Quantum Mechanics - I
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Lecture - 16
The Quantum Beam Splitter

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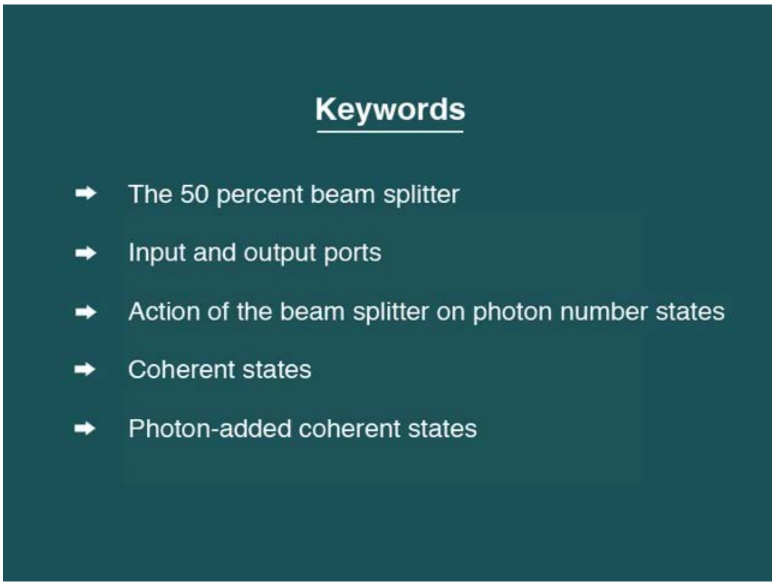

In an earlier lecture, I had considered a composite system, made of two oscillators. We wrote down the creation and destruction operators for one of the oscillators as a dagger and a respectively. And then for the other oscillator we had creation and destruction operators, photon creation and destruction operators, if you spoke about it in the language of photons and ladder operators if you spoke about it in the language of simple harmonic oscillators. We call them b dagger and b and they satisfied commutation relations. Commutator a a dagger is identity. Commutator b b dagger is identity. All other commutators vanish. Now that was a composite system. It had two oscillators and we used these operators to provide the angular momentum algebra.

So, we had j plus the angular momentum raising operator, which takes the 3rd component of angular momentum, increases it in steps of 1 at a time and that was given by a dagger b. Its Hermitian conjugate b dagger a did the job of lowering the 3rd component of angular momentum by 1 and then there was j z the Hermitian operator which was a dagger a minus b dagger b, apart from a constant. So, now I want to consider another physical situation, where I would use two sets of operators a a dagger and b b dagger. This is the quantum beam splitter.

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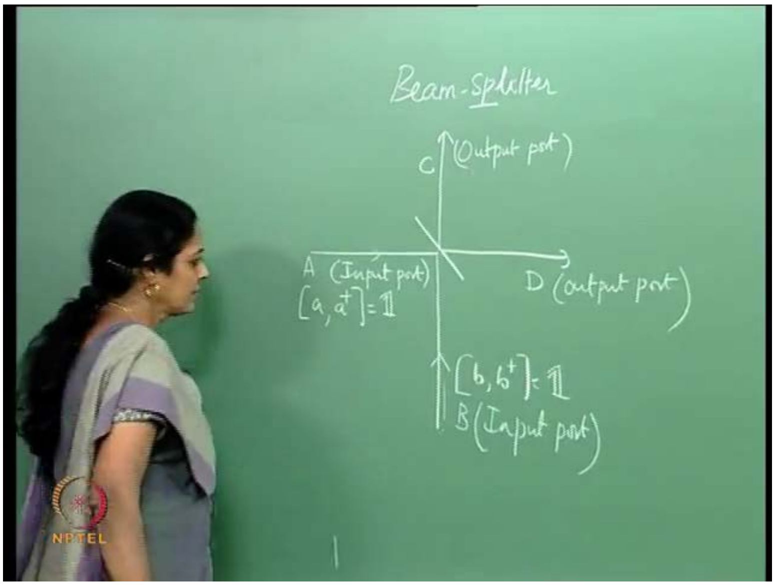

So, I have this beam splitter and it is quantum. In the sense, I am going to talk of a device which is a partially silvered mirror, on which photons can strike either through this direction or from this direction. Any set of photons that strike on the beam splitter, will be partly reflected and partly transmitted. Similarly, photons that come from here would be partly reflected and partly transmitted. Since the photons can come in either in this direction or in this direction and strike the beam splitter. A and B are called input ports. And, then for the sake of nomenclature I would like to call this C and I would like to call this D and these are output ports. So, there are two input ports and there are two output ports and photons that come either in this direction, or in that direction get partially transmitted partially reflected. So, that is the picture that we have of the beam splitter.

The photon creation and destruction operators, for photons that travel in this direction are given by a dagger and a respectively and they satisfy this commutator algebra. Similarly, we have b and b dagger here. The beam splitter itself does the operation of partly transmitting and partly reflecting the photons.

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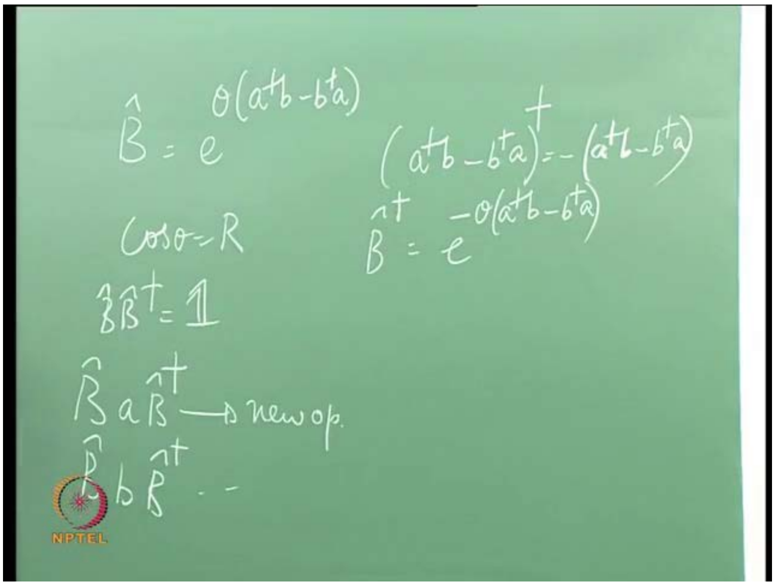

The operator is represented by B. This is the beam splitter operator and B is a unitary operator given in this fashion. Normally, we associate cos theta with the extent of reflection which I call R. So, the extent of partial reflection R is simply denoted by cos theta.

