Quantum Mechanics- I

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Lecture - 26

The Square Well and the Square Potential Barrier

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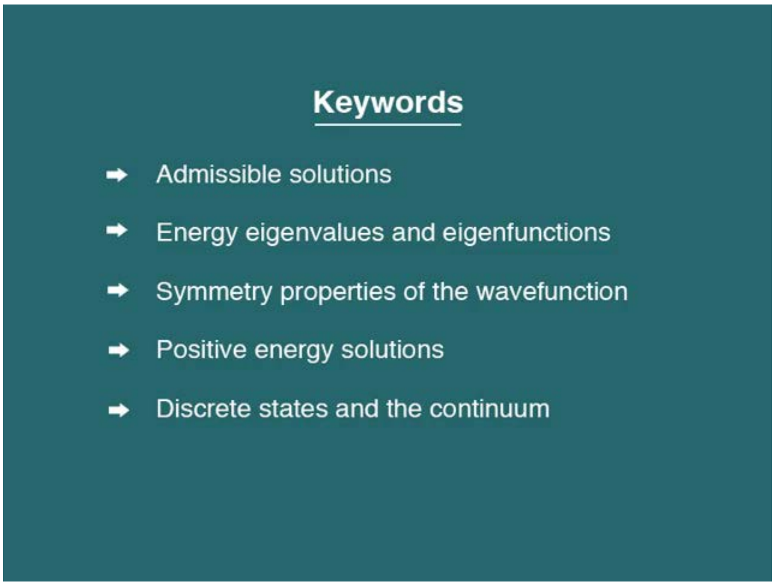

In the last lecture we were discussing the square well potential problem. This is a one dimensional problem where we were discussing stationary states basically the energy Eigen states. Let me quickly recapitulate some of the steps that we worked out in the last lecture so, that we can continue from there and discuss the square well problem in greater detail in this lecture.

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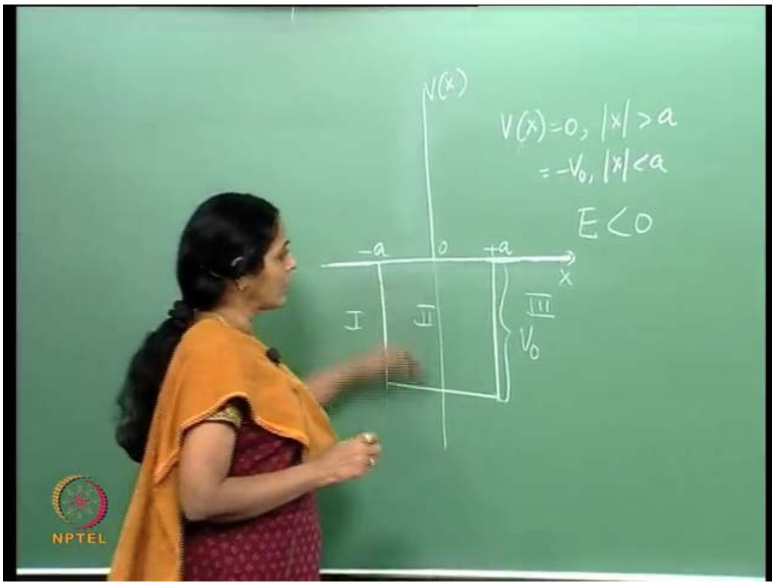

So, we had this square well potential. This was V of x versus x and in the negative region we have this square well. So, out there and out here at minus a and plus a the well began. The potential itself was minus V naught where V naught is a positive quantity. This was in region 2 which is inside the well then there were regions 1 and 3 outside the well and the square well potential was given by 0 for mod x greater than a and was equal to minus V naught for mod x less than a.

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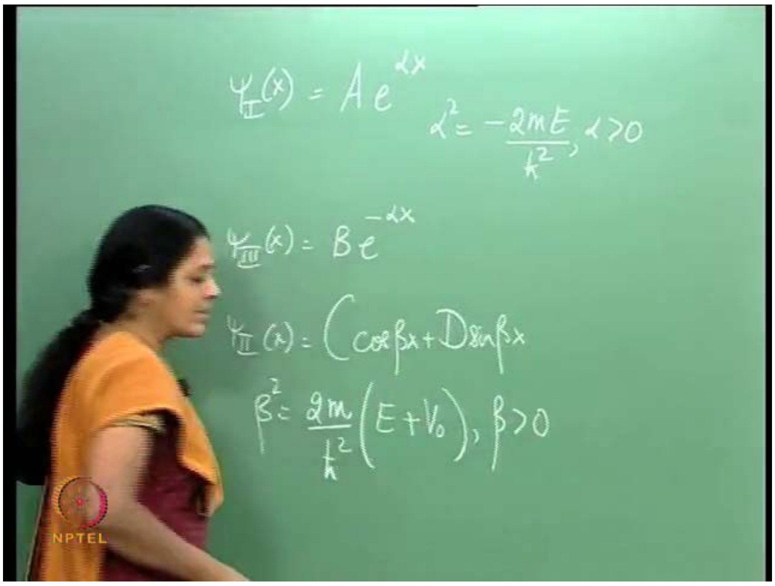

Now, in the case we wrote out the wave functions in the 3 regions and basically because the wave function has to vanish its spatial infinity we had psi 1 of x, that was in region 1. That was simply A e to the alpha x we were considering bound states and therefore, (Refer Slide Time: 00:42) it is important to note that the energy was less than 0. The whole idea was to see if classically forbidden regions are accessible in quantum physics.

(Refer Slide Time: 01:54) So, this came because alpha square was a positive quantity which was given by minus 2 m E by h cross square and we also said that alpha was greater than 0. Since E is negative minus 2 m E by h cross square is positive. (Refer Slide Time: 00:42) So, this is for an energy which is less than 0 somewhere out there psi 3 of x was B e to the minus alpha x. This was put down so that the wave function vanished at x equals plus infinity and we had psi 2 of x which was C cos beta x plus D sin beta x where beta square was 2 m by h cross square E plus V naught. (Refer Slide Time: 00:42) And since E is less than V naught, V naught plus E is a positive quantity and therefore, beta square is positive with beta greater than 0 that was our definition.

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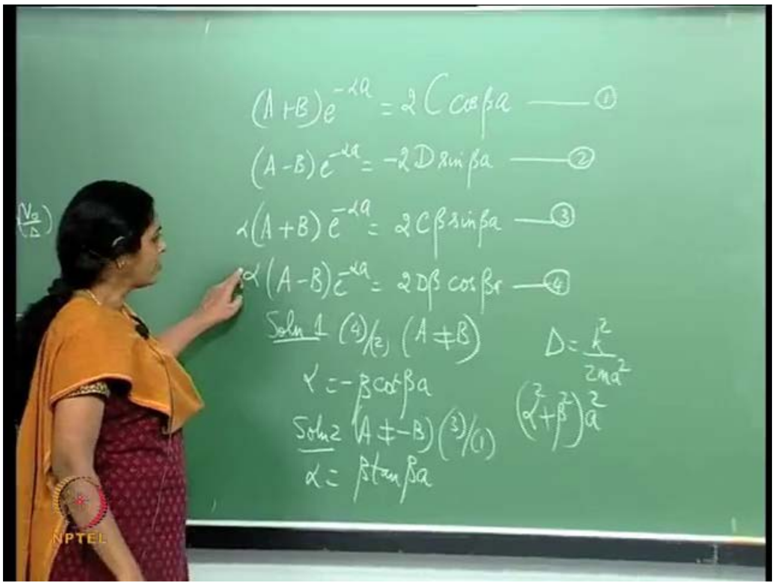

Now, given this we match the wave functions and the derivatives at x is plus a and x is minus a and then we found that we could write the following equations: A plus B e to the minus alpha a was 2 C cos beta a let me refer to that as equation 1. A minus B e to the minus alpha a was minus 2 D sin beta a and then of course, the derivatives when you differentiate it would pull down an alpha or a beta depending on the situation. And you have alpha times A plus B e to the minus alpha a was 2 C beta sin beta a and alpha times A minus B e to the minus alpha a was 2 D beta cos beta a. So, these were the 4 equations that we have written down.

