analogous to the rho that we wrote there. This is what you would have in the absence of magnetic monopoles. In order to worry about whether a monopole exist or not, let us follow Professor Dirac's logic. Suppose I have a point A in space time so the coordinates are x, y, z and t and I have a wavefunction psi which is a function of x, y, z and t. This wavefunction satisfies a normalization condition, psi star of x integral psi star psi over the volume is 1.

So, that allows for a phase in psi. So, instead of psi I could have well written psi e to the i alpha, then psi star would be psi star e to the minus i alpha and the normalization condition is still satisfied. So, there is certain arbitrariness in the phase of the wavefunction. Now, suppose the system moves from point A to point B, notice that here psi is a function of x, y, z and t and therefore, I can write this as some psi 1 of x, y, z, t, e to the i alpha of x, y, z, t. So, there are two objects here which change is space time and then you go from A to B what you can measures is the change in phase of the wavefunction, because the phase changes with space time points.

So, while absolute phase is arbitrary there is certain arbitrariness in the phase. Phase differences between two neighboring points, phase difference in the wavefunction are not arbitrary and in general it can be nonzero. Now, suppose you took the wavefunction from A to a somewhat distant point B, clearly you can think of it as being made up of small paths which take you there. But, since the phase changes from point to point the net change in phase: delta alpha is going to depend upon the path that is taken. Over each of these paths delta alpha is going to be different, because alpha itself varies from point to point.

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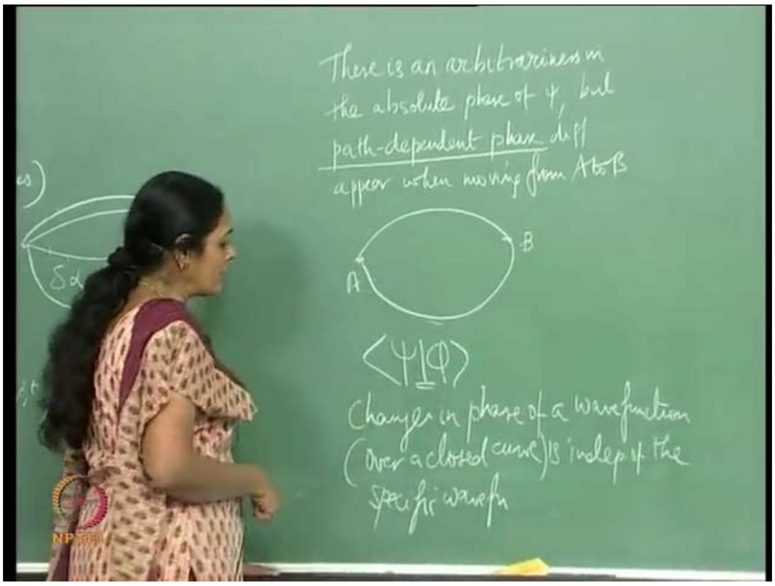

Therefore, we have the following conclusions. There is arbitrariness in the absolute phase of the wavefunction, but phase differences which are path dependent appear when moving from A to B. And therefore, when one goes from A to B and comes back from B to A over a close path we need not expect the net phase to be zero, because this is certain path and the total phase accumulation on this path need not cancel out the phase accumulation on this path. So, that is what I meant by a path dependent phase difference. On the other hand, we know that if you have a wavefunction psi and another phi, this object is the overlap between psi and phi and in fact when mod squares it tells me the probability of agreement of the two states psi and phi.

And therefore, that is ambiguously fixed for a given point which means that the change in phi over a closed path from A to B and back to A the change in phi should be equal to the change in psi. So, that you see this is psi star phi and since it is a phase change it comes as e to the minus i alpha here and e to the plus i alpha there or vice versa and cancels out. So, the change in the phase of wavefunction after traversing a closed curve should be independent of the wavefunction itself. It should be the same for all wavefunctions. Change in phase of a wavefunction over a closed curve is independent of the specific wavefunction.