Notice that a dagger b minus b dagger a is anti Hermitian, is minus a dagger b minus b dagger a. And therefore, B dagger is e to the minus theta, a dagger b minus b dagger a. This clearly means that B B dagger is identity and it is a unitary operator. What is the action of B on a a dagger b and b dagger? Now, this we can find out easily because we know that unitary operators act in the following manner, taking it to a new operator. So, we can compute B a B dagger, B b B dagger and so on. So, we first do that. I want to emphasize the fact (Refer Slide Time: 01:59) that this theta has nothing to do with the geometry of the beam splitter in general. It has to do with the extent of partial reflection.

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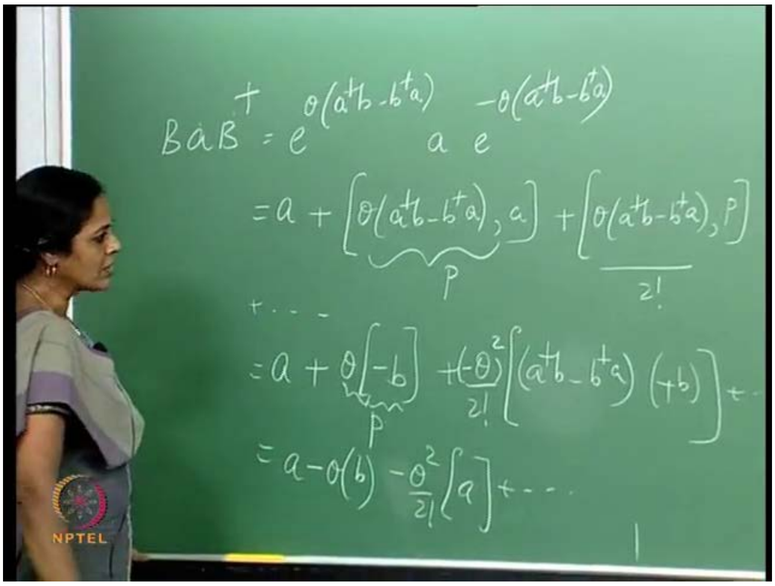

So now, I wish to find out B a B dagger. I deliberately avoided putting this over head caps on top of B a and B dagger. So, we need to compute this. Of course, I will use the Baker Campbell Hausdorff formula. And, this would just be a plus commutator of theta a dagger b minus b dagger a with a. And suppose I call this commutator p, plus theta a dagger b minus b dagger a with p by 2 factorial plus so on.

So, let us first write this down as a plus theta, the 1st term is commutator of a dagger b with a, which gives me a minus b. The 2nd term vanishes because a commutes with itself. So, this is p. This whole object is p. So, I need to find plus theta by 2 factorial minus theta b with a dagger b minus b dagger a, with minus theta b. So, let me write minus b out here and so on. So, this is a minus theta times b plus the minus theta square by 2 factorial. The 1st term does not give me anything. I have put minus theta squared there. The 2nd term will give me a 1 and therefore, that multiplies a and so on.

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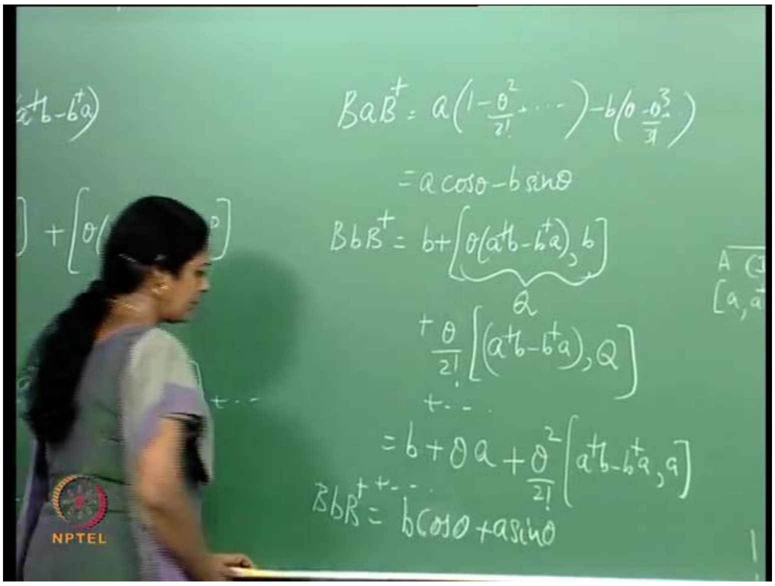

Now, we can write out all the terms and it is easy to check, that B a B dagger, is simply a 1 minus theta square by 2 factorial plus theta to the power of 4 by 4 factorial plus so on. Minus b theta minus theta cube by 3 factorial plus the rest of the sign seems. So, this is a cos theta minus b sin theta. Similarly, we can compute B b B dagger and that object is just b plus theta a dagger b minus b dagger a commutator with b. And, if I call this Q, the next term is plus theta by 2 factorial theta square by 2 factorial commutator of a dagger b minus b dagger a with Q and so on.

So, this object is just b. I told you the 2nd term that contributes and that gives me a plus sign, plus theta a this gives me a theta squared by 2 factorial, commutator of a dagger b minus b dagger a with a. It is easy if we do this series expansion. To see that B b B dagger is b sin theta, b cos theta plus a sin theta. So, this is what we have for the action of the beam splitter. So, let me put that down.

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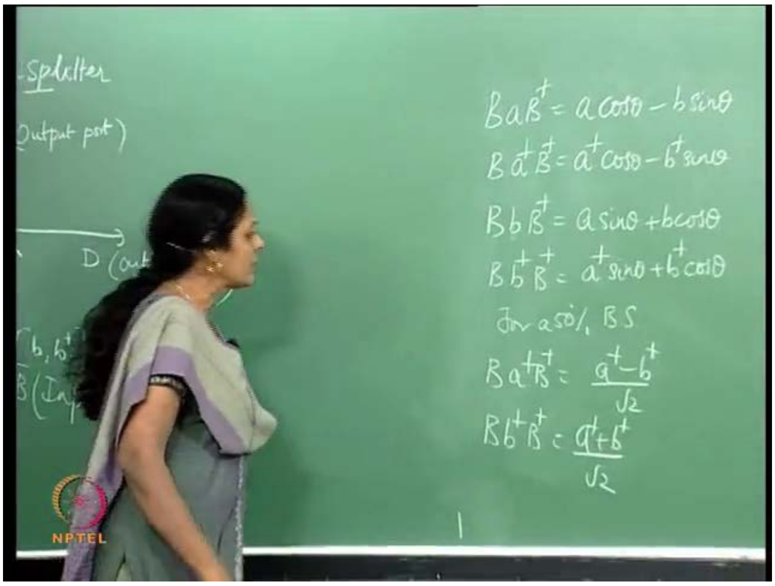

I have B a B dagger is a cos theta minus b sin theta. This is the effect of the beam splitter on a and therefore, B a dagger B dagger, is a dagger cos theta minus b dagger sin theta. I can take the Hermitian conjugate, or I can work it out again once more and this is what I would get.

