Quantum Mechanics - I

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Lecture - 14
Exercises on Quantum Expectation Values

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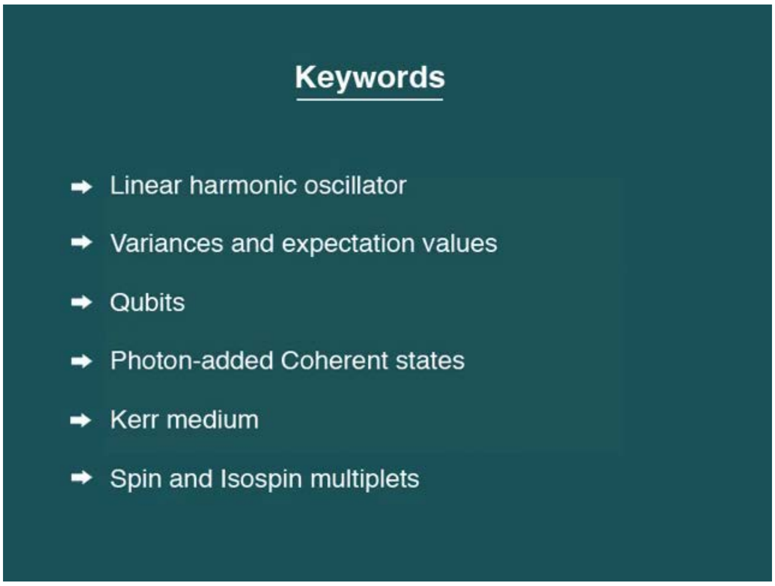

In the last couple of lectures, I have worked out a series of exercises for you pertaining to operator algebras and uncertainty relations and so on. Today, I will continue to work out some more exercised problems. These will be problems of direct relevance to various physical situations and therefore, I will explain the importance of the problems that I will attempt to work out as I go along.

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So, again we have exercises. The 1st exercise pertains to the harmonic oscillator. The linear harmonic oscillator for which if I set m equals omega equals h cross equals 1 for simplicity I can always put them back later, once I do the calculation. Then X would simply be a plus a dagger by root 2 and p would be a minus a dagger by root 2 i and of course, the commutator X P is i times the identity operator. I have set h cross equals 1. This would also correspond to a a dagger equals 1 the identity operator again.

So, I will work with this example where m is equal to omega equals h cross equals 1 and what I will attempt to find out are the following: Expectation value of x in any state n of the oscillator, expectation value of p in any state n of the oscillator. Recall that n takes value 0,1,2 and so on n equals 0 is the ground state of the oscillator then we can build up the excited states from the ground state. We also want to know what is the expectation value of X P plus P x in the state n. So that would be the 1st problem that I will attempt I wish to calculate these 3 objects.

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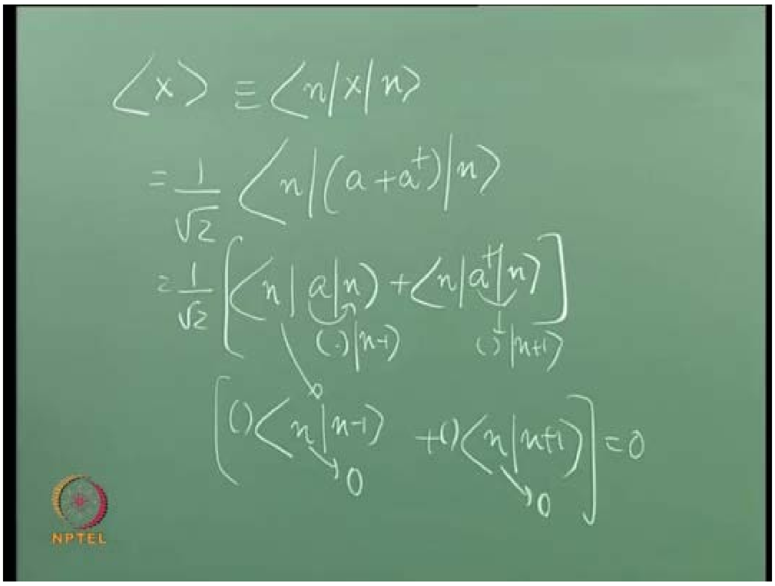

So, let us start with the expectation X. This is identical to n X n and that is equals 1 by root 2 n a plus a dagger n that simply n a n plus the expectation value of a dagger in the state n, a n takes this apart from some coefficient to the state n minus 1 and a dagger takes this apart from a coefficient to the state n plus 1. But the states are orthonormal to each other they are orthogonal to each other and therefore, this would simply be n n minus 1 apart from this coefficient and therefore, this is 0. Similarly, this would be n n plus 1 and that is also 0 and therefore, the expectation value of X in any state of the harmonic oscillator. By state I mean one of the natural basis states in that linear vector space or the fock states. In quantum optics language you would refer to them as the photon number states and X as a quadrature variable. So expectation X is 0.

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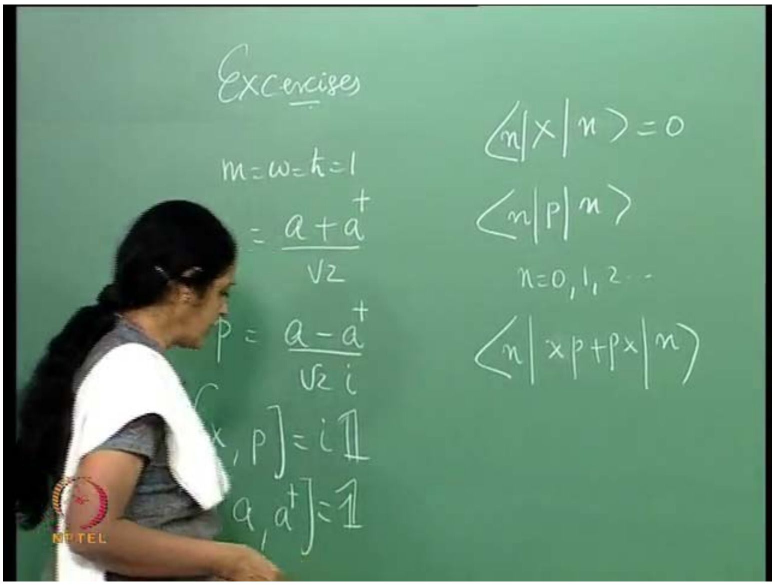

Now suppose I wish to find out expectation P in any of these states. I would repeat this substituting P is a minus a dagger by root 2 i.