(Refer Slide Time: 24:43) If you go back to this example, that is the kind of thing that is being told the change in phase has to be integer multiples of two phi so that there is single valuedness of the wavefunction and so on and so forth. But now, that is because of arbitrariness in phase, I am now talking about a different issue: how could this change in phase of the wavefunction could have happened? It could happen for the following reason and that is what I will illustrate now.

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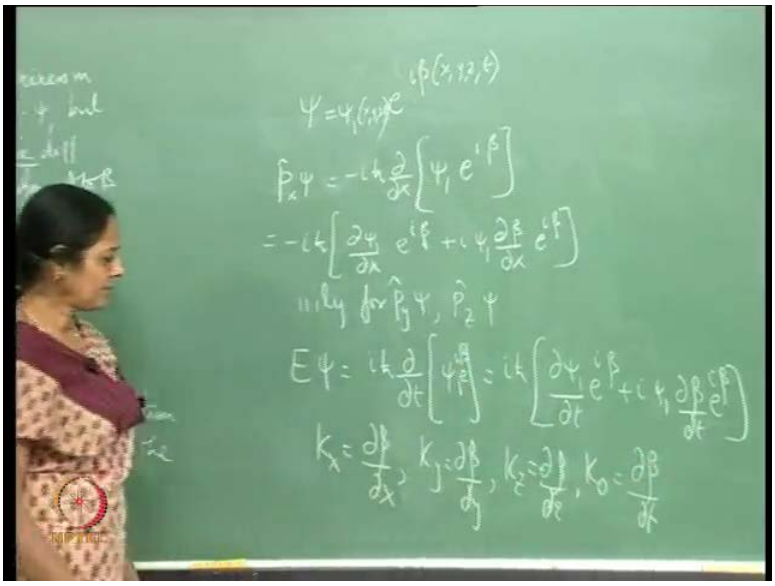

Let me start of in general by writing a wavefunction as some psi 1 and any change in phase happens, because of psi 1 of x, y, z and t and any change in phase which I am going to add up here and call it e to the i beta. Suppose, I consider the x component of the momentum operator P x psi, in quantum mechanics this is minus i h cross delta by delta x psi 1 e to the i beta and that object is minus i h cross delta psi 1 by delta x e to the i beta plus i psi 1 delta beta by delta x e to the i beta. Similarly, I can write down expressions for P y psi, P z psi and also E psi similarly, for P y psi, P z psi and E psi is i h cross delta by delta t psi, psi 1 e to the i beta and that quantity is i h cross delta psi 1 by delta t e to the i beta plus i psi 1 delta beta by delta t e to the i beta.

Now, since the wavefunction is continuous, the derivative of beta the phase should exist. Snd I am going to refer to delta beta by delta x as K x, delta beta by delta y as K y, delta Beta by delta z as K z and by K naught I refer to delta beta by delta t.

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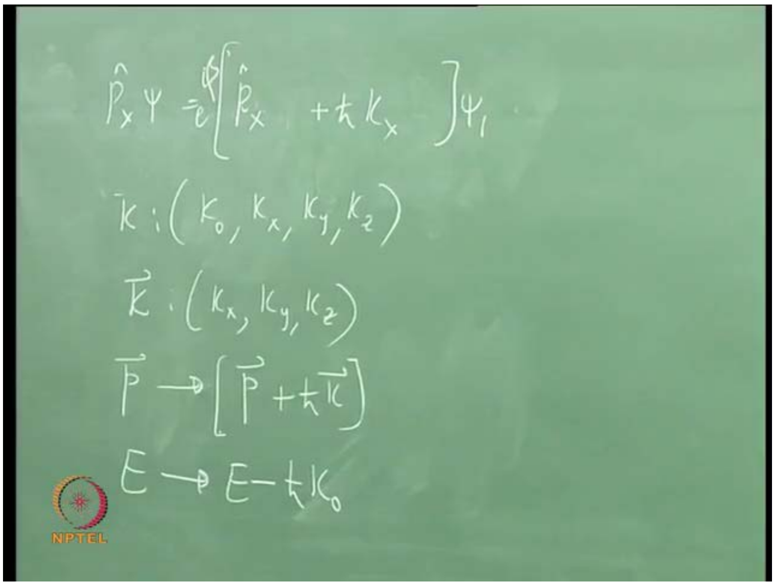