And finally, we discussed 2 possible solutions to this problem depending on whether A was not equal to B. If A is not equal to B then A minus B is not 0 and therefore, I could divide 4 by 2 and that gave me one solution for alpha. If A is not equal to minus B this does not disappear and I could divide equation 3 by equation 1 and that gave me another solution for alpha. So, we had solution 1 and 2 so, let me just write that down. Solution 1, came by dividing equation 4 by equation 2 so, it automatically assumed that A was not equal to B and therefore, I could do this and that told me that alpha was beta. This was equation 4 divided by 2 so, it gave me a minus beta cot beta a, that was solution 1.

Then we have solution 2 and that assumed that I would be able to divide equation 3 by equation 1 which means A is not equal to minus B and the solution itself came by dividing equation 3 by equation 1. And when you do that you get alpha is equal to beta tan beta a so, this was the 2nd solution. So, these are possible solutions and then we use this to also find out the allowed values of beta a for the different cases because if alpha is greater than 0 and beta is greater than 0 beta a is restricted to certain values. So, let us look at just what we had in the last lecture.

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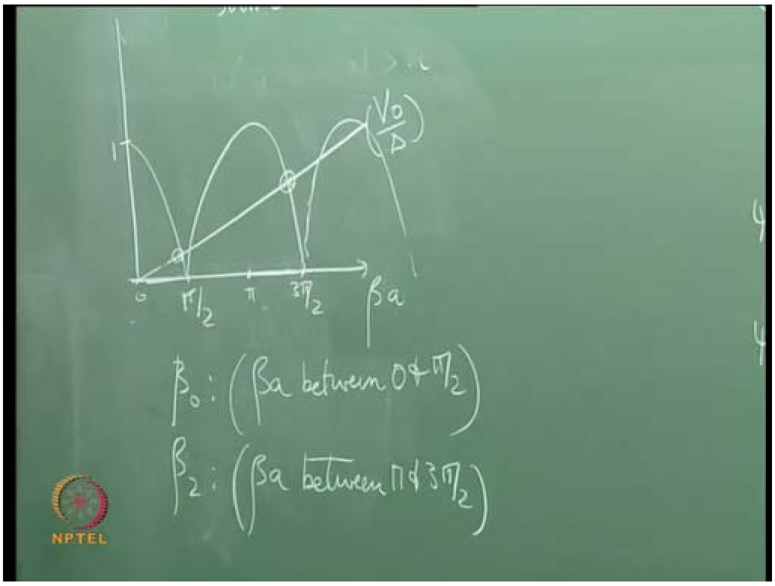

So, if I plotted beta a there were only some allowed regions. Let me look at solution 2. Solution 2 says that alpha is equal to beta tan beta a. (Refer Slide Time: 03:50) A very crucial input was this that we had a parameter delta which was h cross square by 2 m a square and that led us to an equation alpha square plus beta square times a square in terms of V naught by delta. So, once you substitute for alpha square plus beta square from there, you find that there is an expression for V naught by delta which tells you what is the value of V naught in terms of the strength of the potential and we realised that all we have to do was plot mod cos beta a versus beta a.

And therefore, we could choose various values for V naught by delta and that gave me the slope of the equation delta by V naught to the power of half beta a. And, because alpha was equal to beta tan beta a and alpha and beta are both positive, beta a was allowed to have values only from 0 to pi by to pi to 3 pi by 2 and so on. There were certain regions where you could have admissible solutions.

And since this was a plot of modulus of cos beta a versus beta a we found that this was the cos function. The crucial point is there is one point of intersection in this and therefore, there was one possible value of beta between 0 and pi by 2. Let me call that beta naught that lies between 0 and pi by 2 may be it is better to write beta a between 0 and pi by 2. But then you see the next solution happens between pi and 3 pi by 2 there is a cut there and that is an allowed solution (Refer Slide Time: 03:50) and that comes from alpha is equal to beta tan beta a, that is solution 2.

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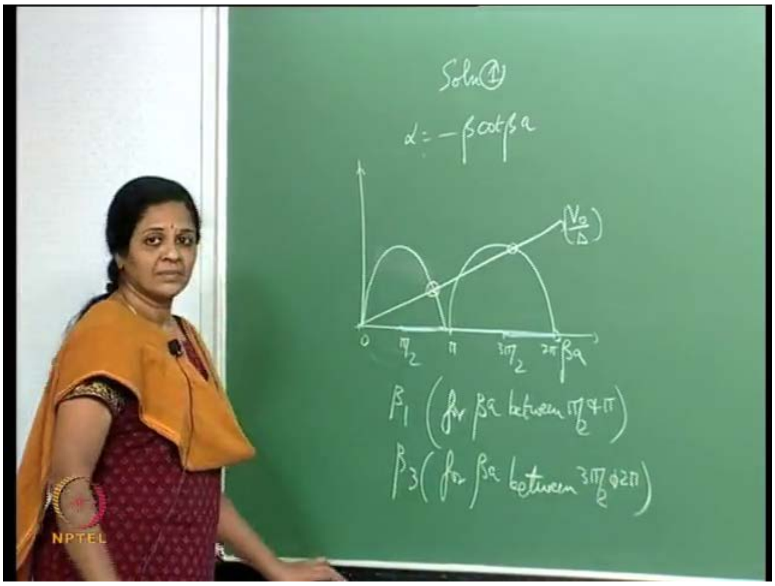

As I said in my last lecture you could have looked at solution 1. If you had done that alpha is equal to minus beta cot beta a that was solution 1 and this time, the allowed values of beta a because alpha is greater than 0 and beta is greater than 0 the allowed values of beta a are between pi by 2 and pi, 3 pi by 2 to 2 pi and so on. In this case, you end up plotting the function modulus of sin beta a. I fix a V naught by delta that would give me the slope and if it is modulus of sin beta a that needs to be plotted I have the sin function. And notice that there is one solution out here for beta a between pi by 2 and pi. This is very crucial so, since that solution happens between pi by 2 to pi let me call that beta 1 for beta a between pi by 2 and pi.

(Refer Slide Time: 07:02) So, if you compare beta naught the 1st solution corresponds to beta a lying between 0 and pi by 2. Beta 1 comes from the other solution alpha is minus beta cot beta a and that is for beta a lying between pi by 2 and pi. (Refer Slide Time: 07:02) Then beta 2 the next solution is here. This corresponds to beta a between pi and 3 pi by 2 and then look at this solution between 3 pi by 2 and 2 pi that comes from here. Because the allowed value of beta a in this case is pi by 2 to pi and 3 pi by 2 to 2 pi. Might be useful to have a coloured chalk that and this and I have a solution here. So, I will call that beta 3 between 3 by 2 and 2 pi and so on. So, you get a set of discrete values for beta we can call them beta sub n, n is equal to 0 2 4 etcetera came from alpha is equal to beta tan beta a, n is equal to 1 3 5 and so on came from alpha is equal to minus beta cot beta a.