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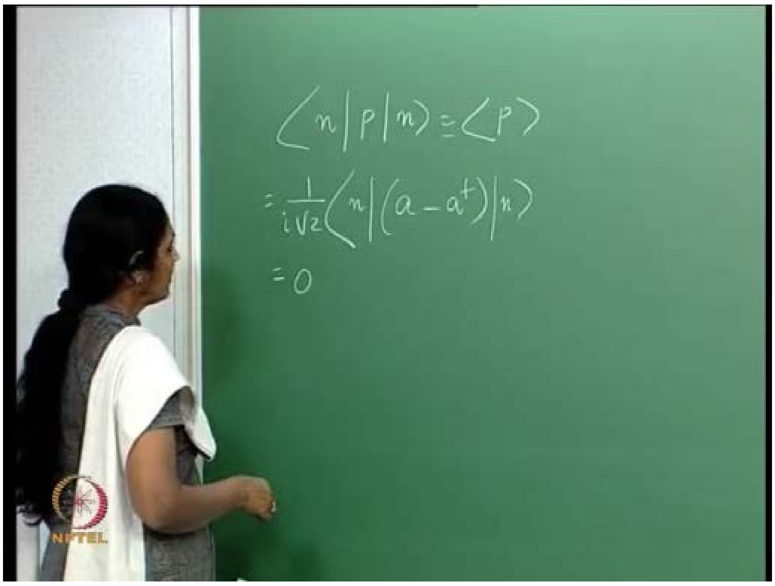

So, I wish to find this is identical to expectation P and that is given by 1 by root 2 a minus a dagger there is also an i n. Once more I use the same argument a acts on n to lower it to n minus 1, but ket n is orthogonal to ket n minus 1. Similarly, a dagger raises this to n plus 1 and the 2nd term also is 0. I use the same arguments as I used earlier and therefore, expectation value of P is 0.

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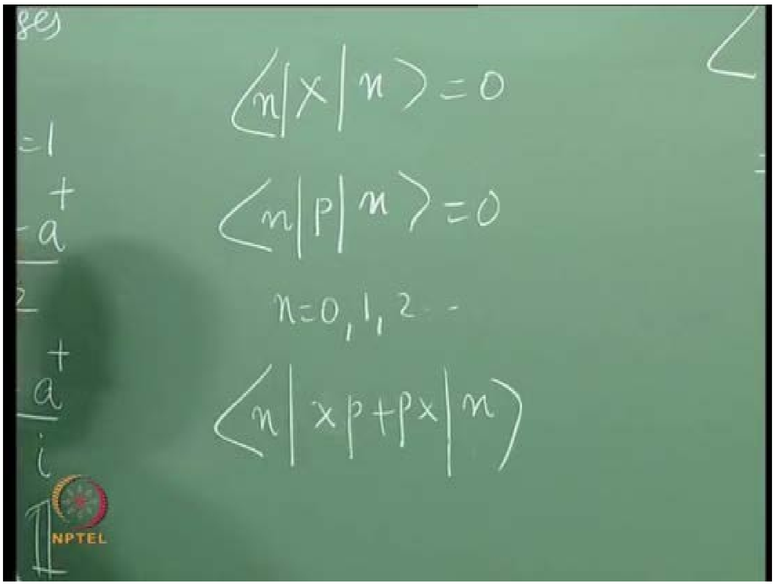

This can be explained well when one goes to the possession representation and the momentum representation, which I will subsequently. It has to do with the fact that all the energy Eigen states when written as functions of X, when written as wave functions that is functions of X they have a definite parity. It is rather early in the day to discuss that, but I will come back to this problem when I discuss the harmonic oscillator in the position representation later on. The next thing I wish to prove is that this object is also equal to 0. So, let me work that out.

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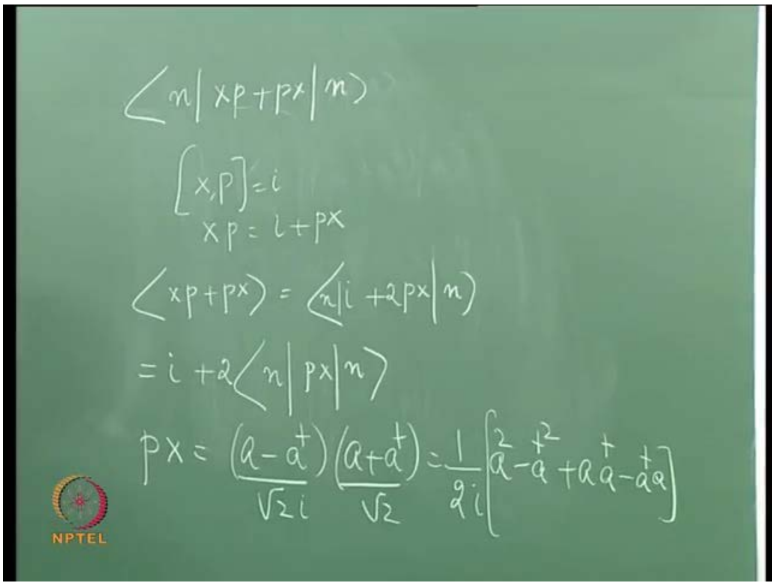

So we start with n x p plus p x n, but we know that the commutator x with p is i therefore, x p is i plus p x. Actually it is i h cross times the identity operator out there, but we would set h cross equals 1. So, x p is simply i plus p x and therefore, I can write this as i plus 2 p x, but I want the expectation value in the state n. The 1st term simply gives me i, the 2nd term is really this. I now expand p and x in terms of a's and a daggers. p x is a minus a dagger by root 2 i times a plus a dagger by root 2. This object is half i a square minus a dagger square plus a a dagger minus a dagger a. I use this and I find out expectation x p plus p x.

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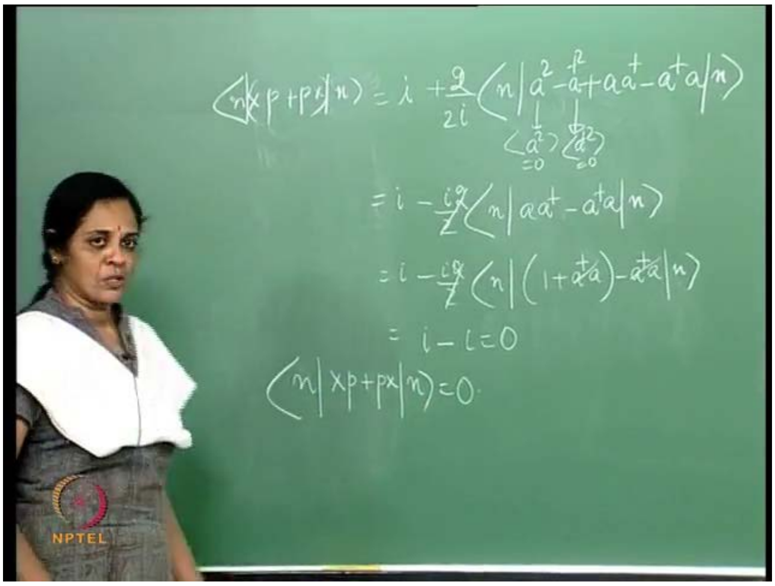

So, I get expectation x p plus p x in the state n is i plus 1 by 2 i expectation a square that is a minus a dagger square plus a a dagger minus a dagger a in the state n. If you look at the 1st when a square acts on ket n, when the 1st term around a reduces n to n minus 1 then the 2nd time it reduces n minus 1 to n minus 2. So, an a square acts on ket n I get ket n minus 2 and again ket n is orthogonal to ket n minus 2. So, this term does not contribute. So this expectation value is 0. Similarly, a dagger square's expectation value is 0. What I would be left with would be objects like this. So this is i minus i by 2 expectation value of n a a dagger minus a dagger a n, but I can always use the fact that a a dagger is 1 plus a dagger a.