So, this is what I have when I work with the momentum and the energy operator on psi. So, let me look at the structure P x psi is simply minus i h cross. I can write this as P x psi 1 (Refer Slide Time: 38:30) because, minus i h cross delta by delta x psi 1 is P x psi 1. There is an e to the i Beta anyway and then I have plus h cross psi 1 K x so I can also pull this psi 1 out, e to the i Beta is here and I can write psi 1 like this.

Similarly, P y and P z which tells me that if I define an object K which has components: K naught, K x, K y and K z and three vector K which I will refer to as K with components K x, K y and K z. Then definitely this is true. It is P goes to P plus h cross K. Similarly, E goes to E minus h cross K naught. What does this mean? This is how psi 1 behaves when I write psi as psi 1 e to the i Beta. So, whenever there is a wave equation which involves the momentum operator P and the energy operator E and if that wave equation is satisfied by psi, psi 1 satisfies the corresponding wave equation with these replacements.

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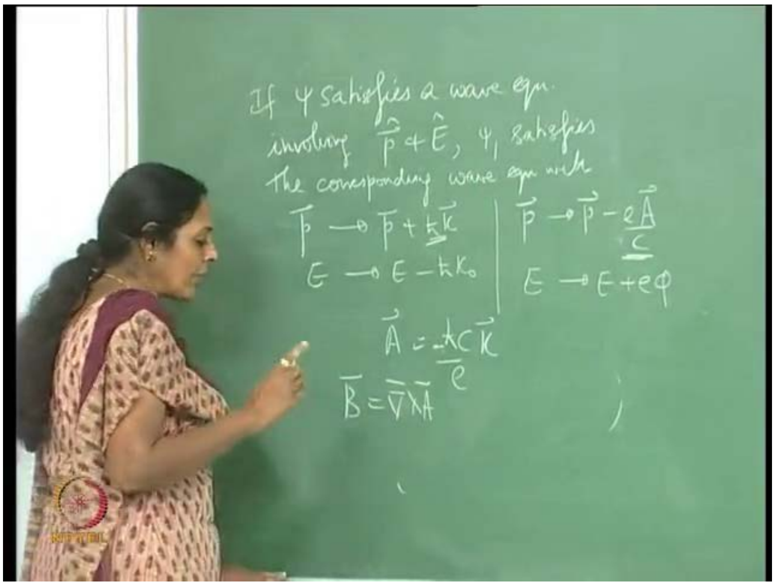

Let us put that down here. So, if psi satisfies a wave equation involving P and E; these are operators remember. Psi 1 satisfies the corresponding wave equation with P going to P plus h cross K and E replaced by E minus h cross K naught. But this is very reminiscent of something that you know. When you learnt about Gaussian variance gauge theory and so on, you realize that when a charged particle is put in the presence of a homogeneous magnetic field, there is a minimal coupling that happens between the charged particle and the field. The minimal coupling prescription was like this. Except, that if the vector potential was a, P was replaced by P minus e A by c and E was correspondingly replaced with the scalar potential.

So, given a vector potential A and a scalar potential phi, this change, this replacement is very reminiscent of what happens when a charged particle is put in the presence of a homogenous magnetic field. Now, by sure identification I know that A can be written as h cross c by e K. Similarly, phi can be written in terms of K naught, but this is of relevance to me right now. So, A is minus h cross c by e K. What is that mean? I can define a magnetic field B which is del cross A. From where does this field appear? Let us imagine that this field is because of a nonzero isolated magnetic charge and let say this strength is g, sitting somewhere in space.