Similarly, this object is a sin theta plus b cos theta and B b dagger B dagger is a dagger sin theta plus b dagger cos theta. So, this is what we have. Now, a 50 percent beam splitter is one which sense 50 percent of, transmits 50 percent of the intensity of the light. Of course, if it comes from this port (Refer Slide Time: 01:59) it transmits 50 percent this way and reflects 50 percent that way. And therefore, for a 50 percent beam splitter cos theta is 1 by root 2 and so sin theta. I just have the following, a dagger minus b dagger by root 2. And similarly, this object is a dagger plus b dagger by root 2. So, these are basic preliminaries that I need, in order to see what happens to various photon number states, (Refer Slide Time: 01:59) which are put through these input ports.

As you can see this is a unitary transformation that is being done on a’s and b’s, a’s and b’s. The abstract of the whole thing is the following.

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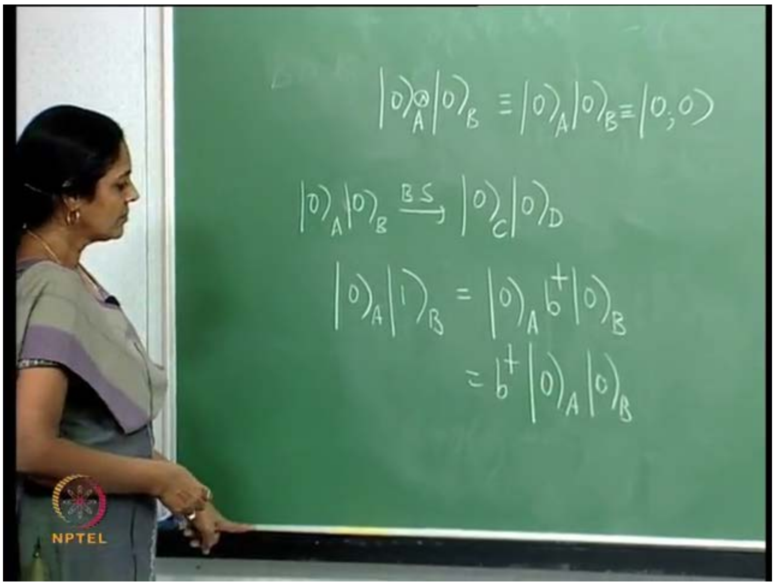

Suppose, in terms of states, I sent 0 photons through the input port A and 0 photons through the input port B. My initial state is simple 0 A direct product 0 B and you will recall that in my notation, that is the same as 0 A 0 B and it would also be the same as 0 0. So, these are just various short hand notations that I would use. But, in any case if the input is 0, 0 photons through the port A and 0 photons through the port B. You do not expect to see photons in the output ports.

So, you would expect 0 photons in port C and 0 photons in port D. So basically, 0 A 0 B goes through the beam splitter, to give me 0 C 0 D and that is common sense. Now I will use this fact, suppose I have 0 photons through the port A and 1 photon through the port B, I can well write this as 0 A b dagger 0 B. But that is the same as b dagger 0 A 0 B. Remember that this is a product state here and b dagger acts only on ket 0 B and leaves 0 A untouched. This is passed through the beam splitter. Now, I can look at this situation, from the framework of the output ports C and D. In which case, I need to write b dagger in terms of c dagger and d dagger.

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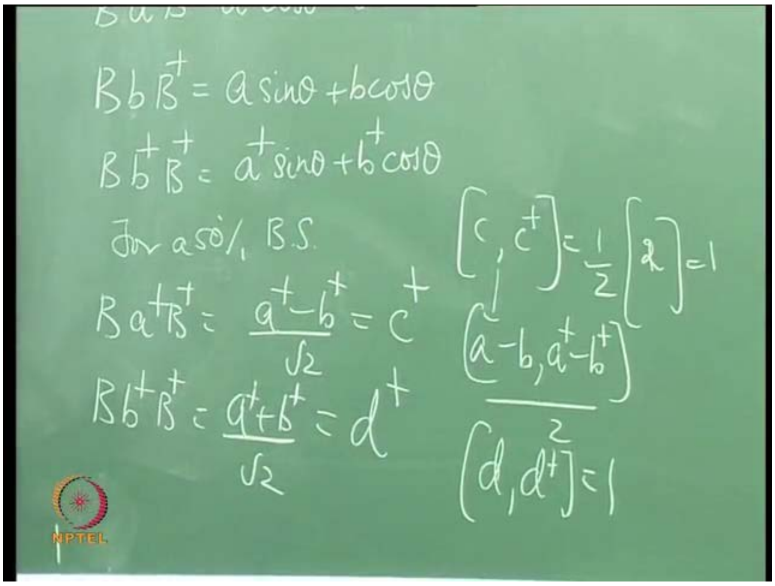

And what are c dagger and d dagger, b a dagger b dagger is a dagger minus b dagger by root 2, I am going to call that c dagger and b b dagger b dagger is a dagger plus b dagger by root 2 and I call that d dagger. Now, it is important to check if indeed we have done a unitary transformation. In the sense, is the algebra preserved. So, I need to find, I need to check the commutator of c with c dagger, c with c dagger is simply half, a with a dagger which is 1. And, then I have b with b dagger and that object is simply going to give me a commutator 1.

So, I have a minus b with a dagger minus b dagger by 2 and that is what I have written there as 1. Similarly, d with d dagger in the commutator is 1. And therefore, I know that I have gone from the algebra a a dagger is 1, b b dagger is 1 in the commutator, commutator algebra, to c c dagger commutator is 1 d d dagger commutator is 1. So, I can well look at this state, (Refer Slide Time: 13:02) this state ket 0 A 1 B which I have written in this manner. From the point of view of the output ports, which means, I write b dagger in terms of c dagger and d dagger. Of course, ket 0 a ket 0 b is identified with ket 0 c ket 0 d. That is what happens if you send nothing through the beam splitter.

I can solve for b dagger and a dagger from these equations and then I can put in the expression for b dagger there.