(Refer Slide Time: 06:14) I have x p plus p x and this gave me a 2 p x. So there was already a two outside and therefore, I need to put in x p plus p x. I need to put a 2 there.

So the twos cancel and I am left with i minus i because the inner product of n with n, bra n with ket n is 1 and therefore, the answer is 0.So, you see the expectation value in any state n of the oscillator of the object x p plus p, x that operator x p plus p x is 0.

So, what we had established in the 1st exercise is that expectation x expectation p and expectation x p plus p x all vanish in any state of the oscillator. I now move on to next exercise to find out delta x and delta p in any state n of the oscillator. We have already shown that delta x, delta p take on minimum values in the ground state of the oscillator and if I set h cross equals 1, delta x is 1 by root 2 and delta p is also 1 by root 2 and therefore, the product delta x delta p takes its minimum value which is half.

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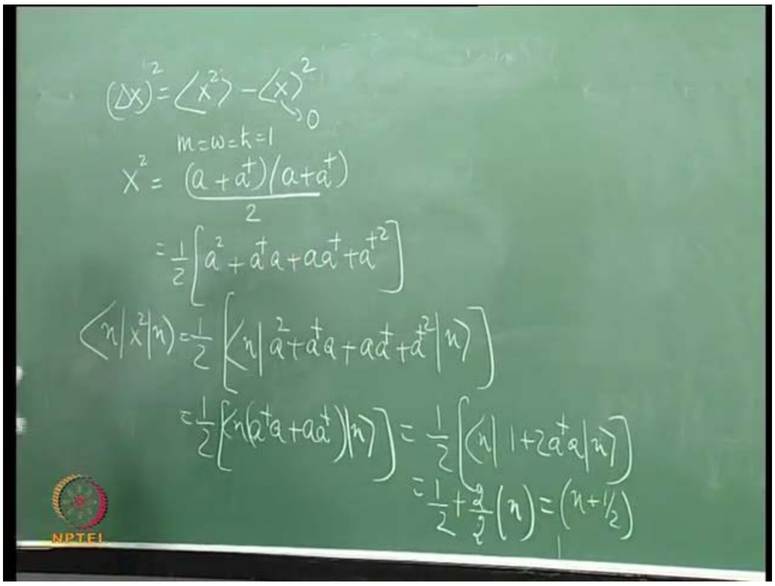

So now in any state of the oscillator, I wish to find out delta x delta p. Recall that delta X square is expectation X square minus expectation X the whole square, that we have just shown that expectation X is 0 in any state of the oscillator. So, I need to calculate expectation X square in units where m omega and h cross are 1 is simply a plus a dagger the whole square by 2, which is the half a square plus a dagger a plus a a dagger plus a dagger square.

So, if I need to find the expectation value of x square in the state n. I need to calculate the expectation value of these four objects and then sandwiched between ket n and bra n. Look at the 1st term, as I argued before the last term will not make a contribution. So I have half a dagger a plus a a dagger. The expectation value of just this operator once more, I can write a a dagger as 1 plus a dagger a and I need to compute this. I have used a fact that a a dagger is 1 plus a dagger a and that is why I have got that. The 1st term just gives me a half the 2nd term gives me there was a 2 from here and a 2 from there so that cancels out it gives me n and therefore, I get n plus half.

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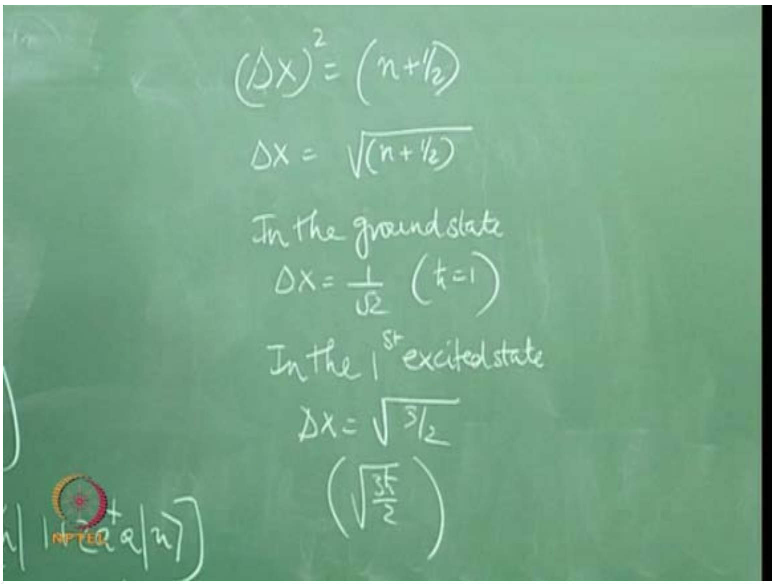

So, the delta X the whole square is n plus half in the state n of the oscillator. And that is why in the delta X in the ground state is just the positive square root of n plus half and that is why in the ground state of the oscillator delta X was 1 by root 2. I have set h cross equals 1 you could put it back and then you will have root of h cross by 2. In any other state delta X is root of n plus half. So, in the 1st excited state delta X is equal to root of 3 by 2, again in units of h cross equals 1 and therefore, if I put back the h cross it is root of 3 h cross by 2 and so on. So, you have the ground state which is the minimum uncertainty and delta X keeps increasing as the state label increases.

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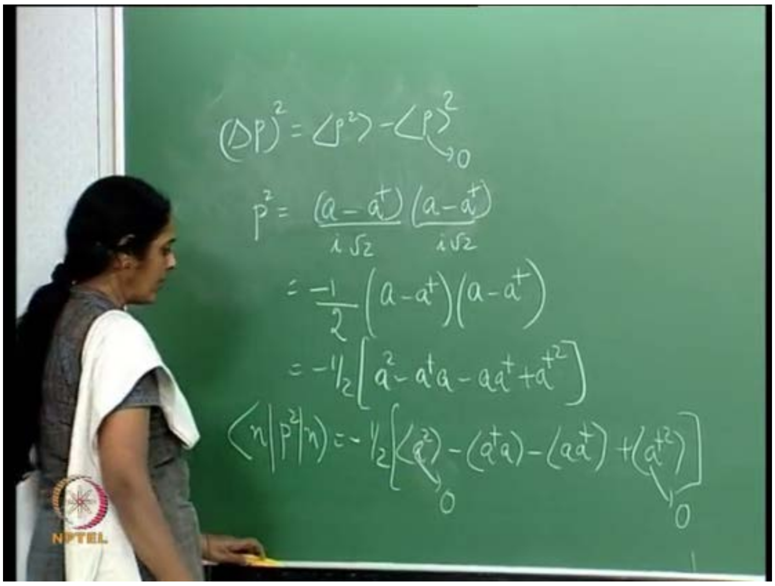

Now, let us look at delta P once more I use the same procedure. Delta P the whole square is expectation P square minus expectation P the whole square. We have just shown that this object is 0 at any state n of the oscillator and therefore, I need to compute expectation P square P square is a minus a dagger by i root 2, a minus a dagger by i root 2. And this object is a minus half a minus a dagger times a minus a dagger. So, I can expand it and write it in this manner. I need to find n P square n and that is minus half expectation a square minus expectation a dagger a minus expectation a a dagger plus expectation a dagger square all expectation values obtained in the state n. As before this is 0 and so is this I need to only worry about the contributions from a dagger a and a a dagger.