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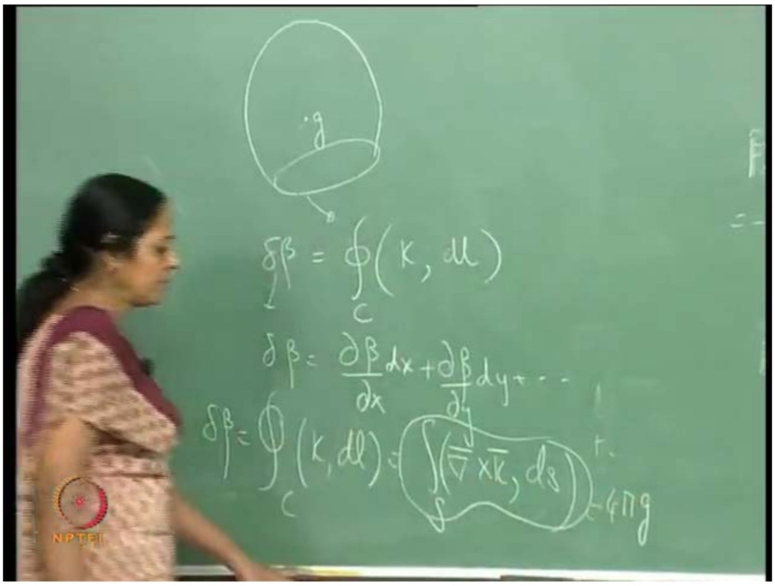

So all I am doing now is to take a wavefunction of a charge particle over a circle and what is change in phase of the wavefunction? The change in phase of the wavefunction can be written as delta beta which is simply integral over the circle. Let me call that C of K naught d t plus K x d x plus K y d y plus K z d z. So, if I defined a K (Refer Slide Time: 40:45) which had components that I have put down here and I define a d l which has components d t, d x, d y and d z. Then, in the notation used by Dirac in this paper, this would be the line integral of K. But this delta beta because I know that delta beta is this object and that is how I got that expression, because K x is this and K y is that and so on.

But I also know from Stokes theorem that I can write this object as an integral over the surface that it encloses, the surface that it bounds of del cross K d s, again in the notation used by Dirac in his paper. Because this is like the line integral related to the surface integral Stokes theorem and d s is the surface that I am talking about. So, this is what I have and this object is delta beta. But I also know the following: I know that this is the flux due to this charge that is sitting there, because del cross k is like the magnetic field. When I do this surface integral it is the flux and analogous to Gauss's theorem in electrostatics since, d is a constant this flux is going to be equal to four pi g. If I had a charge q it would have been an integral over the surface of the field due to this charge and analogous to that I have four pi g here.

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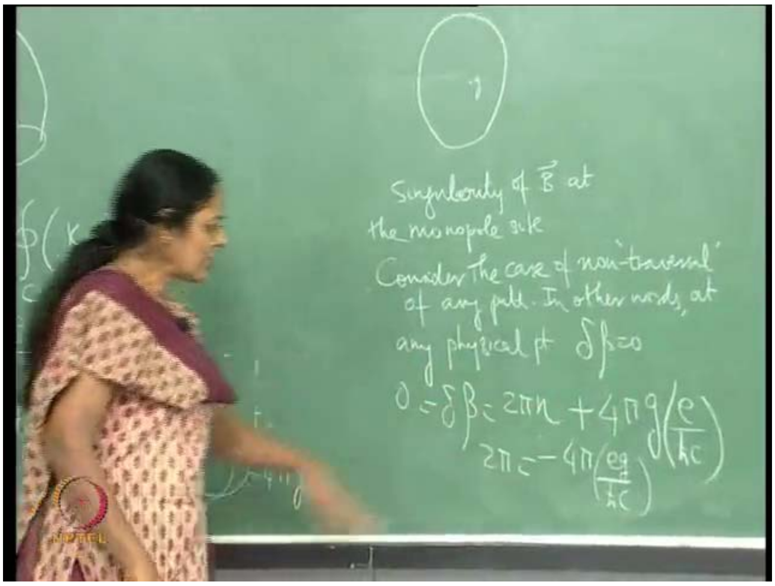

Now, let me examine the various surfaces that are possible here. Consider for instance, the limiting case which is of interest to me. When the curve that bounds the surface, shrinks to a point because the surface becomes a closed surface so it just shrinks to a point.