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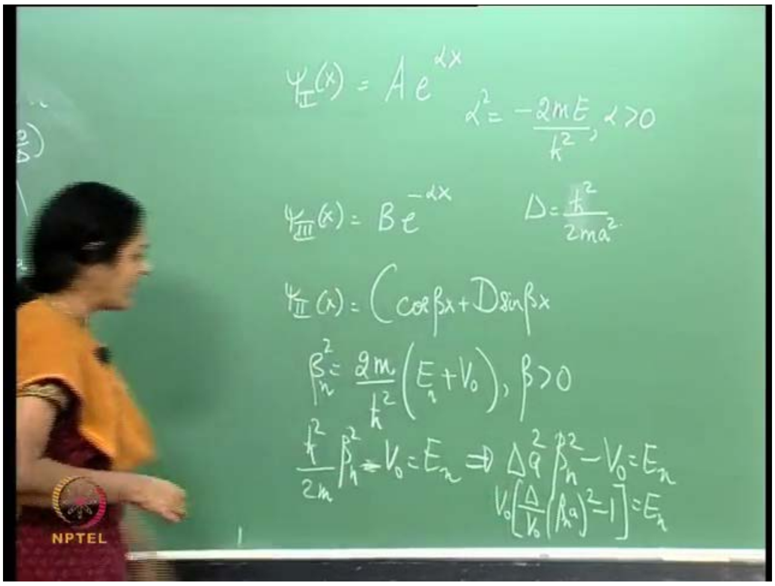

Now, once I have beta sub n I know that the energy is quantized, because look at this here I have beta sub n square is 2 m by h cross E n plus V naught and therefore, h cross square by 2 m beta n square minus V naught is equal to E n. So, I can find out the corresponding values of the energy. The important thing which I also emphasized last time was that however, shallow the well may be however small the strength of the potential may be you find that there is definitely one solution and that solution comes for beta between 0 and pi by 2. There is always a solution. That is an important feature of the quantum system.

If you look at this and from here and if we try to find out what is an E n? There is a nicer way of doing this problem. Please recall that delta was h cross square by 2 m a square and therefore, I can write h cross square by 2 m as in terms of delta and I can write that as delta a square. So therefore, this implies that delta a square beta n square minus V naught is equal to E n. In other words, I can pullout the V naught and I just have delta by V naught beta n a the whole square minus 1 is E n. What is nice about writing it in this fashion is that the, when I have put down inside the square braces delta by V naught times beta n a the whole square.

(Refer Slide Time: 07:02) This can be read off from here and therefore, this is a more instructive way of writing it. The quantity within the square braces can be read off directly from the plot and therefore, I get various values of E n once I know V naught and V naught is something that I have decided on to begin with decided on the strength of the potential. In other words, I know V naught by delta. So, this is what we have in terms of the energy Eigen values. Let us look at the energy Eigen functions.

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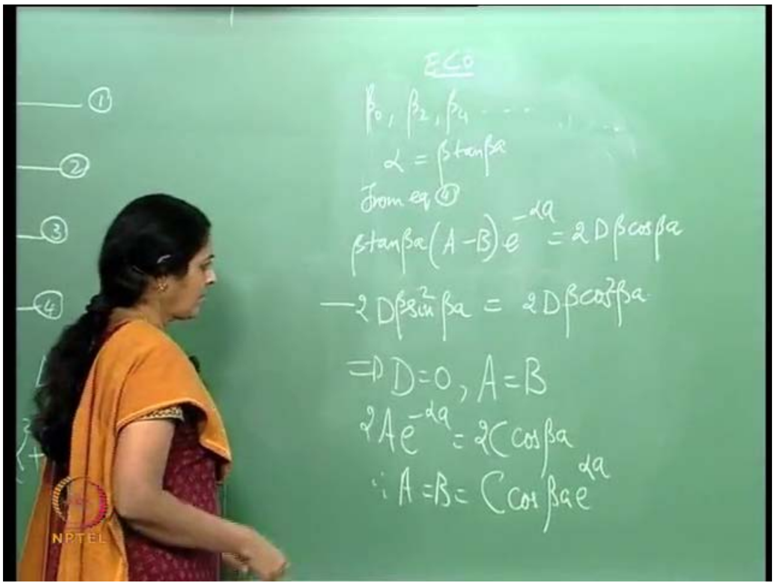

Now, as far as the energy Eigen functions are concerned we will take up solution 2 first. Recall that the solution 2 gave us beta naught, beta 2, beta 4 and so on. This is all for E less than 0 we are looking at the bound states and alpha was beta tan beta a; this was solution 2. (Refer Slide Time: 03:50) In order to get alpha is equal to beta tan beta a; I had divided equation 3 here by equation 1. So, now I would like to take this value of alpha and substitute it in equation 4. So, from equation 4 beta tan beta a times A minus B e to the minus alpha a is 2 D beta cos beta a. (Refer Slide Time: 03:50)

So, I have already used equation 4 now, I am left with equation 1 and I would like to substitute equation 2 and I would like to substitute for A minus B e to the minus alpha a from equation 2 and therefore, I have minus 2 D beta sin square beta a is equal to 2 d beta. From the tan I can take the cos there and that gives me a cos square beta a. So, this cannot happen because it tells me that sin square beta a is minus cos square beta a not possible this implies the D is equal to 0.

(Refer Slide Time: 03:50) So, going back to these equations, if D is equal to 0 from equation 2 I know that A is equal to B, started off by saying that A was not equal to minus B. That is okay. We would just now seen that A is equal to B plus B so, if A is equal to B and D is equal to 0 I can find A in terms of C. So, that tells me that 2 a E to the minus alpha a from equation 1 is 2 C cos beta a and therefore, A itself which is also equal to B is C cos beta A e to the alpha a. So, now I can write out the wave function in the different regions.

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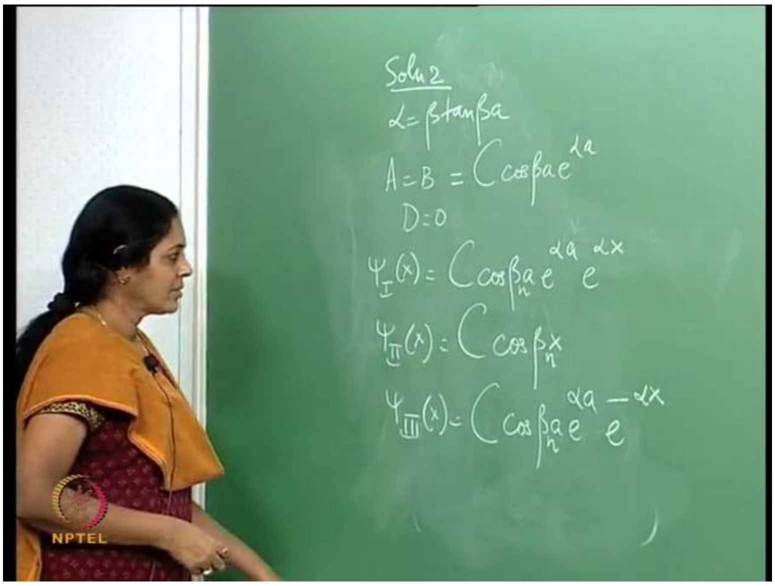

So, let us see psi 1 of x is A e to the alpha x. So, we are looking at solution 2 alpha is beta tan beta a. It gave me A equals B and that can be written in terms of C as C cos beta a e to the alpha a, D was equal to 0. It means that psi 1 of x which was A e to the alpha x can be written as C cos beta a e to the alpha a e to the alpha x. Boundary condition is satisfied when x goes to minus infinity the wave function goes to 0. Psi 2 of x the term with d as a coefficient dropped out so, you just have C cos beta x. Psi 3 of x a was equal to B and therefore, you had C cos beta a e to the alpha a e to the minus alpha x.