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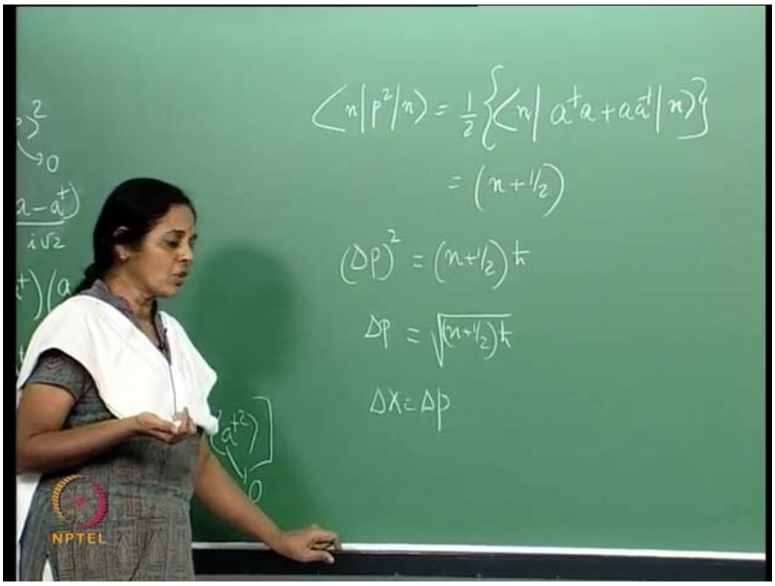

So n P square n is half a dagger a plus a a dagger, but this is precisely what we computed earlier and this will give me an answer n plus half. Because once again I will write a a dagger as 1 plus a dagger a. Therefore, delta P the whole square is n plus half of course, if I put in the h cross I get this and therefore, delta P is square root of n plus half the positive square root. While in the ground state, I have the minimum value in the 1st excited state it is 3 h cross by 2 the square root of 3 h cross by 2 and so on and we find that delta X is equal to delta P in any state ket n of the oscillator.

So this is what the 1st exercise was about and we have established that while the ground state is a minimum uncertainty state, in all other state delta X and delta P do not vanish. Because, expectation X square and expectation P square are non zero, but we have shown explicitly that delta X equals delta P and the uncertainty product gets larger and larger as we go to higher values of n.

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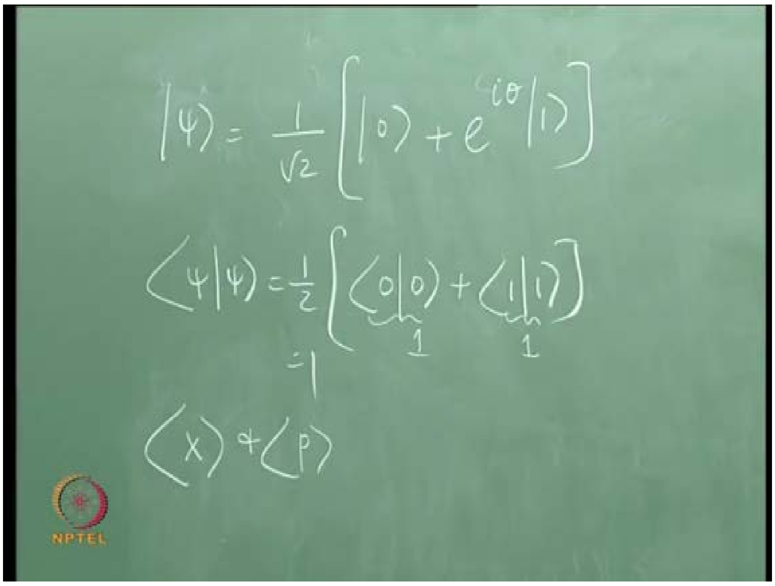

My next exercise concerns the qubit. I now wish to look at the state psi which is ket 0 plus e to the i theta ket 1.The theta is e to the i theta is just a phase it is normalised. I can check that out it’s normalised to unity, because this object is 1 and so as this. And therefore, I take this normalised state ket psi, it is a qubit. It is a superposition of ket 0 and ket 1 and I wish to find expectation X and expectation P in the state psi.

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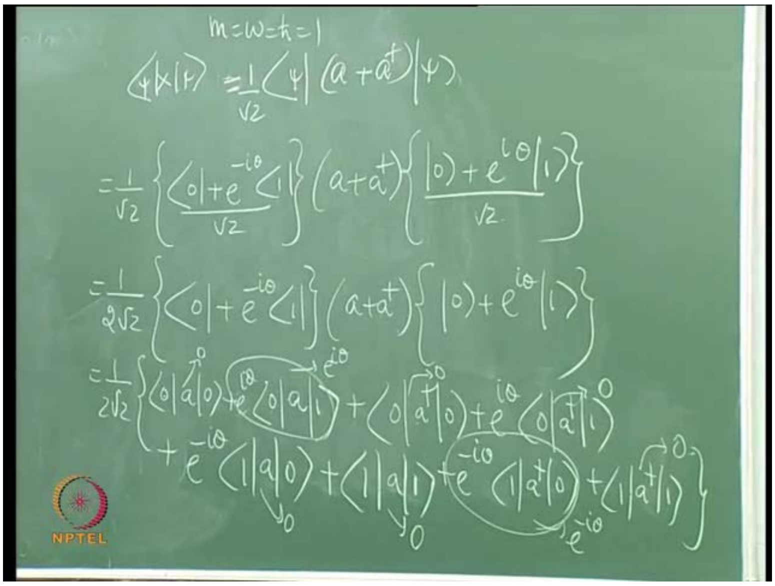

So, expectation X in the state psi implies computing this. I have set m equals omega equals h cross equals 1, but that object is the same as 1 by root 2 bra 0 plus e to the minus i theta bra 1 by root 2 a plus a dagger ket 0 plus e to the i theta ket 1 by root 2 and that gives me 1 by 2 root 2 and I have to compute this. So, this is what I have. This involves the following terms: 1st I have 0 a 0 then I have 0 a e to the i theta ket 1, that is what I have worked with 1st, then I have 0 a dagger 0 plus e to the i theta 0 a dagger ket 1.

