Quantum Mechanics - I
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Lecture - 13
Exercises on Angular Momentum Operators and their algebra

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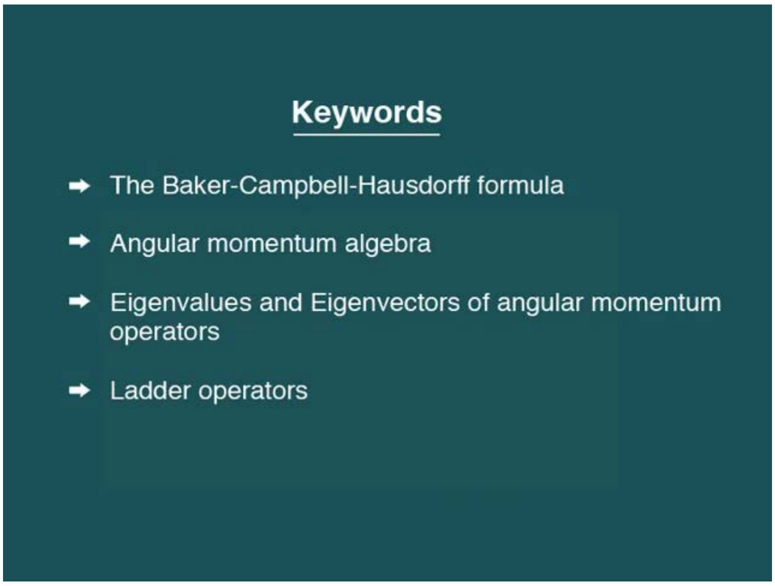

In the last class, I had worked out certain exercises. Essentially pertaining to Hermitian operators, their Eigen values their Eigen vectors. And also established that, if there are 2 commuting Hermitian operators, equivalently 2 commuting Hermitian matrices, you are guaranteed that there is a complete set of common Eigen states of these 2 matrices, this can be extended, to more than 2 matrices as well.

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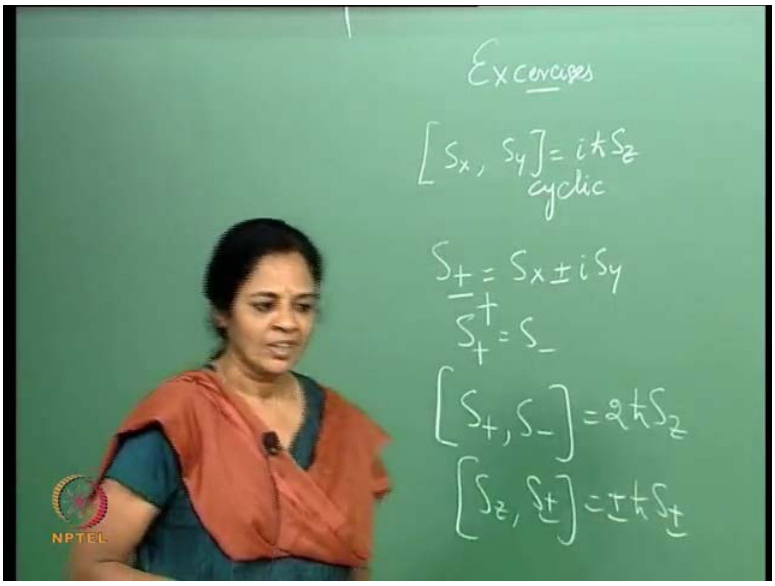

Today, I will continue to work out further exercises and today’s exercises would largely be related to operators and their algebras. You are already aware of the angular momentum algebra. Where, you have the spin generators, satisfying the commutation relation. Commutator S x with S y is i h cross S z and that is a cyclic relation, because you could have a z here and x there and y there and so on.

You could have written the commutator this way and therefore, the algebra in the fashion, or you can define, S plus and S minus as S x plus or minus i S y depending upon the context and S plus dagger is S minus. In that case, the equivalent way of writing this algebra, is S plus S minus is 2 h cross S z and S z with S plus or minus is plus or minus h cross S plus minus. So this is by way of recapitulation. We certainly use this algebra extensively, in the study of the spin doublet, the 2 level atom problem and also the 3 level atom problem.