The minus here was selected was chosen by us so, that when x goes to plus infinity, the wave function vanished. There is another check, when x is equal to minus a psi 1 of x matches with psi 2 of x which is the way it should be and when x is equal to plus a: psi 2 of x matches with psi 3 of x. You can also check that the derivatives match. There is a very interesting feature that comes out here. This wave function, the total wave function has 3 parts psi 1 in the region x less than minus a, psi 2 in the region minus a to plus a for x and psi 3 plus a to infinity.

Now, we find that this wave function is a symmetric function of x because we should be really writing cos beta n out here. But these solutions correspond to n equals to 0 2 4 and so on because, we got it from alpha is equal to beta tan beta a. This wave function is a symmetric wave function because when x goes to minus x, psi 1 goes to psi 3 and psi 2 is left invariant. So, what we have seen is that under the parity transformation, that is when x goes to minus x the wave functions corresponding to the Eigen values beta naught, beta 2, beta 4 etcetera. In other words, energy Eigen values E naught, E 2, E 4 etcetera have even parity, they are symmetric about x is equal to 0.

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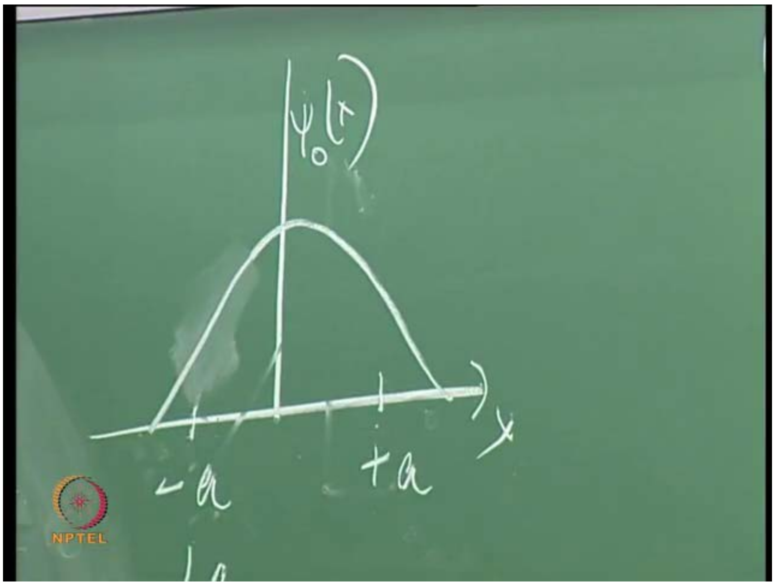

This means that if I plotted for instance, the ground state wave function. The wave function would look something like this. That is x is equal to plus a, that is x equal to minus a. Notice that it is a cos function here, but then it does not taper off there it goes all the way here. There is an exponential fall off so, there is a certain penetration or a leakage into the classically forbidden regions. This is psi 0 of x for instance there is a penetration into the classically forbidden regions.

In other words, there is a non zero probability of seeing the particle with mass m outside the square well potential although its energy is less than 0. But, then that is an exponential fall off which comes out from this term e to the alpha x here and e to the minus alpha x there. So, certainly two important features of quantum physics emerge 1: In a bound state the energy levels are discretized. 2: Although the energy is less than 0 quite in contrast to classical physics, I can see a non zero probability of leakage into the classically forbidden regions of the wave function.

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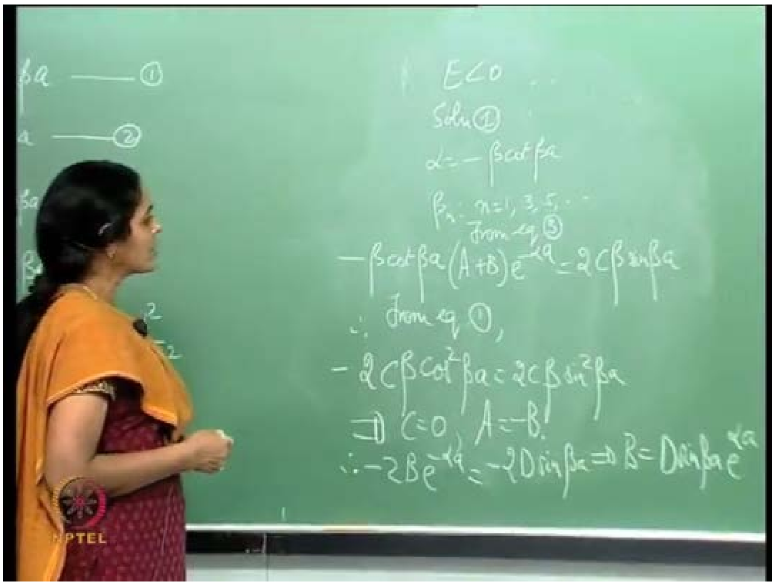

So, this is as far as solution 2 is concerned. So, let us look at solution 1. Solution 1 again e is less than 0 and we repeat our argument. Now for solution 1 that was alpha is equal to minus beta cot beta a and remember the values that beta could take where beta n with n equals 1, 3, 5 and so on. (Refer Slide Time: 03:50) I got the equation alpha is equal to minus beta cot beta a by dividing equation 4 by 2 so, I can well substitute for alpha now in equation 3. So, if I did that minus beta cot beta a so, from equation 3 minus beta cot beta a A plus B e to the minus alpha a is 2 C beta sin beta a. So, I have used equations 4, 3 and 2, but I am left with equation 1 from which I will substitute for A plus B e to the minus alpha a.

Therefore, from equation 1 minus 2 C beta cos beta a so that gives me a cos square beta a is 2 C beta. I can take the sin from here there sin square beta a and that is not possible because as in the earlier case this time again cos square beta a is equal to minus sin square beta a. Therefore, this implies that C 0. (Refer Slide Time: 03:50) So, going back to these equations if C is 0, it means that A is minus B. Recall that we got this solution by saying A is not equal to plus B and what we have seen is that A is minus B.

So, if A is equal to minus B and C is 0 I can substitute here and I have minus 2 B e to the minus alpha a is minus 2 D sin beta a. So, I can solve for B in terms of D and this implies that B is equal to D sin beta a e to the alpha a. So, A is minus d sin beta a e to the alpha a. So, I have got the solution corresponding to the various wave functions now.