Now I move to the 2nd term here, so that gives me plus e to the minus i theta 1 a 0. There is an e to the minus i theta times e to the i theta so I just have 1 a 1. I work with the dagger now, so that gives me an e to minus i theta 1 a dagger 0 and the last term which is simply 1 a dagger 1. This is what I need to compute. It is clear that this is 0 because a and ket 0 is 0 so this will make a contribution because a and ket 1 is ket 0 and therefore, this object simply turns out to be e to the i theta a dagger on ket 0 is ket 1 and by the fact that ket 0 and ket 1 are orthogonal, that is 0, this is ket 2 and I have a 0 here so this is also 0. This annihilates the vacuum or the ground state and therefore, it is 0 this brings it down to the ground state and because ket 1 and ket 0 are orthogonal, this is 0. This contributes because a dagger on ket 0 is root 1 ket 1 so this whole term is simply e to the minus i theta. This is 0, because this takes it to k 2 and ket 1 and ket 2 are orthogonal to each other.

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So I have psi x psi where psi is the qubit that I have selected. This object is 1 by 2 root 2 e to the i theta plus e to the minus i theta and that is what I have. (Refer Slide Time:

20:10) There was an e to the i theta here and there was an e to the minus i theta there. The whole thing is multiplied by 1 by 2 root 2.

So, this is here cos theta by root 2. What about psi P psi? I can repeat this argument and expand putting in the appropriate values the appropriate operators for P. I leave that as an exercise that is again non 0. So, here is a state where expectation X is not 0 and expectation P is not 0. Again as an exercise, one can compute delta X and delta P in this state; they are not minimum uncertainty superpositions of 0 and 1. The uncertainty product is higher than what you would expect for the ground state of the oscillator. Calculations like this, exercises like this would become relevant in quantum computation where qubits are used and expectation values of various operators in a qubit state become important.

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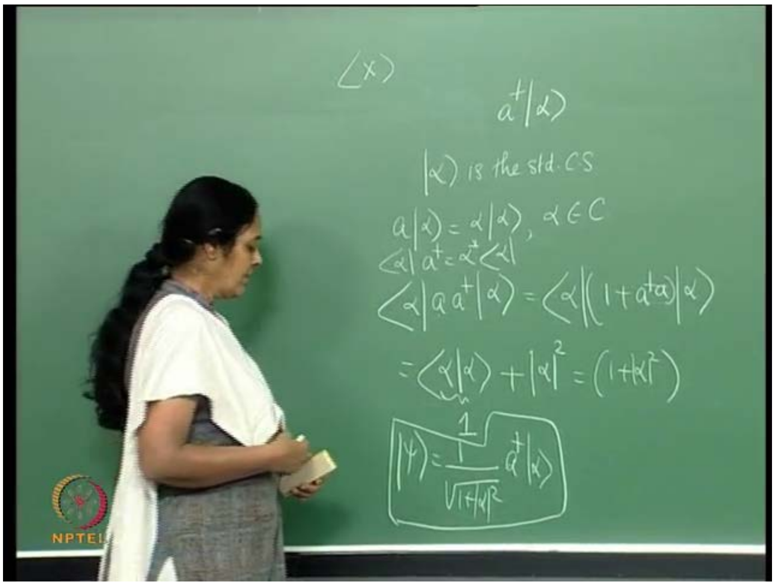

My next exercise is to compute the expectation value of X. In a very interesting state I define that state as a dagger acting on ket alpha. I have to normalise this where ket alpha is the standard coherent state and as you know this is an Eigen state of a with Eigen value alpha, alpha being a complex number. Now, first of all let us normalise that state. So, consider this object I can always write this in terms of a dagger a. The idea is to get a to this side, because then I know that I can use this property and therefore, this property. So, a a dagger is 1 plus a dagger a and I need to find this and this is something I know.

The first term is simply one because alpha is a normalised state, normalised to 1 plus mod alpha square, that comes from the expectation value of a dagger a in the state alpha.

So, this object is just 1 plus mod alpha square and therefore, the state psi which is 1 by root of 1 plus mod alpha square a dagger on ket alpha. This state you can easily check is a normalised state. Now, in the parlance of quantum optics what is it that I have done. Taken these standard coherent state of lights acted ones with the photon creation operator so this is a photon added coherent state suitably normalised for the single photon added state, coherence state.

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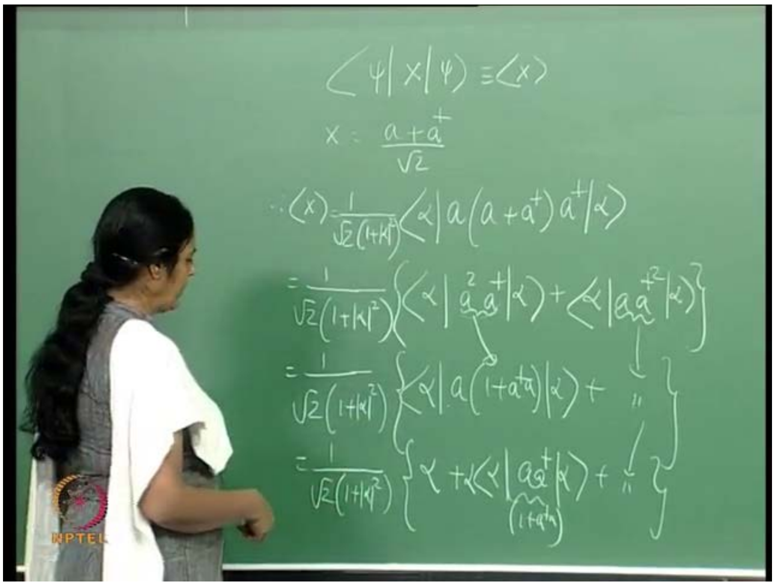

Now, I want to calculate expectation X in this state. This is what I wish to find but, X itself is a plus a dagger by root 2 and therefore, the expectation value of X in this state is 1 by root 2. I have just taken to the bra psi of course, that is going to be a normalisation and that is 1 plus mod alpha square. So, a 1 by root 2 that came from X and a 1 plus mod alpha square that comes from the normalisation of the state, a plus a dagger a dagger on ket alpha so this is what I have. Now, that is just 1 by root 2 mod 1 plus mod alpha square. I have two terms here. The 1st is the expectation value of a square a dagger in the state alpha. The next is the expectation value of a a dagger square in that state.

Now, it is clear that this operator here its dagger is here and therefore, if I know this expectation value I can always take the complex conjugate and get the other expectation value. So, I have 1 by root 2 1 plus mod alpha square. Look at this once more it is good to shift a to that side. So, I have a times a a dagger which is 1 plus a dagger a. So, I have just written this operator out here plus this operator. So, this number is the complex conjugate of that number and what do I get here? That is 1 by root 2 1 plus mod alpha square.

The 1st term is just the expectation value of a in this state alpha and that just pulls out an alpha, because of this property. The next is expectation value of a a dagger this a acting on alpha pulls out an alpha so that is what I have. By this I mean just take down this term and put it there. Again I write a a dagger as 1 plus a dagger a and now I know what is happening.