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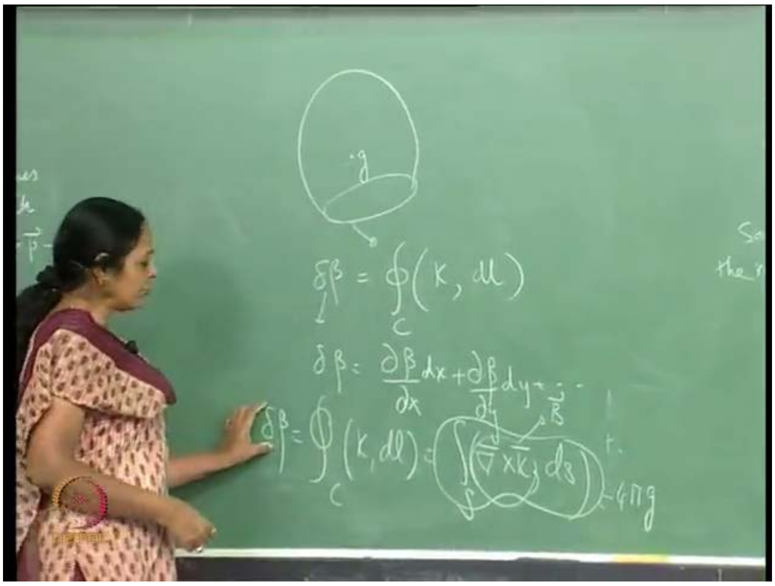

Then I know the following: this integral goes to 0 because d l is 0. But, on the other hand there is nonzero magnetic charge there and therefore, the flux is nonzero. If the flux is nonzero, the only way by which I can have a nonzero quantity with this whole thing going to 0 is if B becomes singular at that point. And therefore, I have a magnetic field which picks up a singularity at one point on the closed surface and suppose, the close surface is simply the point g itself, if the close surface is shrunk. Then I can well imagine that there is a singularity of the magnetic field at the monopole site.

So, I have a singularity of B at the monopole site. That is one aspect that if I define a B, I can still get away with it provided, I have a singularity at one point on any closed surface around the magnetic charge which I consider here. (Refer Slide Time: 49:04) Now, more through the point if the line integral, if the line, the curve is simply reduced to a point there is nothing like a delta beta, delta beta is 0 because I never really went over a path to accumulate a phase. In fact there is a no movement, it is simply a point.

So, let us consider the case of non traversal of any path. In other words, at any physical point delta beta is equal to 0, there is no phase accumulation because no path was traversed. As a digression I should say that if you did traverse a path and somehow accumulated a phase that too can happen in certain contexts and the Berry phases is an example, I will comment about it later. But returning to this, as it is there is arbitrariness in the phase of 2 pi n. In my earlier example I call that m, magnetic quantum number 2 pi m, but now I am referring to it as 2 pi n, where n is an integer 0, 1, 2, 3 anything plus minus 1 and so on. We have to put in that arbitrariness plus 4 pi g.