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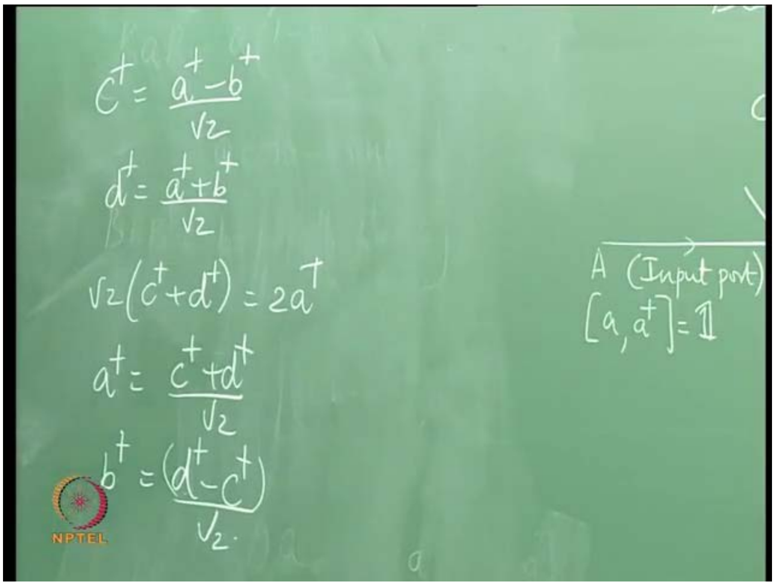

So, I have c dagger is a dagger minus b dagger by root 2 and d dagger is a dagger plus b dagger by root 2. So, I can substitute for b dagger and get my answer here.

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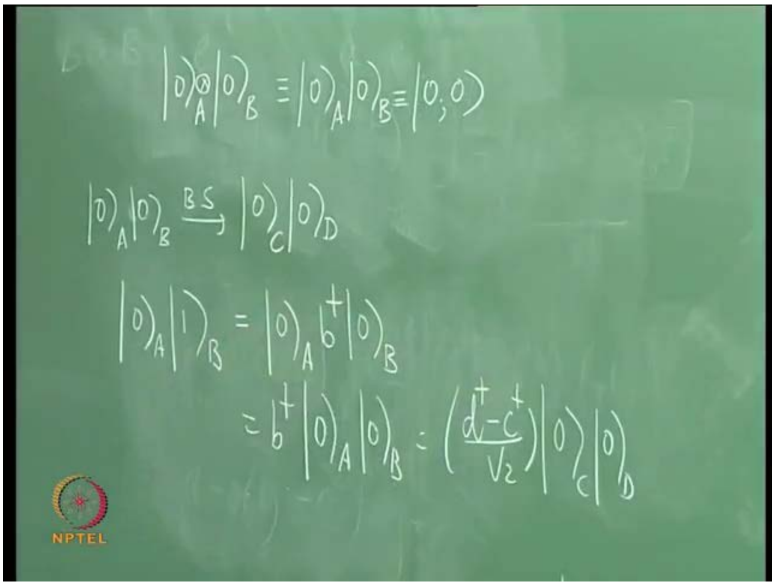

And therefore, I have this object, to be given by d dagger minus c dagger by root 2 0 C 0 D. But I know what this does.

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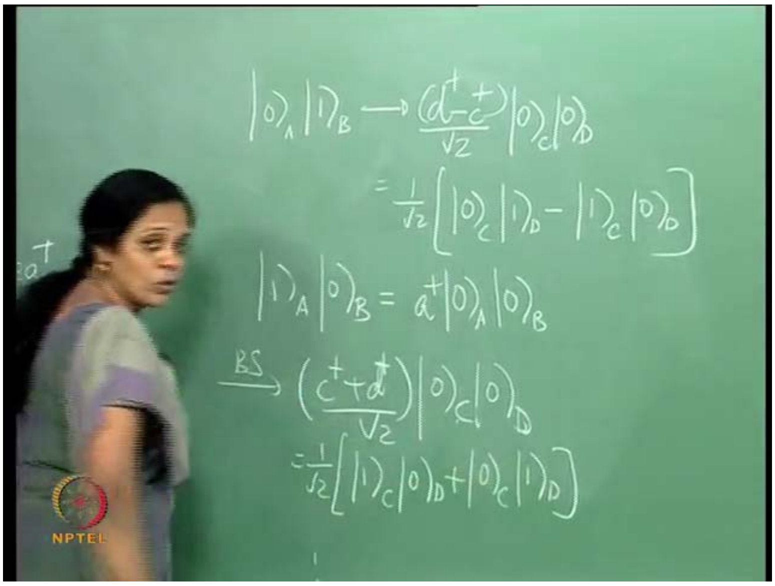

I really have 0 A 1 B. Thus same as d dagger, that was my input state, this goes to d dagger minus c dagger by root 2 0 C 0 D. But that is 1 by root 2. The 1st term is d dagger acting on 0 D giving me 1 D leaving 0 C alone. Then the 2nd term is c dagger acting on 0 C leaving 0 D alone and this is what I get. Now, this is a very interesting and important result because it tells me that if I had sent in 0 photons through the input port A and 1 photon through the input port B. I might expect to see the single photon that I sent in through the input port B, either coming out through port C, or through port D.

But, this tells me that there is a 50 percent probability, that it comes out through port D and 50 percent probability that it comes out through port C. This is a very interesting result. So, it looks like the probability is really divided between port C and port D. I could see the single photon here 50 percent of the time, or in port C 50 percent of the time. So, this is an interesting result and it has come about because of superposition. Now, let us look at a situation where we have 0 A, or 1 A 0 B. I could repeat this argument. 1 A 0 B is the same as a dagger on 0 A leaving 0 B alone.

And, writing a dagger in terms of c dagger and d dagger. That is the same as (Refer Slide Time: 17:12) a dagger minus b dagger by root 2 is c dagger and therefore, a dagger is c dagger plus d dagger, by root 2 acting on 0 A 0 B. So, this is what happens, when it goes to the beam splitter and this is the same as 1 by root 2. But, 0 A 0 B is the same as 0 C 0 D and therefore, this gives me 1 C 0 D plus 0 C 1 D. Once more I have verified that if I had sent a single photon through the input port A and nothing through the input port B. There is a 50 percent probability of seeing the single photon either through the output port C or the output port D.

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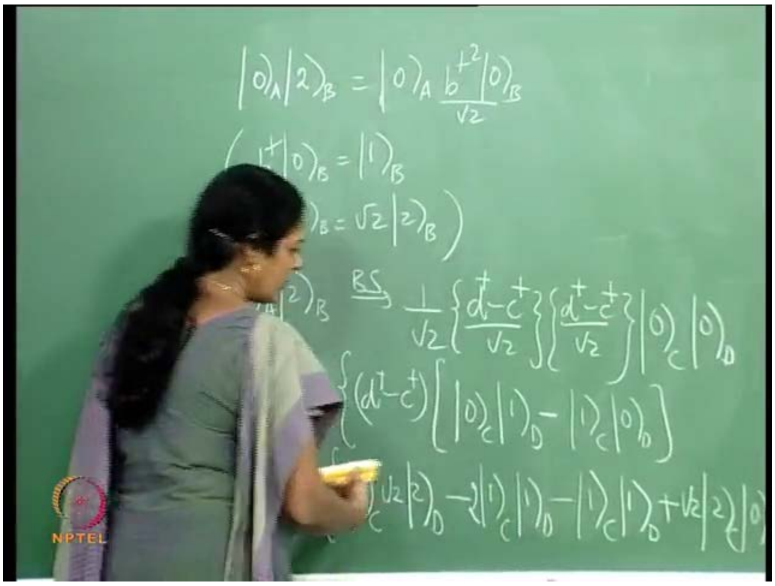