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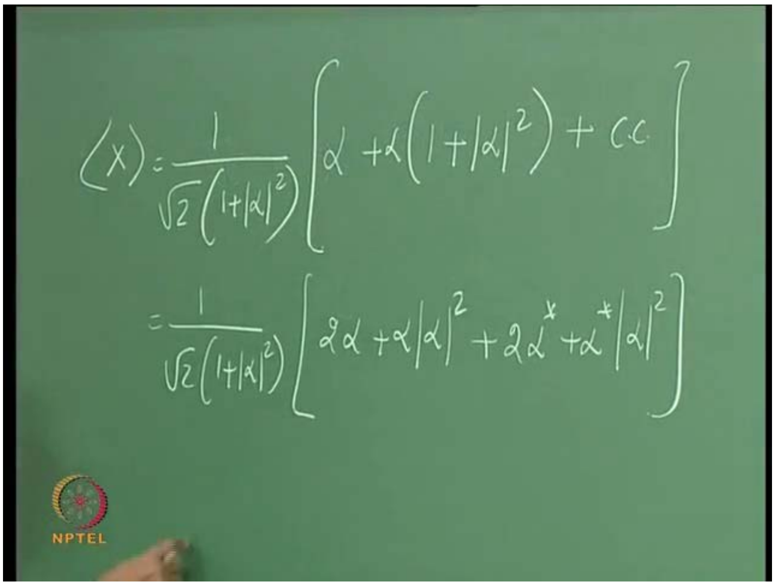

This expectation value therefore, becomes 1 by root 2 1 plus mod alpha square. The 1st term gave me an alpha and here there is already an alpha. (Refer Slide Time: 28:38) So, I just have to look at this operator and what it gives me. So the 1st term gives me an alpha times a 1 plus mod alpha square and so I have alpha from the 1st term plus alpha times 1 plus mod alpha square and then of course, its complex conjugate that came from the 2nd term. So, basically I can now simplify and write this as 2 alpha plus alpha mod alpha square and from there 2 alpha star plus alpha star mod alpha square.

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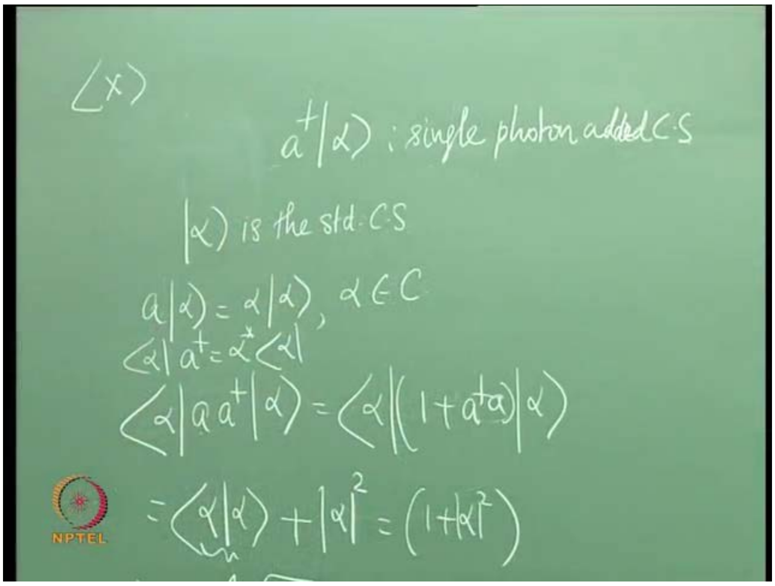

This can be further simplified. I leave it to you to put this in a better form and realise that a dagger on ket alpha the single photon added coherent state is a very important state, because 1st of all they are produced in the laboratory about 5 years ago and it is not a perfect coherent state, but it departs from coherence quantifiably added 1 photon to the coherent state and to that extent created a departure from the coherent state.

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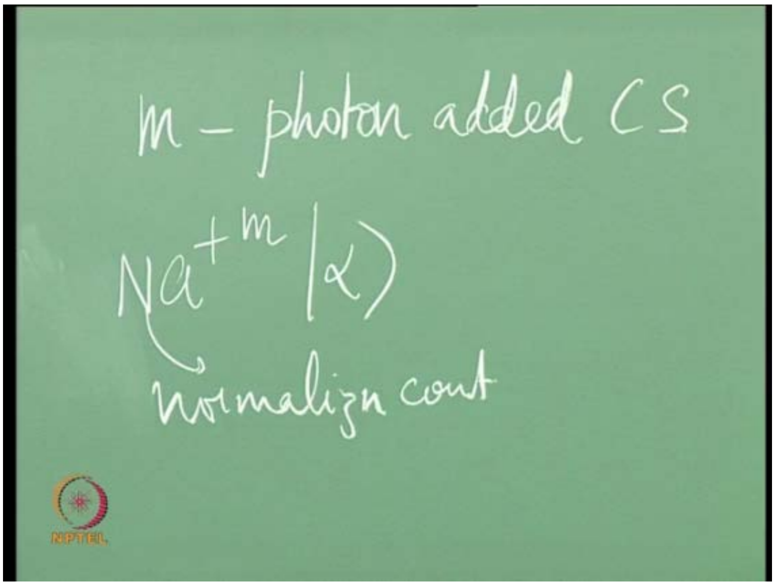

Now, I could have a in general created an m photon added coherent state in principle and that is got by repeated application of a dagger m times on ket alpha suitably normalised.

This is the normalisation constant. And I would call this state the m photon added coherent state. Such states become important in various context. I now move on to a nonlinear Hermitian operator which is nonlinear in a and a dagger.

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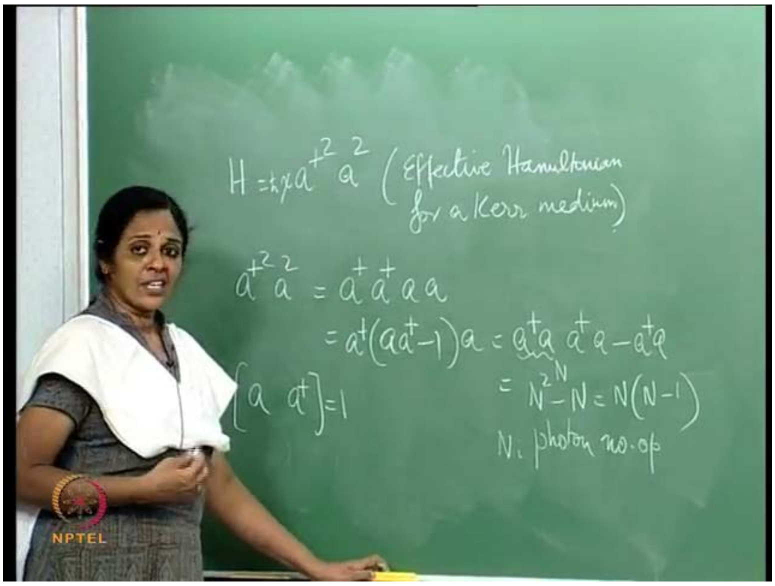

Consider the operator a dagger square a square. I have deliberately used H there because if I put in some overall constants outside. I find that this Hamiltonian is the effective Hamiltonian for nonlinear optical medium specifically, the Kerr medium. This is the effective Hamiltonian for a Kerr medium, for a nonlinear optical medium. I wish to find out the Eigen spectrum of this Hamiltonian so let me proceed to work out that exercise. I consider a dagger square a square. I can write this. The idea is to write it a dagger a a dagger a so I write a dagger a in terms of a a dagger. Recall that a a dagger is 1 and therefore, a a dagger is 1 plus a dagger a so I can write a dagger a as a a dagger minus 1 and there is an a there. So, this object is simply a dagger a, a dagger a minus a dagger a.