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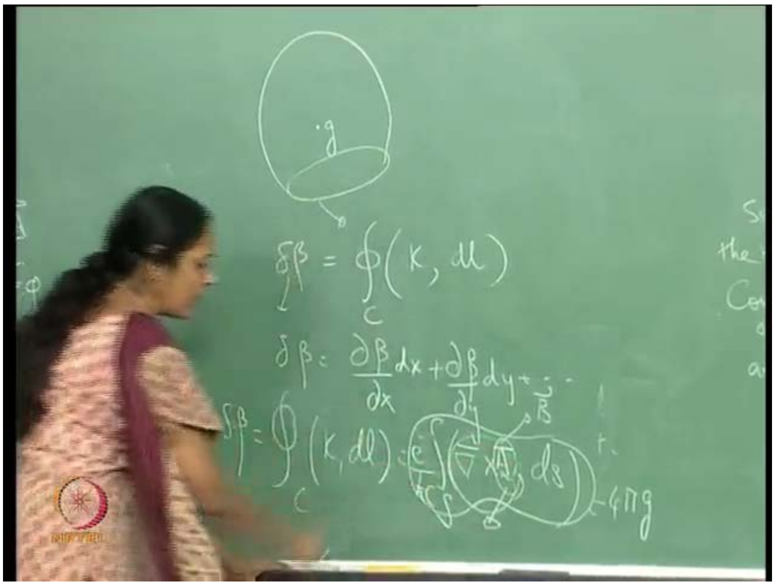

But, it is not just 4 pi g because, this is del cross K and therefore, this is not quite the flux I should be able to write this K as (Refer Slide Time: 43:00) e A by h cross c. So, I have an e by h cross c multiplying A. Instead of K, I write it as e A by h cross c and this is in fact the magnetic field so I have a constant e by h cross c to reckon with. So, it is 4 pi g e by h cross c. So, look at a situation when delta beta is 0. What does that mean? Let us set n equals 1 that means I am talking about a fundamental pole analogous to e the electrons charge. So, 2 pi is equal to minus 4 pi e g by h cross c.

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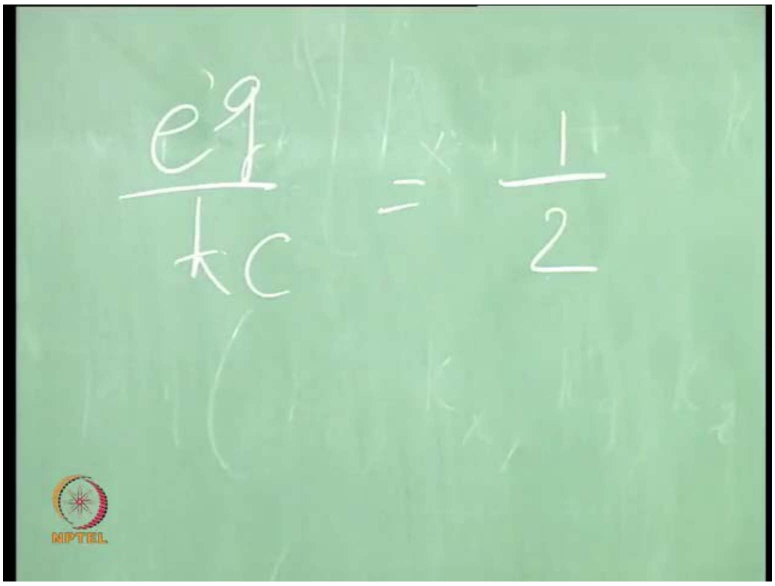

So, I have e g by h cross c getting quantized now, because e g by h cross c is now equal to half (Refer Slide Time: 48:30) I have set n equals 1 otherwise I will have an n by 2. And suppose I look at this as the electrons charge so if I am talking about an electron so I would have had a minus e and that would have canceled the minus sign here. So, in fact I just have e g by h cross c equals half. Suppose by this I meant the electronic charge.