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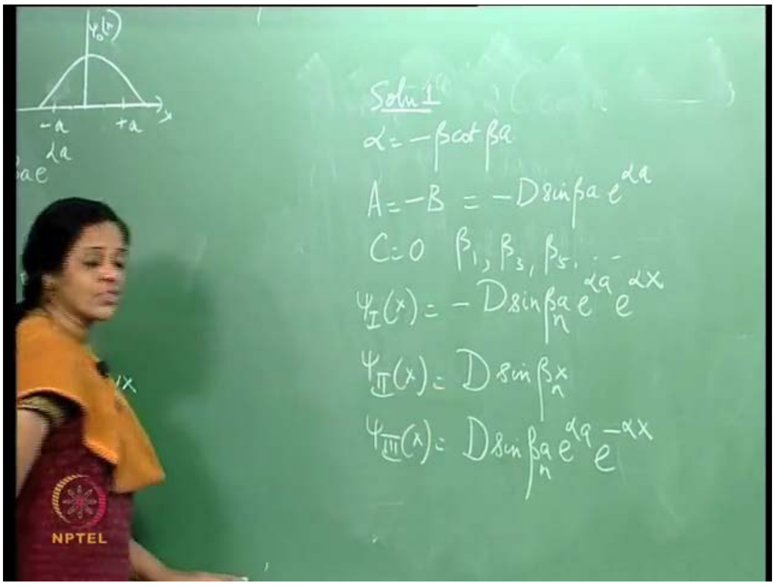

So, psi 1 of x, psi 2 of x and psi 3 of x can be written for this case. So, let me do that here. So, this is solution 1: alpha is equal to minus beta cot beta a. This tells me that A is minus B and that quantity is minus D sin beta a e to the alpha a. It also tells me that C is 0 and therefore, I can write various wave functions now. Psi 1 of x is a e to the alpha x so, it is minus D sin beta a e to the alpha a e to the alpha x. Psi 2 of x since, C is 0 it is just D sin beta x and psi 3 of x is D e to the minus alpha x. So, I can write that down as D sin beta a e to the alpha a e to the minus alpha x. This corresponds of course, to quantized values of beta, beta 1, beta 3, beta 5, and so on.

And you can see that this wave function also has a definite parity because if you take x to minus x, psi 1 of x goes to minus psi 3 of x and psi 2 of x again is an odd function of x. So, these states have odd parity, they are odd parity states. So, what we have come across now is that the ground state has even parity, the 1st excited state has odd parity the 2nd excited state is even parity and so on alternately. The point is this: we saw the same thing in the case of the linear harmonic oscillator where the states came with definite parity, the energy Eigen states, once more alternating between even and odd parity, the ground state having even parity.

In that context I had explained that this feature arose because the Hamiltonian commuted with the parity operator. It is the same reason that I would give here because the square well itself if you look at form of the potential, the potential was a symmetric function of x went from minus a to plus a and it was symmetric about the origin.

And therefore, the Hamiltonian versus symmetric function of x as a result of which it commuted with the parity operator and hence you could find a complete set of common Eigen states of the Hamiltonian and the parity operator. Now, these are energy Eigen states of the Hamiltonian therefore, they should be Eigen states of the parity operator. This is indeed the complete set of common Eigen states between H the Hamiltonian and p the parity operator and since they are Eigen states of a parity operator and the only Eigen values allowed are plus or minus 1. They are states with definite parity either symmetric under x going to minus x or antisymmetric under x going to minus x. So, that is what we have seen in this problem. So, so much for the bound state solution.

They are energy Eigen states with definite parity even for energy less than 0. There is a possibility of leakage into the classically forbidden regions and also the energy states are discretized. So, this is a good place to look at what happens if you increase e and if the energy were positive. So, let us look at the same problem, but now the energy takes positive values.

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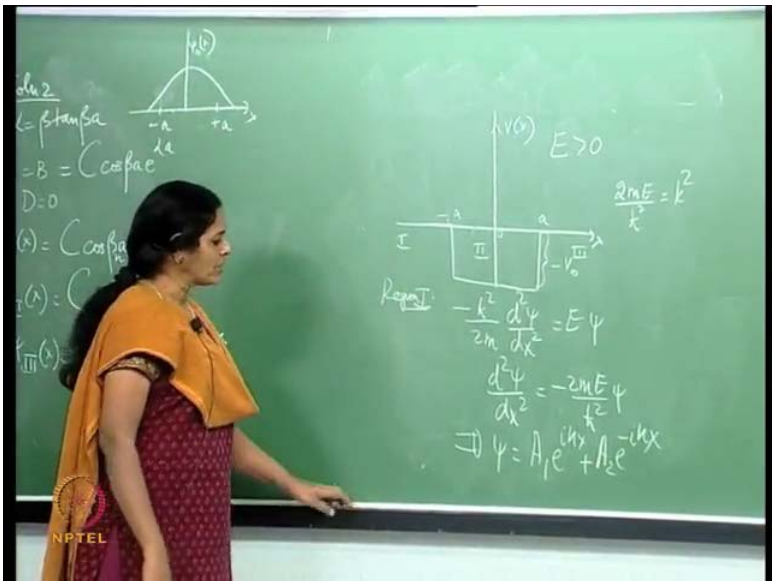

So we have the square well potential. This is V of x versus x and this is x is plus a and that is x is minus a. This is minus V naught where V naught is positive except that now we look at energies greater than 0. This is the reference level, this is the origin. So, the energy is greater than 0 and you have regions 1, 2 and 3. Even from a very classical viewpoint it is clear that the particle with mass m can move all over. It is not as if regions 1 and 3 are forbidden so, this region 1 similarly, region 3 are accessible regions to the particle simply because it has enough energy to go across. So, this is not a situation where we would be taken by surprise if quantum mechanically also. The particle behaves like as if it is free because there is enough energy to overcome the potential.

In other words, it will not be surprising if the solutions that we have for the energy Eigen states are plane wave solutions because it is as if particle is a free particle. Although there is a potential, there is enough energy to overcome the potential in region 2 and of course, in regions 1 and 3 it is genuinely free. So, let us look at the wave functions in the 3 regions. If you look at region 1, the Schrodinger equation there is no V, but there is an E and the energy is positive. Since, the energy is greater than 0 I define an object 2 m E by h cross square as a some positive value k square. So, it is greater than 0 and therefore, the solution would be psi is A 1 e to the i k x plus A 2 e to the minus i k x in region 1. So I have plane wave solutions in region 1.