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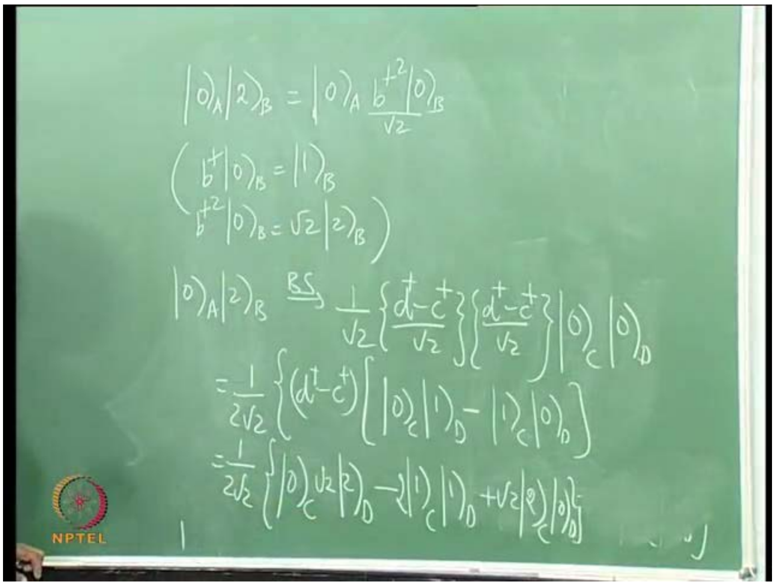

Now, let us look at a slightly more complicated situation. By way of exercise, let us consider an input state 0 A 2 B. I would write this, as 0 A, when b dagger acts on 0 B it gives me 1 B and when b dagger square acts on 0 B, it gives me root 2 2 B. And therefore, I will write 0 A 2 B as ket 0 A 2 B apart from a root 2. So, this is what I have.

Now, this object can well be written as 0 A 2 B after beam splitting, goes to 1 by root 2. I will have to write b dagger, b dagger squared in terms of d dagger and c dagger and therefore, I have d dagger minus c dagger by root 2 twice d dagger minus c dagger by root 2.

So, this multiplies that, 0 C 0 D because I will use the fact that 0 A 0 B is the same as 0 C 0 D. After beam splitting I get nothing. So, this is 1 by 2 root 2, d dagger minus c dagger. The first time this d dagger minus c dagger acts on ket 0 C ket 0 D, I get 0 C 1 D minus 1 C 0 D. Now, once more it does the job that is 1 by 2 root 2. The 1st term gives me 0 C d dagger on 1 D gives me root 2 2 d, c dagger on 0 C, gives me 1 C and leaves the 1 D alone. When d dagger acts on this term it gives me minus 1 C 1 D. And of course, there is a c dagger which acts on this, which gives me root 2 2 C 0 D. Of course, I can add these two terms and put minus sign out there and this is what I have.

So, what does that tell me? It tells me that, if I had sent 2 photons through the input state, there is a non 0 probability of finding 2 photons in the output port D, or 2 photons in the output port C, or 1 photon in each of the output ports. The photon number is conserved. It needs to be conserved in any case. We should also check that the total probability is 1 and that happens really, because the 1st term comes with the coefficient half, that is the probability amplitude, that gives me a probability quarter for 0 C 2 D. Similarly, a probability quarter for 2 C 0 D and a probability half for 1 C 1 D. And therefore, the total probability of finding the photons in the output port happens to be 1, which is the way it thought to be. So, this is what I have if I sent 2 photons in, is easy to generalize this to n photons through an input port.

This also tells us about the power of the vacuum state because despite the fact that nothing was sent in through port A. You find that the output ports have a certain distribution of photons. There is a non-zero probability of seeing the photons in both C and in D and that is because of the power of the vacuum, the ket 0 a, a very interesting and important result now. Where we simply look at a single photon going through each port.

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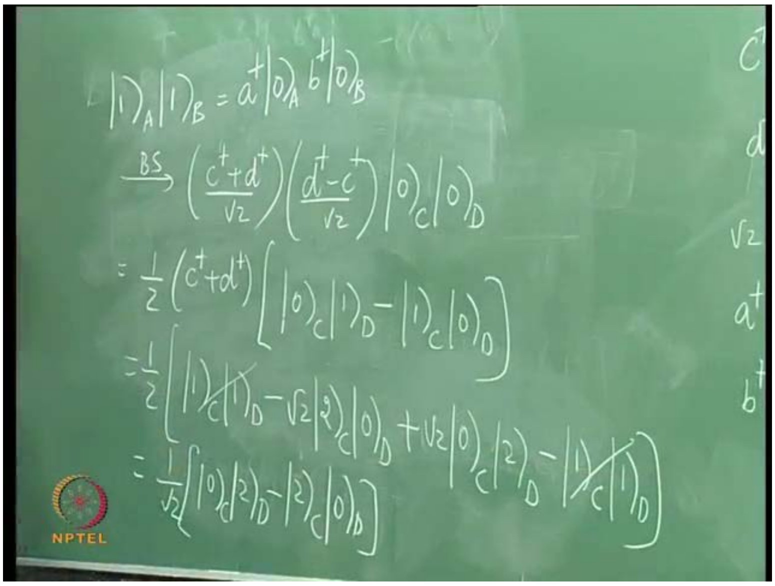

So, let us look at this state, 1 A 1 B. Now, let us before I write this as, a dagger 0 A b dagger 0 B. So, after passing through the beam splitter I substitute for a dagger and b dagger in terms of c dagger and d dagger. So, a dagger is simply (Refer Slide Time: 17:12) c dagger plus d dagger by root 2 and b dagger is d dagger minus c dagger by root 2, and this acts on 0 C 0 D. And that is a half c dagger plus d dagger. The action of d dagger on 0 C 0 D is 0 C 1 D and the action of c dagger is 1 C 0 D.

Now, once more the action of c dagger on the 1st term is 1 C 1 D and on the 2nd term is minus root 2 2 C 0 D. That is very nice, certainly I am conserving the total number of photons, that is a 1 plus 1 and this is a 2 plus 0. But, now look at this, d dagger on this gives me a root 2 0 C 2 D, which is fine and then the last term is minus 1 C 1 D. There is a cancellation between, these two terms and I just have, my final answer as 1 by root 2 0 C 2 D minus 2 C 0 D.