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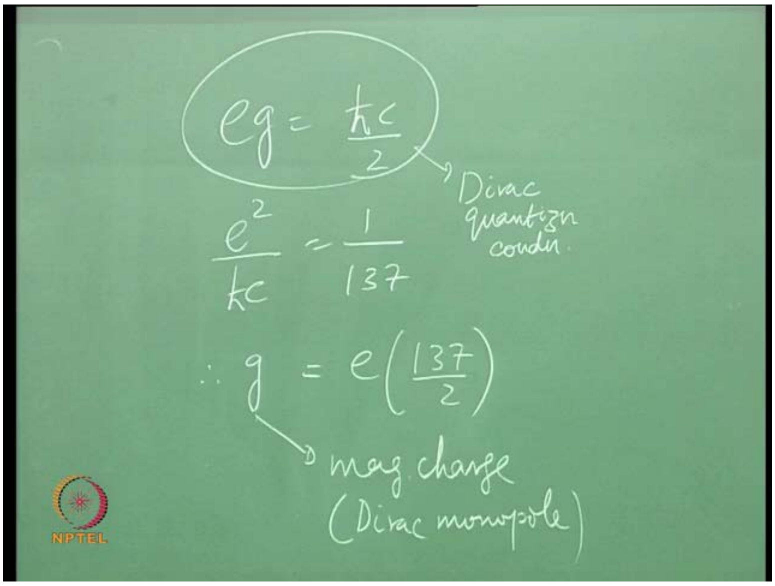

Therefore, I have e g by h cross c is half or e g is h cross c by 2, but e square by h cross c is a very small number it is 1 by 137. So, substituting for h cross c into the Dirac quantization condition that is e g is equal to h cross c by 2 into that if I substitute for h cross c. I find that g is very large g is 137 by 2 e and that is the large number.

And therefore, not that quantum mechanics precludes the existence of isolator monopoles. We have just now gone through the entire machinery, as proposed by Dirac. However, it is very difficult to isolate a monopole because monopoles are very strongly interacting with each other, very strongly coupled to each other. So, these are the contents of the significant part of the contents of the paper by Professor Dirac in the proceedings of the royal society. Quite apart from what I have told you here, there is another matter that comes out very clearly and that is this. (Refer Slide Time: 52:03)

If there is a method of producing a net change in phase apart, from the arbitrariness 2 pi n which I will not talk about really. (Refer Slide Time: 48:30) That change in phase should come because of effectively a field sitting in the appropriate space. In this case we are talking about the change in phase in physical space, when a wavefunction goes around and that is why the single valuedness of the wavefunction came into effect. (Refer Slide Time: 52:03) However, in a general setting the change in phase would really appear because of an effective field force in an appropriate space and that is what happens in the case of the Berry phase.

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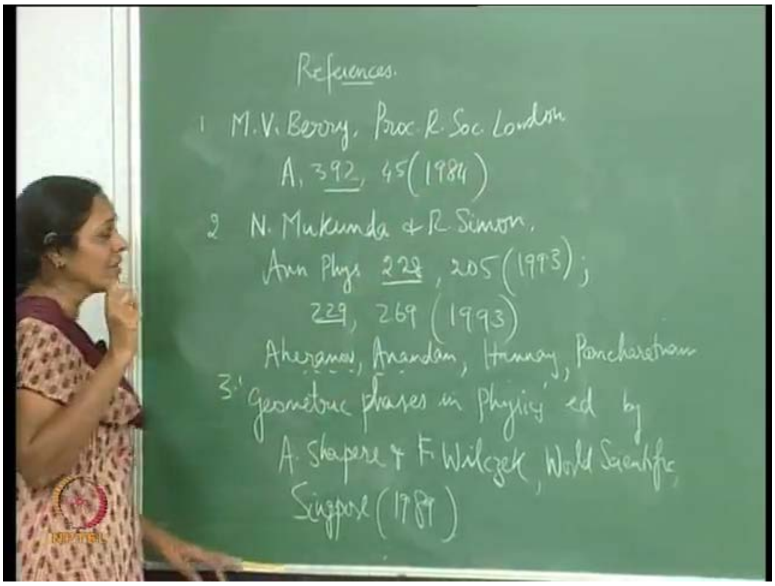

This is very interesting, again very useful and important seminal paper by Professor Berry. I have the reference here. It is M V Berry Proceedings of the Royal Society of London again and this is volume 392 in the year 1984 page 45. Now, it led to an ((Refer Time: 56:03)) of papers, but before I talk about some of the at least or mention some of the at least I want to point out how exactly the Berry phase appears.