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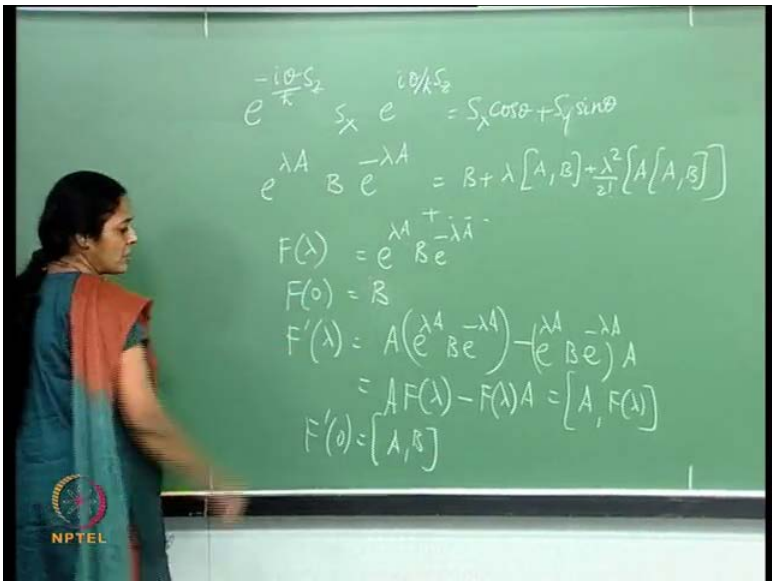

Further at one stage I derived the following relation, e to the minus i theta by h cross S z S x e to the i theta by h cross S z, is S x cos theta plus S y sin theta. Now, this is something that tells us that, S x transforms like the x component of a vector. Similarly, if you had S y here you will have minus S x sin theta plus S y cos theta out there, to tell you that S y transforms like the y component of a vector and so on. I did this explicitly in one of my earlier lectures. By expanding out the exponential and then multiplying S x across that and showing that indeed you get this as the answer.

So that S x transforms, like the x coordinate where x prime goes to x cos theta plus y sin theta under rotation by an angle theta like that. On the other hand, later on I did put down an identity, of this form e to the lambda A B e to the minus lambda A is B plus lambda commutator A B plus lambda square by 2 factorial commutator of A with A with B plus so on. We use that in the context of the displacement operator and the squeezing operator as well.

At that time this identity was not proved. And I want to prove this identity now as an exercise. So I define F of lambda, as e to the lambda A B e to the minus lambda A. Lambda is the parameter, it is clear that F of 0 is simply B. It is assumed that A and B are Hermitian matrices that is what is of relevance to us. So, F prime of lambda, that means the differentiation with respect to lambda, gives me A e to the lambda A B e to the minus lambda A, minus e to the lambda A B. I bring down the A there e to the minus lambda A but I bring down the A there. And therefore, this object is A F of lambda minus F of lambda A. It is a commutator of A with F of lambda it is clear that F prime of 0 is simply the commutator of A with B because F of 0 is B.

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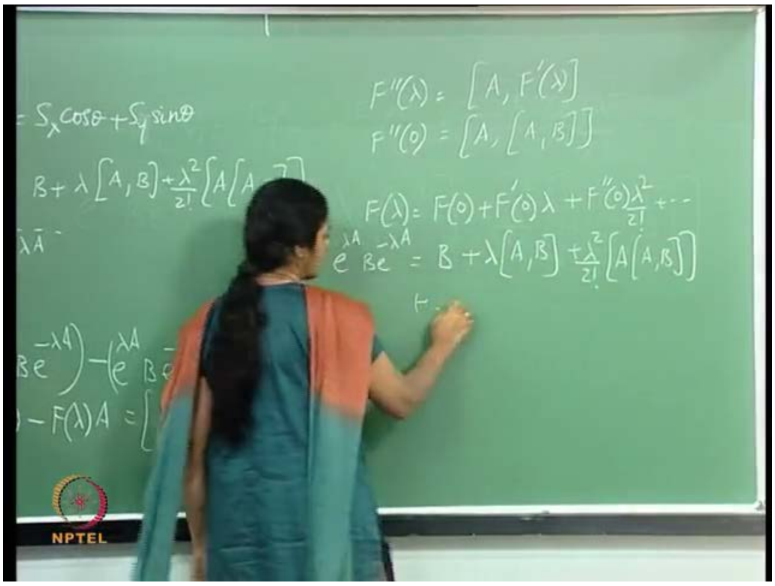

Similarly, I can find out F double prime of lambda. So, F double prime of lambda. Clearly from here if F prime of lambda is the commutator of A with F of lambda, F double prime of lambda is the commutator of A with F prime of lambda. And that is the same as so F double prime of 0 is A with A with B and so on. So, I can do the higher derivatives of F. And this is what I get for every term. Now, F of lambda is of this structure, it is a smooth function of lambda and therefore, can be expanded as a Taylor series in lambda. So, I can write F of lambda as F of 0 plus F prime of 0 times lambda, plus F double prime of 0 lambda square by 2 factorial and so on.