There is 0 probability of seeing a single photon in the output port C and a single photon in the output port D. And, this is an important statement because at the phase of it, we might imagine that given 2 photons, as in the earlier example. I should be able to get a term which says ket 1 c ket 1 d in my final state. But, such a thing is missing and this is what I have. Both the photons are seen either in the output port D, or in the output port C. Now, let us consider another situation, where I send a coherent state, through one of the ports and the vacuum in the other.

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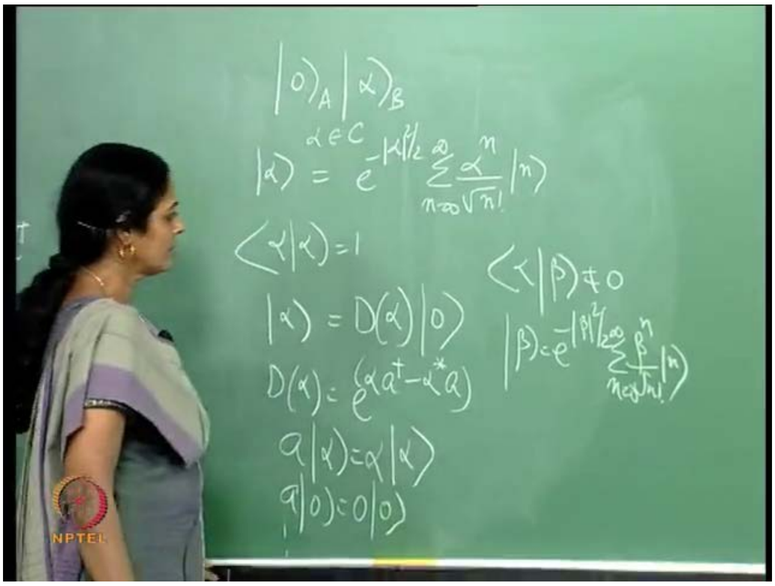

So, my input state 0 A alpha B. Just by way of recapitulation, alpha is any complex number in the Fock state representation that means, in terms of the photon number states. This is written as follows and therefore, it is an interesting quantum superposition of the 0 photon state the 1 photon state, the 2 photon state and so on. This is a normalized state. You can get it from the vacuum, by the action of the unitary operator D of alpha, where D of alpha itself is e to the alpha a dagger minus alpha star a.

Further, alpha is an Eigen state of the annihilation operator a with Eigen value alpha. Recall that the 0 photon state is also an Eigen state of a, but with an Eigen value 0. That is a non trivial state, Eigen state of a. Further, alpha beta is not equal to 0, where beta is another coherent state. So, this is what we had shown earlier. Now, let us consider this input state, d f alpha itself is a unitary operator, satisfying d dagger of alpha, is d of minus alpha.

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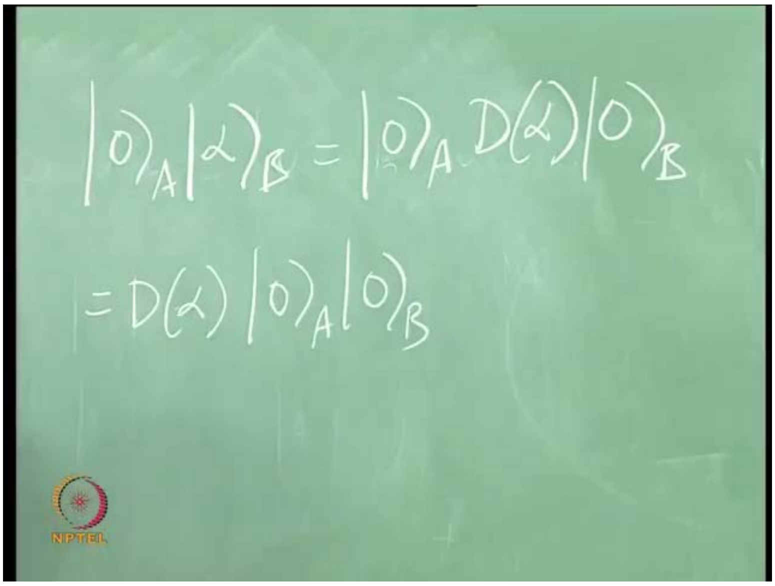

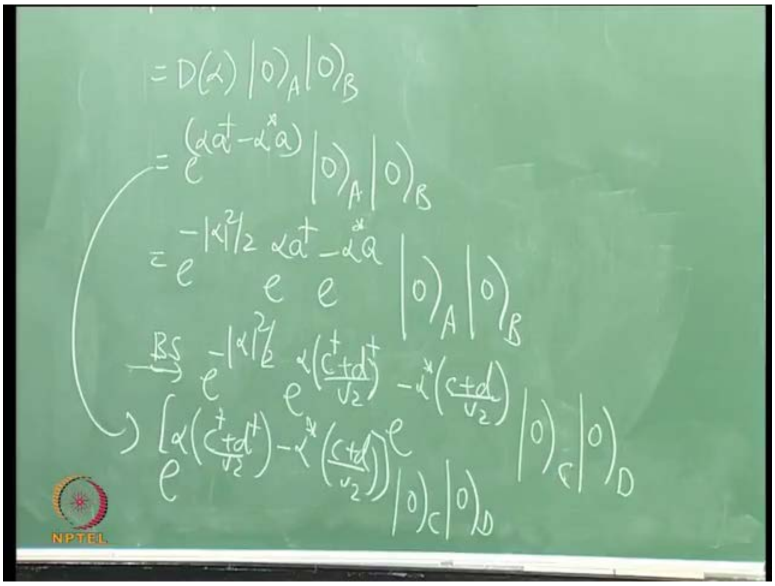

So now, I have an input state 0 A alpha B and this is passed through the beam splitter. I can write this 0 A D of alpha acting on 0 B; by 0 B I mean the vacuum state through the input port B. So, that is the same is D of alpha 0 A 0 B, ket 0 A ket 0 B. But, this is e to the alpha a dagger minus alpha star a, ket 0 A ket 0 B. I know what happens to a dagger and a and how they appear if I look at it from the framework of the output ports. I know that (Refer Slide Time: 17:12) a dagger is c dagger plus d dagger by root 2 and a is c plus d by root 2.

Now incidentally, this is just using the disentanglement formula. This is e to the minus mod alpha square by 2, e to the alpha a dagger, e to the minus alpha star a, acting on 0 A

0 B. So, this basically, once it goes to the beam splitter, it is e to the minus mod alpha square by 2, e to the alpha, (Refer Slide Time: 17:12) a dagger is c dagger plus d dagger by root 2, minus alpha star e to the minus alpha star a and this acts on 0 A 0 B but that is the same as 0 C 0 D.