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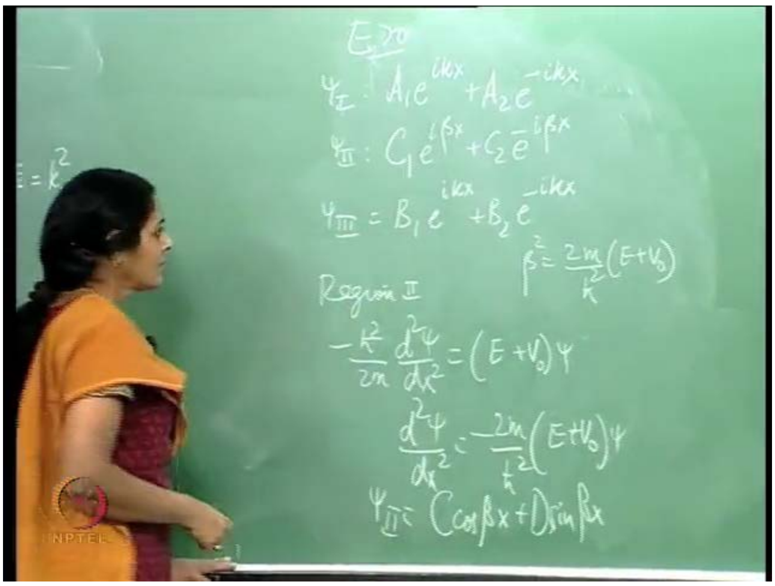

The same equation holds in region 3 and therefore, I have the following solutions in region 1, 2 and 3. So region 3 again the same type of solutions. Now, look at region 2. In region 2 we have said that the energy is greater than 0, the equation itself in region 2 would have been minus h cross square by 2 m d 2 psi by d x square plus V psi is e psi. So therefore, I had an E minus V psi out here, but V was minus V naught and I have this quantity E plus V naught psi. But this is all right because then d 2 psi by d x square is minus 2 m by h cross square E plus V naught psi and 2 m by h cross square E plus V naught is a positive quantity so, I define beta square as 2 m by h cross square E plus V naught as before and therefore, in region 2 I have my old solution C cos beta x plus D sin beta x.

Now, in order to make a comparison it would be useful to write this also as exponential of i beta x and exponential of minus i beta x. So, I can write it as C 1 e to the i beta x plus C 2 e to the minus i beta x. They are the old familiar solutions that we had earlier except that I would like to write it all in a plane wave form to show that essentially it behaves like as if it is a free particle. But then if you look at these equations if I matched the wave functions and derivatives at x equal to plus A and x is equal to minus A. I will get 2 conditions out here at x is equal to minus A and here at x equals plus A if I matched psi 2 and psi 3, I will get two more conditions. So I have 4 conditions, but I have 6 unknowns and therefore, it would not be possible to determine the coefficients in this case.

The coefficients being k and beta in other words, there is no particular relationship between k and beta. In the earlier example there was a relation that said that alpha was either beta tan beta a or minus beta cot beta a. Such a thing is not possible in this case because I do not have a sufficient number of equations to solve for so many unknowns. And since there is no such equation relating k to beta, it is clear that there is no control on the situation. In other words, since k and beta are functions of e it is obvious that any value of e is allowed. There is nothing that constraints the value of e and makes it discrete or even says these are the only possible values of e.

So, all values of e are allowed all values of e greater than 0 are fine. The energy is therefore, not discretized which means that you get a continuum of states. The word continuum is used in contrast to the word discrete. So, instead of a discrete set of energy Eigen states I have a continuous set of energy Eigen states. So, that is what happens in this problem, not very surprising it is a kind of thing I would expect even in classical physics not as stunning a result as there is a classically forbidden region and there is leakage.

On the other hand, instead of looking at this problem in detail it is worth looking at another problem which also deals with plane waves and that is the problem of a potential barrier. Because in classical physics if the energy is not enough to cross a barrier then there is nothing like tunnelling across the barrier. In quantum physics there will not only be tunnelling across the barrier there will also be leakage across the barrier and therefore, I will discuss the barrier problem now to show the possibility of tunnelling and the possibility of reflection from the barrier.

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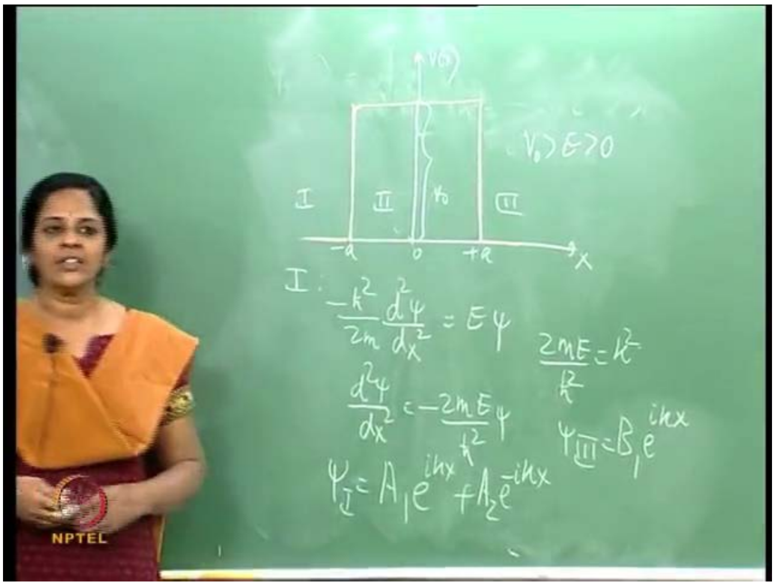

So, let us look at the problem of the potential barrier which is a kind of complementary problem to the well. So, for the potential barrier I have a one dimensional potential barrier and that is like this where this is positive and this is x is equal to minus a and that is x equals plus a. So, it is 0 outside the barrier so, I have regions 1, 2 and 3. So to show you the barrier that is 0 there and 0 here, 0 all the way and this is a square potential barrier. It is to be seen as a square potential barrier.

So, that is where we are. Now, if I write out the equations the problem is the following. I imagine that there is a wave with momentum h cross k which is incident on this barrier from the left. A part of it can be reflected I should 1st of all put in that possibility and see if it is consistent with what I have and the rest of it could be transmitted. The point is in region 3: how much of what went through actually tunnelled across? That is the question that we ask. So, let me write out the Schrodinger equation in the various regions. In region 1 minus h cross square by 2 m d 2 psi by d x square is equal to E psi. The point is this we give a certain amount of energy which is greater than 0, but the height of the barrier is more than the energy that we give.

So, V naught is greater than E and both of them are positive quantities therefore, I have d 2 psi by d x square is minus 2 m E by h cross square psi in region 1, 2 m e by h cross square is a positive quantity because E is greater than 0 I call that k square therefore, the solution for psi in region 1 is clearly going to be A 1 e to the i k x plus A 2 e to the minus i k x.

So, e to the i k x represents a wave with momentum h cross k hitting this barrier from the left, e to the minus i k x has a coefficient A 2 which should give me the probability amplitude for reflection from the barrier because that would correspond to a wave with momentum minus h cross k. So, this is in the region psi 1 of course, in the region psi 3 I will have an analogous equation. So, I will have psi 3 is B 1 e to the i k x. Normally I should also write plus B 2 e to the minus i k x, but I am not because I am not allowing for reflection out here.

There is a wave that just tunnels across if at all it is possible it tunnels across. There is nothing like reflection into the barrier and therefore, in the region 3 the problem that I envisage can only have a solution B 1 e to the i k x corresponding to a wave travelling this way with momentum h cross k travelling along the positive x axis so, the problem posed is this. There is a wave with momentum with plus h cross k in region 1 hits the barrier from this side, part of it gets reflected. The question is: How much of it can be tunnelled across into the region 3? Can you have a non zero value for B 1?