Now, if I substitute I get F of lambda which is e to the lambda A B e to the minus lambda is equal to f of 0 which is B, plus F prime of 0 times lambda which is lambda A B, plus lambda square by 2 factorial F double prime of 0 plus so on. And that proves the point. (Refer Slide Time: 02:39) So this is the identity, this is the general identity e to the lambda A B e to the minus lambda A, is B plus, lambda times this commutator plus lambda square by 2 factorial A with A with B commutator and so on.

So, clearly this infinite series will terminate, if one of these commutator is 0. I use this specifically, in the context of squeezing in displacement operators, when we discuss the harmonic oscillator problem and quantum optics. Now, I would like to use this identity, to establish this relation. So, let me start with e to the minus i theta h by h cross S z S x e to the i theta by h cross S z. (Refer Slide Time: 02:39)

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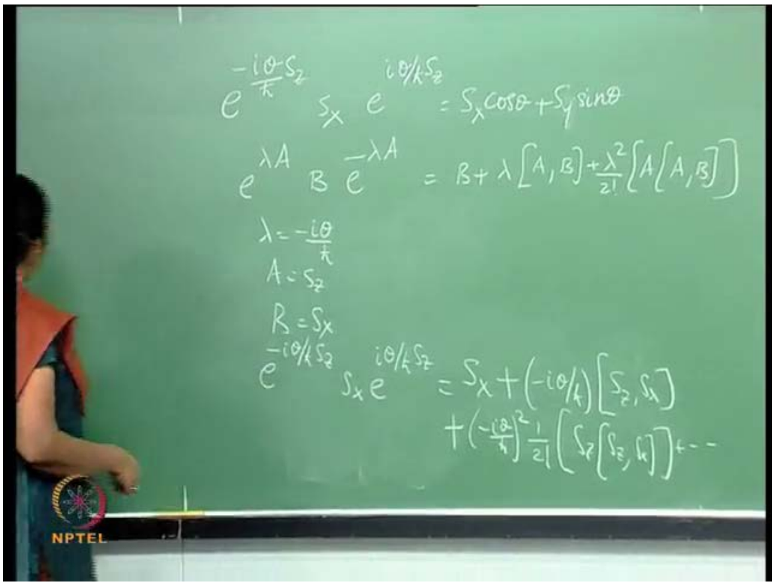

So here lambda is minus i theta by h cross A is S z and B is S x, this is what I have. If I now substitute, I have e to the minus i theta by h cross S z S x e to the i theta by h cross S z is equal to B and B is S x plus lambda times commutator of A with B which is minus i theta by h cross. The commutator of S z with S x, plus lambda square by 2 factorial, the commutator of A with the commutator of A with B, that is S z with S z S x commutator plus so on.

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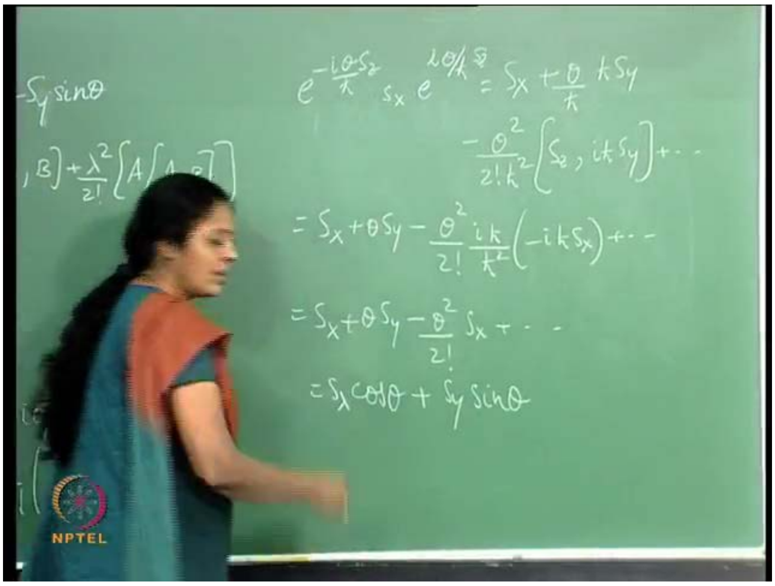

Now, again work out these commutators. (Refer Slide Time: 00:54) Simply because I know, the algebra of the spin matrices and then I have the following: e to the minus i theta by h cross S z S x, e to the i theta by h cross S z is equal to S x, (Refer Slide Time: 08:44) S z with S x is i h cross S y, so that gives me, plus theta by h cross h cross S y. So, that is what I get from this term. Then I have the next term, which gives me a minus theta squared, by h cross square 2 factorial. The commutator of S z with the commutator of S z with S x which is i h cross S y plus so on.