Now, I am looking at the whole thing and the whole system from the point of view of the output ports. That is in terms of operator c d, c dagger d dagger and the basis state there, that means ket 0 C ket 0 D and so on. So, this is what I have. Now of course, I need not even have disentangled this. I could have just written this in toe toe, as e to the alpha c dagger plus d dagger by root 2, minus alpha star c plus d by root 2, without using the disentanglement theorem, acting on 0 C 0 D. And then I can group terms in the exponent which is what I will do now.

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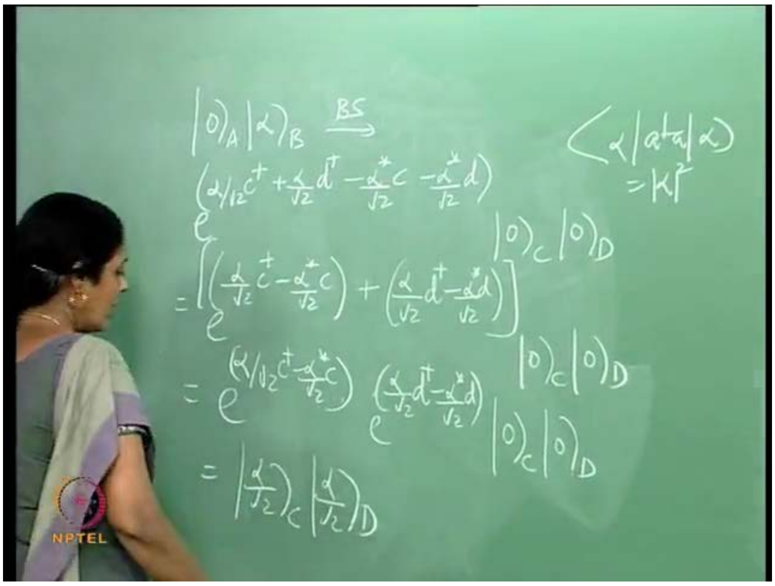

So, 0 A alpha B passes through the beam splitter and looks like this, e to the alpha by root 2, c dagger plus alpha by root 2, d dagger minus alpha star by root 2 c, minus alpha star by root 2 d 0 C 0 D. And, I can combine terms in the exponent and write this as e to the alpha by root 2, c dagger minus alpha star by root 2 c, that is the 1st term, plus alpha by root 2 d dagger, minus alpha star by root 2 d that is the rest of the terms in the exponential, acting on 0 C 0 D.

This is an interesting structure, because this is like D of alpha by root 2 and that is another D of alpha by root 2, except that this corresponds to the operator c dagger and that to the operator d dagger. It is also true that these two objects commute with each other because c and d commute with each other. And therefore, I can now use the disentanglement theorem and I just can write this as e to the alpha by root 2 c dagger minus alpha star by root 2 c, e to the alpha by root 2 d dagger minus alpha star by root 2 d, acting on 0 C 0 D. Well, that is just going to give me, when this object acts on 0 D and this object acts on 0 C. That just gives me alpha by root 2 C alpha by root 2 D.

In other words, I started with a coherent state, going through one input port and ket 0 in the other. The output ports have two coherent states. But, instead of the original alpha, it is alpha by root 2, sitting in each of these kets, inside each of these kets. And therefore, the intensity of the coherent state has gone down. Remember that the expectation value of the photon number operator in the state alpha is mod alpha square. And therefore, now the intensity has gone down. Instead of alpha I have alpha by root 2, and therefore in this state the expectation value of a dagger a has come down. It is half the intensity here and half the intensity there. By intensity I mean the number of photons, the expectation value the mean photon number. So, this is what I have.

So, the power of the vacuum is again well demonstrated, because it helps you to produce 2 clones of the original coherent state. Initially, I had one coherent state but now I have two of them, each one in one of the output ports. With the intensity down, but still, the coherence properties are preserved. So, that is one of the uses that I have, when I sent ket 0 through one of the input ports of the beam splitter and ket alpha through the other input port of the beam splitter.

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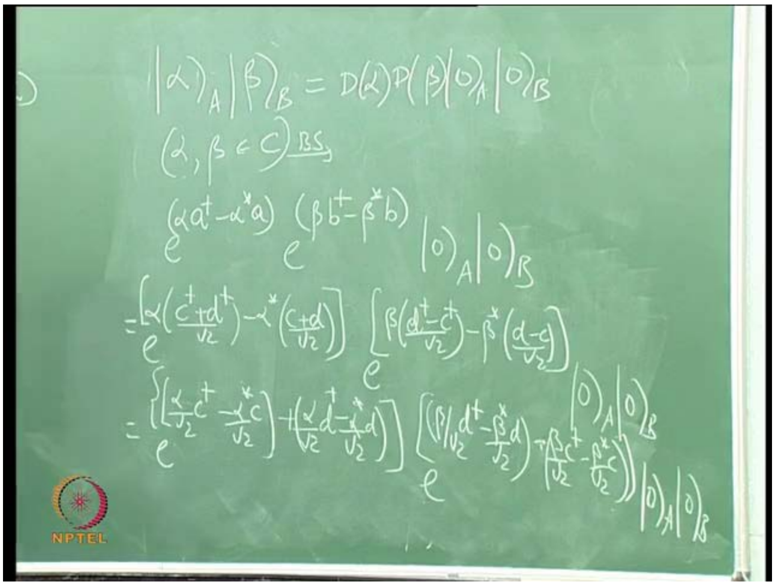

Finally, let us look at the following situation, alpha A beta B, where alpha beta are complex numbers. ket alpha is a coherent state ket beta is another coherent state. So, I have sent in two coherent states, one through each of the input ports. And so what do I have by way or output? So, this can be written as D of alpha D of beta 0 A 0 B. So, this is pass through the beam splitter. By way of output I have the following.

This object is e to the alpha a dagger minus alpha star a, e to the beta b dagger minus beta star b, 0 A 0 B. Once more, I can write this as e to the alpha a dagger c dagger plus d dagger by root 2 minus alpha star c plus d by root 2 that is my 1st term. Then, I have e to the beta b, which is beta d dagger minus c dagger by root 2, minus beta star d minus c by root 2, because (Refer Slide Time: 17:12) b dagger was d dagger minus c dagger by root 2. I have just put that in, ket 0 A ket 0 B.

Now, it is clear that these two objects do not commute with each other. But, I could use the following. I could use the fact; I can now group terms for instance. This is e to the alpha by root 2 c dagger, minus alpha star by root 2 c plus alpha by root 2 d dagger minus alpha star by root 2 d. That is what I have in the 1st exponent. The 2nd exponent has e to the beta by root 2 d dagger minus beta star by root 2 d, plus minus beta by root 2 c dagger, minus beta star by root 2 c and this acts on 0 A 0 B.