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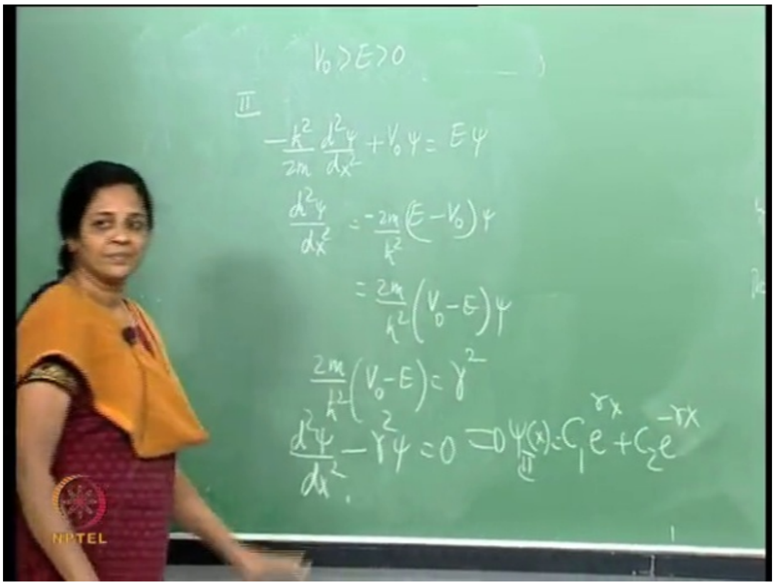

Now as far as region 2 is concerned we can always write out the equations for region 2. In region 2 this is the barrier problem, but we should remember that V naught is greater than E and E itself is greater than 0. You have minus h cross square by 2 m d 2 psi by d x square plus V psi is equal to E psi and therefore, d 2 psi by d x square is minus 2 m by h cross square E minus V naught psi which is 2 m by h cross square V naught minus E psi and this quantity is positive because V naught is greater than E. So I define 2 m by h cross square V naught minus E as a positive quantity. We have used alpha and beta so, let me use gamma square.

So, my equation is this d 2 psi by d x square so, I have minus gamma square psi equals 0. The solution psi 2 of x is clearly going to be I have used a 1 A 2 B 1 B 2 so, let me say C 1 E to the gamma x plus C 2 e to the minus gamma x. So, this is my solution. So, in a sense that the solutions are just the opposite of what we had in the case of the square well potential where you had C e to the i k x plus D e to the minus i k x within the square well and outside you had exponentially falling things whereas, here it is other way round and these are my solutions.

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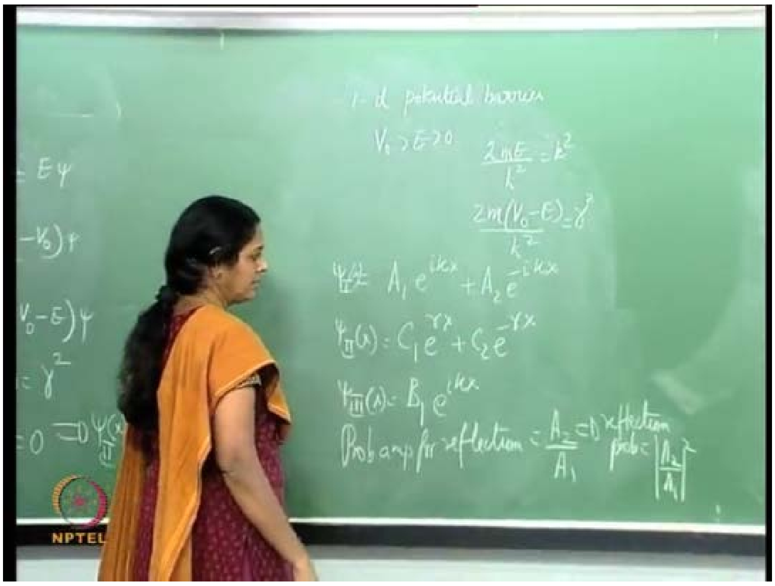

So, let me just write that down clearly in region 1. So, this is the barrier problem one dimensional potential barrier V naught is greater than E which is greater than 0, 2 m E by h cross square is k square and 2 m V naught minus E by h cross squared is gamma square and psi 1 was a 1 psi 1 is of course, is the function of x and that is A 1 e to the i k x plus A 2 e to the minus i k x. Psi 2 of x is C 1 e to the gamma x plus C 2 e to the minus gamma x and psi 3 of x given the kind of problem that we are envisaging is simply B 1 e to the i k x. I want to emphasise the following. It is not as we are considering a wave travelling in time. We are not looking at any dynamics in this problem.

(Refer Slide Time: 39:48) You should not imagine that there is a time period over which the wave travels and start wondering why is it the time does not figure in this equation. We are merely considering a wave of this form so clearly the probability amplitude for reflection would be A 2 by A 1 at this boundary, x is equal to minus A and therefore, reflection probability is modulus of A 2 by A 1 the whole square. Similarly, we can write the transmission probability.

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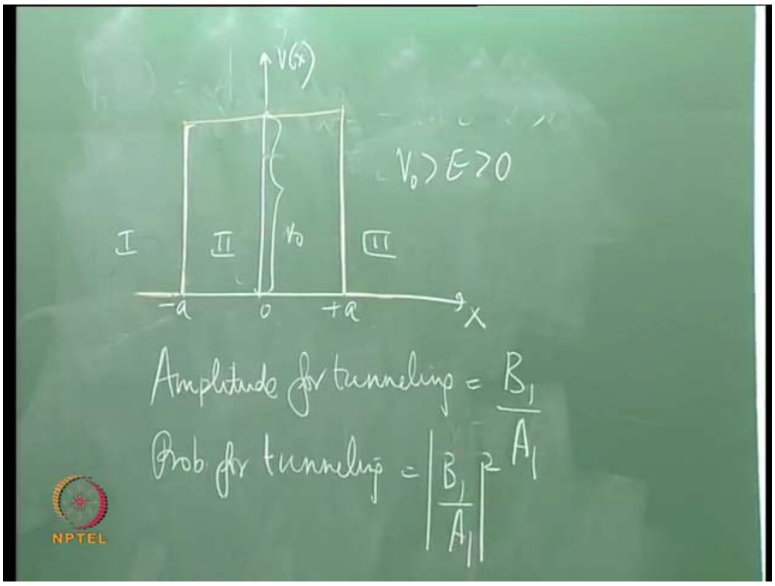

The amplitude for transmission for quantum tunnelling depends on what B 1 is given A 1. So, the amplitude for tunnelling is B 1 by A 1 and therefore, the probability for tunnelling is modulus of B 1 by A 1 the whole square. Question is: how do you solve for these? (Refer Slide Time: 46:16) There is a very systematic way of doing this 1st of all match the wave function and derivatives at x equals plus a and that should give me the ratio C 2 by C 1 that is the 1st step. So there are a series of steps involved which is just a lot of algebra. So I would request you to work out these steps and I will put on the results for you so that you can see them and try to get each of them. I will explain to you how exactly these results are got and then I will discuss the result with you in brief.