In other words, in terms of the photon number operator n this is n square minus n or n into n minus 1, where n is the photon number operator. I therefore, need to find out the Eigen value of Eigen values and Eigen functions of this Hamiltonian h cross chi n times n minus 1.

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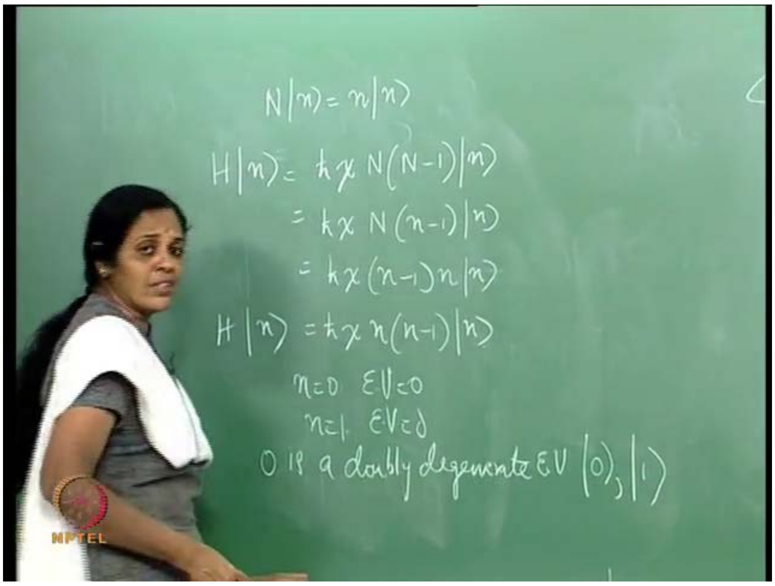

It is evident that the basis set ket n, the n photon state for instance is an Eigen state of this Hamiltonian. Because, h acting on ket n is h cross chi n into n times n minus 1 ket n. This just pulls out a number. Recall that the number operator acting on ket n is simply n times ket n and therefore, this is n minus 1 and again the number operator acts on ket n pulls out an n. So I can just write this as h cross chi n times n minus 1 ket n.

So, all the photon number states are Eigen states of this Hamiltonian and this is the Eigen value n times n minus 1 essentially. So, let us look at what happens when n is 0 the Eigen value is 0 similarly, when n is 1 the Eigen value is 0. Therefore, 0 is a doubly degenerate Eigen value and there are 2 states corresponding to this degeneracy that is ket 0 and ket 1. So, both the 0 photon state and the 1 photon state are Eigen states of this Hamiltonian the Kerr Hamiltonian, because they are Eigen states corresponding to a degenerate Eigen value 0.

All other Eigen values are non degenerate. For instance, if you look at n is equal to 2 that just gives me 2, then n equals 3 gives me 3 times 2 which is the 6 and so on. So, none of the other Eigen values are degenerate, but the ground state is the 0 is degenerate. We have now looked at a large class of states of relevance to qubits quantum computation of relevance to quantum optics. The single photon added coherence states of the relevance to harmonic oscillator, because we looked at fock states and we computed delta X delta P, expectation values of other objects like X P X P plus P X and so on in these states.

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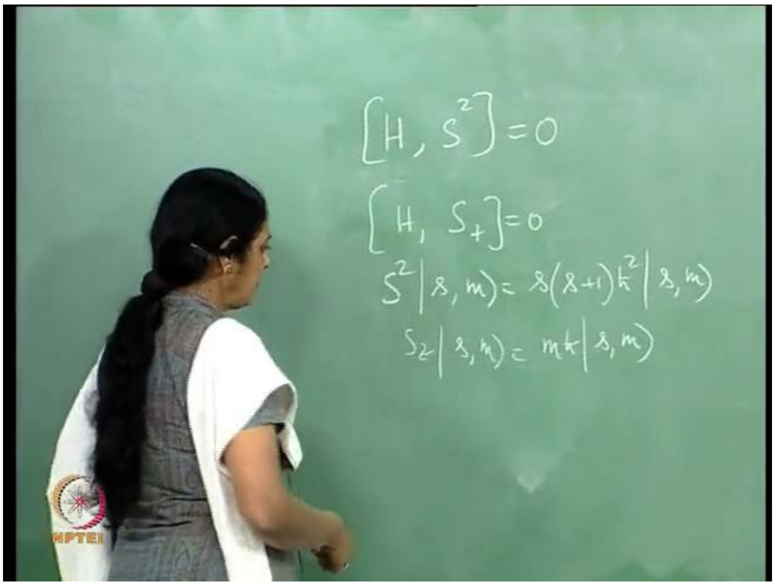

I would now like to move on and work out a problem in angular momentum and this would be of relevance to particle physics. Suppose, I have a Hamiltonian which commutes with the angular momentum operator J square, in particular J is spin so; it is not orbital angular of momentum. So let me look at the spin angular of momentum so h a square suppose I have a Hamiltonian which commutes with this and the Hamiltonian commutes with S x S y S z and therefore, with S plus.

The spin Eigen states are denoted by s m of course, S z s m is m h cross s, m these are the spin labels. In this picture I want to show that an object with a given spin has the same mass independent of whether it exists in the state m, or the 3rd component having a value m or a value m plus 1 or m plus 2 or m minus 1. Independent of the value of m provided, the spin of the object is given to me I want to show that the object has the same mass whether it exists in the state s, m or s, m plus 1 or s, m minus 1 it is independent of m. Provided, the Hamiltonian commutes with S plus in other words, it commutes with S x S y and of course, S z.