I can use the disentanglement theorem; I know for instance, that this commutes with this. Because, this is just a function of c and c dagger that is a function of d and d dagger and they commute with each other. Similarly, this term commutes with that term. Now, if I did that, I would have the following. Out here, n fact, once I do the disentanglement and I combine things I should be able to get alpha minus beta by root 2 c dagger, minus alpha star minus beta star by root 2 c that will be one term. Then I have alpha plus beta by root 2 d dagger minus alpha star plus beta star by root 2 d that would be the other term. So, already I can see where it is going. But, I am going to pick up some phase factors.

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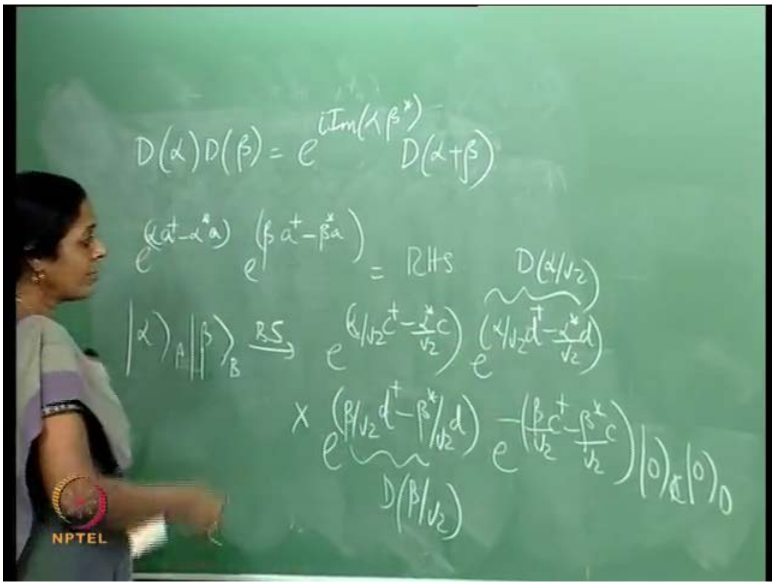
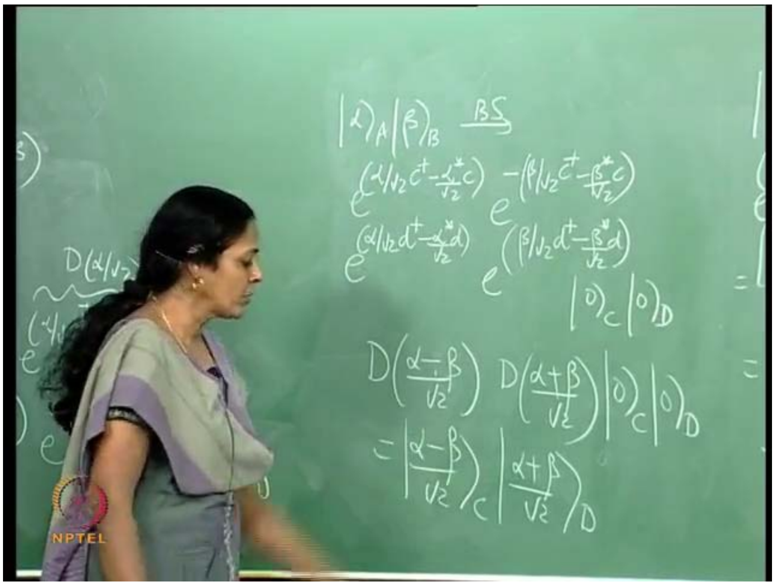


I use the fact, that D of alpha, D of beta is e to the i imaginary part of alpha beta star, d of alpha plus beta. So, this is a crucial input because this is e to the alpha a dagger minus alpha star a, e to the beta a dagger minus beta star a and then that quantity can be written in terms of the right hand side suitably. So, out here, 1st of all I use the disentanglement theorem. And this can be written, (Refer Slide Time: 38:07) as I initially started with alpha A beta B, went through the beam splitter, to give me, times e to the alpha by root 2 d dagger minus alpha star by root 2 d. (Refer Slide Time: 38:07) These two commute with each other. So, I can well split the exponent like this.

Similarly, there I have times e to the beta by root 2, d dagger minus beta star by root 2 d, times e to the minus beta by root 2 c dagger minus beta star by root 2 c. I can well write it in this fashion, acting on 0 C 0 D. We can now group term suitably, for instance these two quantities can be combined. I have an alpha by root 2 d dagger minus alpha star by root 2 d, beta by root 2 d dagger minus beta star by root 2 d. And therefore, using this trick, I can combine them. This object is D of alpha by root 2 and this is D of beta by root 2. Similarly, I can combine these 2 terms. This commutes through this, through that, out here. And therefore, I write the following.

(Refer Slide Time: 45:07)



Alpha A beta B goes through the beam splitter, to give me e to the alpha by root 2 c dagger, minus alpha star by root 2 c, e to the minus beta by root 2 c dagger minus beta star by root 2 c times. (Refer Slide Time: 42:20) I maintain this order, e to the alpha by root 2 d dagger, minus alpha star by root 2 d, times e to the beta by root 2 d dagger, minus beta star by root 2 d and this acts on 0 C 0 D.

I can now combine things and see very easily, that this is to be written as D of alpha plus beta by root 2. The 1st term is alpha minus beta by root 2, then there is an alpha plus beta by root 2 0 C 0 D. There is a phase out here because when I add the two of them, this is D of alpha, D of alpha by root 2 D of beta by root 2. Is D of alpha minus beta by root 2, minus sign coming from here, (Refer Slide Time: 42:20) times an overall phase which is e to the i imaginary part of alpha beta star. But, I pick up exactly e to the minus i imaginary part of alpha beta star from here and so the phase factors, those extra phases cancel out and this is what I have.

But, this is precisely the coherent state and therefore, when I send in 2 coherent states, ket alpha through one and ket beta through the other, I get 2 coherent states in the output ports one in each port. One of them being, alpha minus beta by root 2 the label is alpha minus beta by root 2 and the others, label is alpha plus beta by root 2. So, once more the coherence property is preserved except that there is a change in the intensity of the state. And, I get precisely this kind of combination, which leads to a different intensity compared to the original intensity of the states. So, this is the set up in which a quantum beam splitter is used. One of the main advantages, is that you can produce copies of coherent states.

You will recall that if I had the vacuum in one of the input ports and a coherent state in the other. I finally get 2 coherent states. So, I can produce 2 copies of a single coherent state. Further, if I started with 2 coherent states the output ports also, have a coherent state in each of them, except that the intensity is different. These are the ways in which a quantum beam splitter can be used and this is another example of a system, a physical system, a very relevant system, where I play round the 2 sets of operators, a a dagger and b b dagger and produce some very interesting results.