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The Berry phase as it is called, it is clear that it comes because of dynamics. Because earlier on I told you that all wavefunctions pickup the same phase when they come back to a physical point. But there could be small change in the statement if there is an effective field sitting in the space. For instance, suppose I am able to change the parameters in a Hamiltonian. Take for instance the simple harmonic oscillator certainly, one parameter is omega. Suppose, I cycle omega slowly over time and then get back to the original value of omega in general, there can be more parameters. So, in a parameter space which is appropriate for that given physical system, if I did an adiabatic slow cycling of parameters and brought them back to their original value Hamiltonian cannot change its physically measurable.

But then the wavefunction can pick up an extra phase and that is called the Berry phase. The Berry phase could be different for different wavefunctions simply because their dependence on the parameter, each wavefunction depends on the parameters differently. The Berry phase itself is experimentally measurable in certain settings, because I could have a handle on the parameters and the way I cycle the parameters. While the original work done by Professor Berry talks about adiabatic cycling of the parameters and therefore, there is a monopole sitting in the parameters space now, not in physical space. That has been relaxed and in a very general setting where the cycling need not be adiabatic. In fact, it need not be over a closed path, it can even be over an open path.

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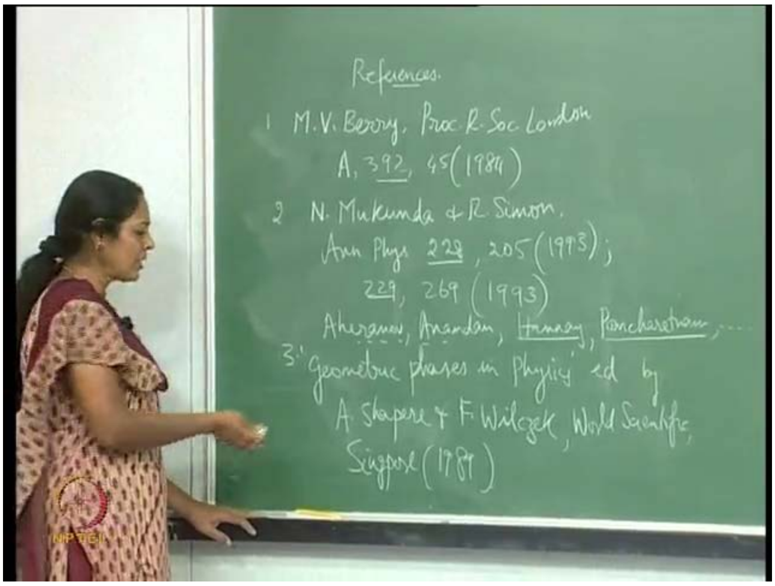

Generalizations have been done there are many papers dealing with that but, I mention one of these papers by Professor Mukunda and Simon, this was done in 93. They have been many names associated with the Berry phase and it is classical counterpart, the Hannay angle is the classical counterpart. The Berry phase in the context of optics is a Pancharatnam phase and subsequently many other names have been associated with it.

I put down some of these names and in fact this is an extremely partial list of names. It is for those of you who are interested in knowing something more about the Berry phase, there is this book Geometric phases in Physics edited by Shapere and Wilczek and we should also include their names here in this list. There are many more names. This is to be treated completely as a very partial list of people who have worked on Berry phases.

And in general it is clear that phases are very important some of them are experimentally measurable and one cannot just very casually put aside a statement like the wavefunction is single valued up to an arbitrary phase. It is not just an arbitrary phase you can generate more components to the phase like the Berry phase, or like the phase that led to (Refer Slide Time: 48:30) the monopole quantization. In the next lecture, I will continue where I left off with orbital angle of momentum, work out the details of the separation variables and we will take it up from there.