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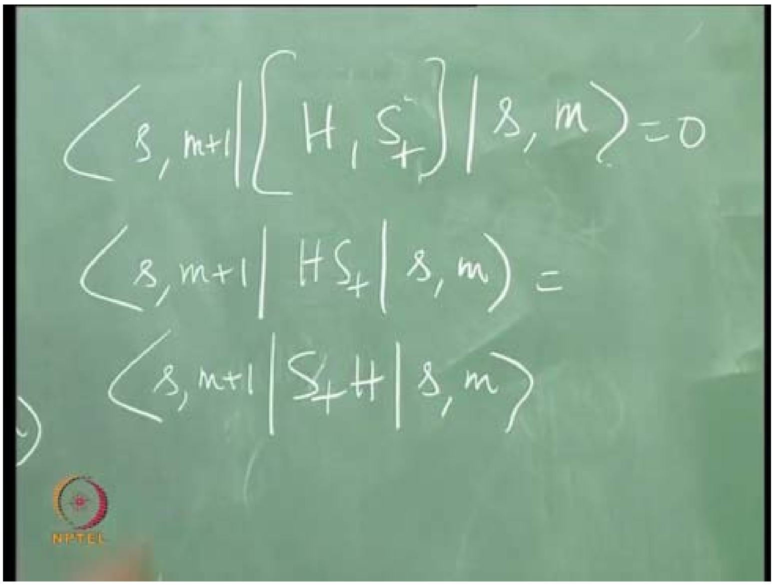

The way I go about it is as follows. I know that this is true, because H commutes with S plus that is given to me. So, I expand this commutator out so this is what I have. But, I know the action of S plus on s, m it raises the value of m by 1. So, let me put that in.

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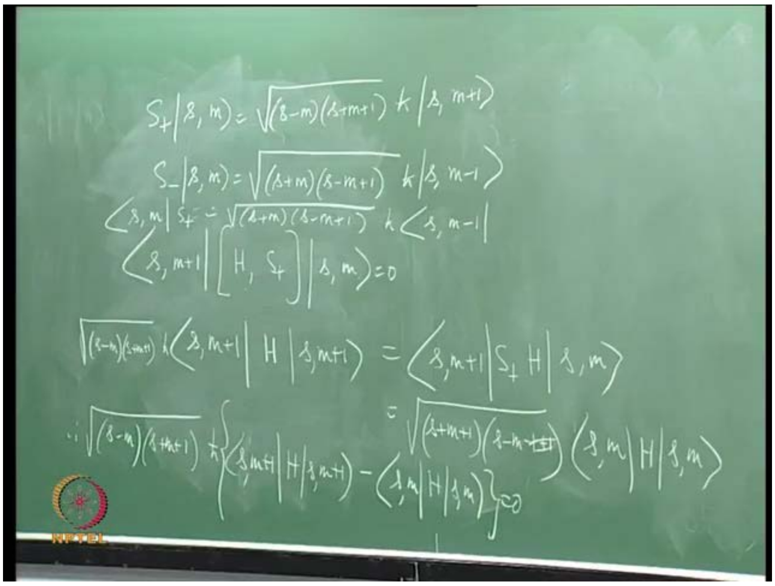

You will recall that S plus acts on s, m to give me root of s minus m into s plus m plus 1 h cross s comma m plus 1, m itself takes 2 s plus 1 values ranging from minus s to plus s in steps of 1. Similarly, S minus 1 s, m gives me the following. I use this in my problem therefore, I started with this object and this implied H S plus 1 s, m gives me H s, m plus

1. Pulling out this coefficient root of s minus m into s plus m plus 1 h cross, h cross is outside the square root. That is the 1st term and this quantity is equal to s m plus 1 S plus H s m, that if S minus acting on s, m gives me this its dagger is this and that is a real quantity therefore, I have the same coefficient. So, I use that here.

So, S plus with an s, m on this side of it except that I have m plus 1 instead of m. So, I use this and I have, it reduces m plus 1 to m. Notice that this coefficient is a same as that coefficient that scores off and I just have the same coefficients. Therefore, root of s minus m times s plus m plus 1 h cross s n plus 1 H s m plus 1. The expectation value of the Hamiltonian in the state s comma m plus 1 minus the expectation value of the Hamiltonian in the state s, m is 0, m is not equal to s and therefore, this coefficient does not vanish.

So it tells me a very important property that the expectation value of the Hamiltonian in the state s, plus 1 is a same as the expectation value of the Hamiltonian in the state s, m. So, let me look at this spin doublet for instance what is this means for the spin doublet.

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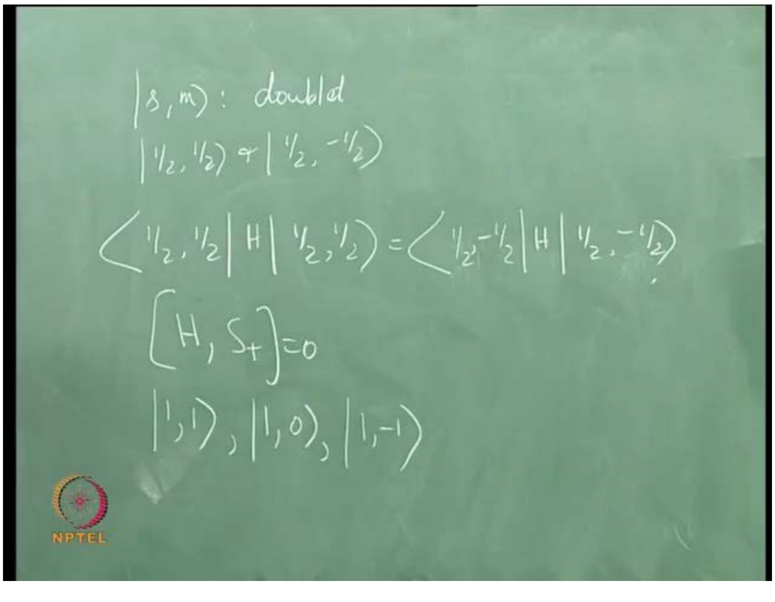

In the case of the spin doublet, I have s is half and m is half and s is half with m minus half. So this boils down to the statement that half half h half half is equal to half comma minus half h half comma minus half that is what it means. In other words, look at the Hamiltonian. The Hamiltonian has in general a kinetic energy plus a potential energy.

For particles of a certain mass, this would mean the kinetic energy of the particle and the rest mass energy of the particle. Let us go with the rest frame of the particle.

So, let us go to the rest frame of the electron which is a spin half of the object for instance or a proton or the neutron. The statement is this: in the rest frame this expectation value would correspond to the rest mass of the particle in question. So, the rest mass of the electron in the spin up state equals the rest mass of the electron in the spin down state. Similarly, for protons and neutrons the rest mass of the proton in the spin up state is the same as the rest mass of the proton in the spin down state. The manner in which I take care of this is by starting off with a Hamiltonian which commutes with S plus.

Once this is given to me it automatically follows that objects in a certain state of spin in the sense that s is fixed, but m has a definite value. Objects in a state m s, m would have the same rest mass as the state with s, m plus 1 or m minus 1. In other words, in a spin multiplet the same holds for a spin triplet. For instance, in a spin triplet the rest mass of the object in the state 1, 1 in the state 1, 0 and the state 1, minus 1 would all be the same. So, within a spin multiplet the rest mass is the same.

We discuss a certain particle in a certain state of spin therefore, but it could exist in different states of m whatever may be the state of m the particle has the same rest mass, that is guaranteed by the fact that H commutes with S plus. So, I have worked out certain exercises for you using commutator algebras. Essentially, the algebra involved in the case of the harmonic oscillator X P commutator is i h cross equivalently a a dagger commutator is 1 and the spin algebra, where I have used the fact that if I have a Hamiltonian, which commutes with S plus certain consequences follow.