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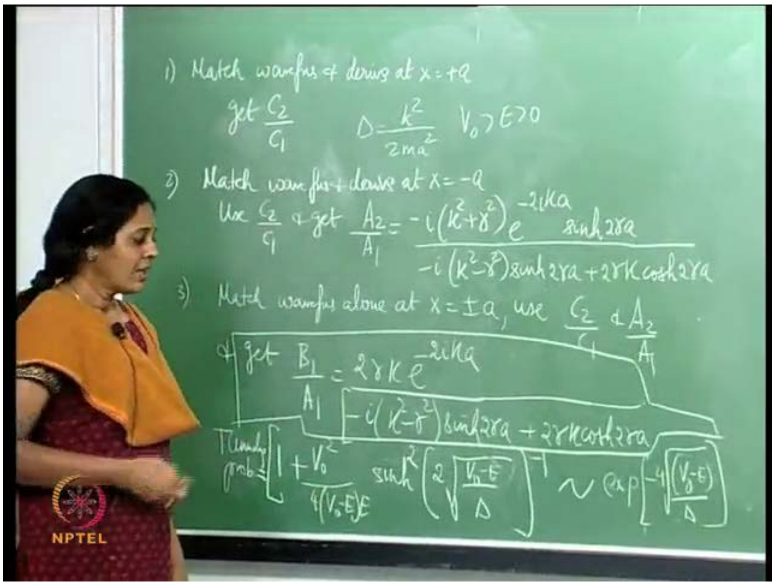

So treat this as a set of exercises 1st of all match the wave function and derivatives at x equals plus a well that would be like matching psi 2 with psi 3.(Refer Slide Time: 46:16) So, that will involve B 1, C 1 and C 2. Since the derivative would involve B 1 and the wave function would also involve B 1 you can divide the derivative matching equation with the wave function matching equation eliminate B 1 and get the ratio C 2 by C 1. And now once that is done you match the wave function and the derivatives and the other boundary x equals minus A.

Now, that will involve C 2 C 1 A 2 and A 1 feeding the values of C 2 by C 1 from here and get this expression for A 2 by A 1. Our aim is to find transmission probability and the reflection probability so, the aim is to find B 1 by A 1. Remember that B 1 is the coefficient of the wave that tunnelled across to region 3 and A 1 was the incoming wave e to the i k x A 1 e to the i k x was the incoming wave and therefore, if you want to find B 1 by A 1 you match the wave functions again at x equals plus minus a using these as inputs: C 2 by C 1 and A 2 by A 1. That would give you this expression for B 1 by A 1 this expression here.

The B 1 by A 1 mod square will give you the transmission probability which I have called t so, this is the tunnelling probability. That quantity can be written in this form you write everything in terms of V naught and V naught minus E because once more it is a question of what is the deficit energy? V naught minus E and how well can the tunnelling happen depending upon V naught minus E. Look at this expression I wish to comment on this expression before I conclude. This is sin hyperbolic square of an object twice root of V naught minus E by delta where delta itself is h cross square by 2 m a square. That was the reference in terms of which we could define the strength of the potential.

Now, this quantity can be written in terms of sin hyperbolic. This argument can be written in terms of e to the y and e to the minus y. Now, as long as the argument y is not far less than 1 this can be well approximated by the exponential and therefore, the square gives me e to the minus 4 root of V naught minus E by delta. So, I will use e to the y and not e to the minus y so, that gives me e to the 2 y where y is this argument, but there is a an inverse here and therefore, it becomes e to the minus 2 y which was e to the minus 4 root of V naught minus E by delta.

So, this is revealing because it tells me that V naught minus E. Remember that V naught is greater than E which is greater than 0 so, depending upon the energy V naught minus E there is going to be an exponential fall off of the tunnelling. So, this quantity has to be scaled by delta because it tells me what the deficit energy is in terms of delta and there is an exponential fall depending on that.

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However, there is a tunnelling, there is a non zero tunnelling probability which is not just quite unlike in start contrast to what you would expect in classical physics where you have an energy in this situation which is less than the potential V naught. So, the crucial point is this for an energy which is less than the potential barrier we do not expect in classical physics the object to tunnel across to region 3 if it came from region 1. In quantum physics on the other hand, there is tunnelling. To summarise therefore, both in the problem of square well potential and in the problem of the potential barrier these are complementary problems.

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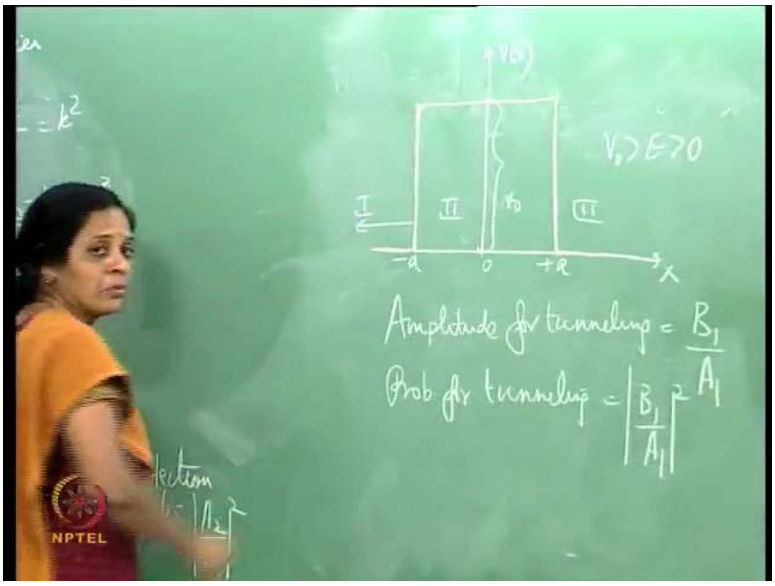

We realised that there were certain stunning aspects of quantum physics leakage into classically forbidden regions of course, discretized energy values for the square well potential when the energy was less than 0 and a non zero tunnelling probability as also a non zero reflection probability, because you will notice that A 2 by A 1, the probability amplitude for reflection is not 0.

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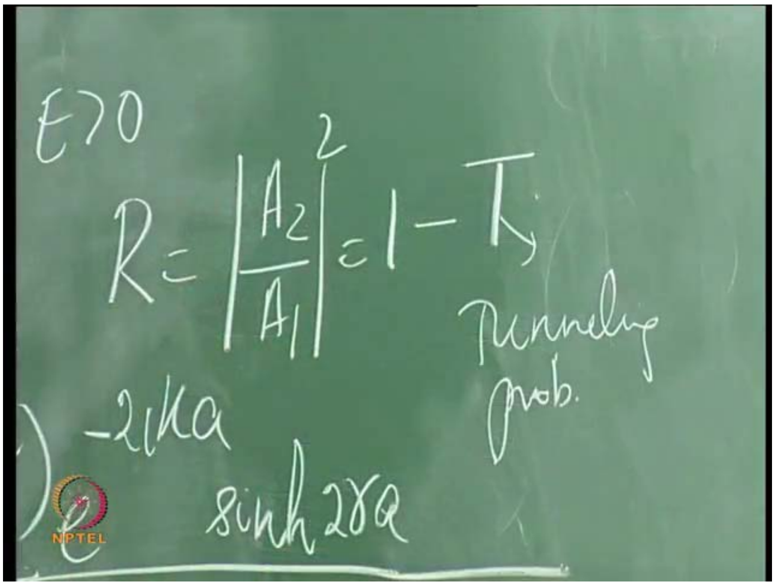

This is the expression for A 2 by A 1 there is a non zero reflection probability, there is a non zero tunnelling probability and you will be able to show that the probability of reflection which is modulus of A 2 by A 1 the whole square is 1 minus the tunnelling probability. Naturally because you are not allowing for stickiness in the barrier or anything like absorption in the barrier therefore, it either reflects or it tunnels. The total probability being 1, the reflection probability plus the tunnelling probability must be equal to 1. So, these are the startling features that emerge in quantum physics quite a contrast to a classical problem of the same kind.