And this object is S x, plus theta S y minus theta squared by 2 factorial, i h cross by h cross squared, that is an S z with S y and that is a minus i h cross S x plus so on. And that is S x plus theta S y minus theta squared by 2 factorial S x plus so on. Now, if I work out the other terms, it will be clear that this will be S x times 1 minus theta squared by 2 factorial plus theta to the power of four by four factorial and so on, which is S x cos theta plus S y the leading term is theta when I have a theta q by 3 factorial theta to the 5 by 5 factorial and so on. And therefore, I get S x cos theta plus S y sin theta.

So this is a simple way of establishing, (Refer Slide Time: 08:44) what I set out to prove by a brute force method earlier on. I got this relation, by expanding out the exponential and doing the entire algebra. This is a neat way of doing this provided I establish this identity. Now having said that, I wish to ask some more questions on the angular momentum algebra. We know that the orbital angular momentum components L x L y and L z also satisfy this relation.

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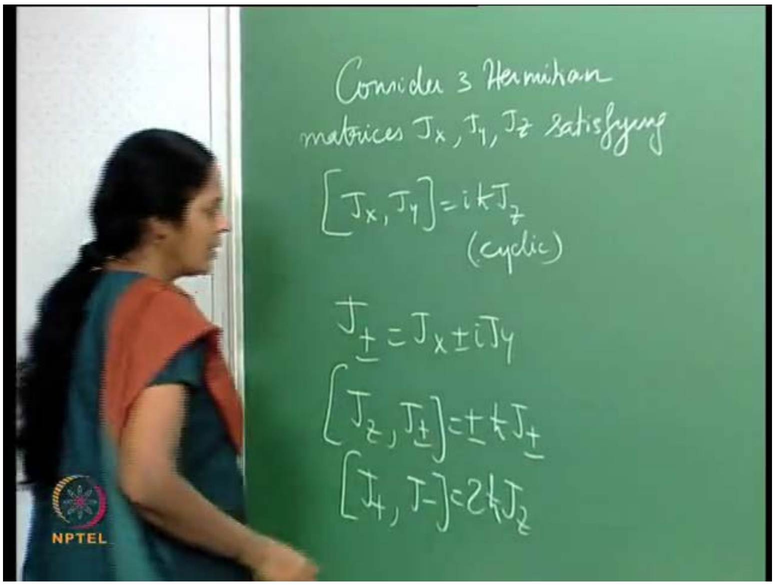

And therefore, in general consider, 3 Hermitian matrices, J x J y J z satisfying the lie algebra J x J y commutator is i h cross J z cyclic relation. Define J plus J minus as J x plus minus i J y, and then surely J z J plus minus plus or minus h cross J plus minus, and J plus with J minus is 2 h cross J z. Now J could represent the spin matrices, J x could be S x J y could be S y and J z could be S z, or if we are discussing orbital angular momentum, J x would be denoted by L x J y by L y J z by L z where l itself is our cross p. The algebra is the same the situation is different. In the one case we are dealing with an intrinsic property called spin. The another case we are dealing with something in physical space called orbital angular momentum.

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The question that I wish to ask is the following. I know therefore, that J squared and J z commute with each other. I have already established therefore, that there is a complete set of common Eigen states of J squared and J z. Let me denote, the Eigen states by J, m. In the case of spin this would be given by the label s and m would be the 3rd component of spin the Eigen value of S z. In the case of orbital angular momentum, this would be replaced by l and that would continue to be m, where now m denotes the Eigen value of L z.

So J square J m I know that there is an Eigen state J, m of J square. And since J square is a positive definite of operator, recall that J squared is J x squared plus J y squared plus J z squared. I can well write this Eigen value this lambda h cross squared, lambda is to be determined and it is clear that, lambda is greater than zero. Now J z, J m I know is m h cross ket J, m m 2 has to be determined given lambda. One thing is clear that since J squared is J z squared plus some positive quantities, lambda is greater than or equal to m squared and m squared itself, by its very nature is greater than 0, greater than or equal to 0. So this is evident, we now need to find out the values, the set of values that lambda can take and the set of values, that m can take. You will recall that when we did the 2 level atom problem. We had two Eigen states of spin. I label them ket s, m and ket s, minus m, in the following sense.

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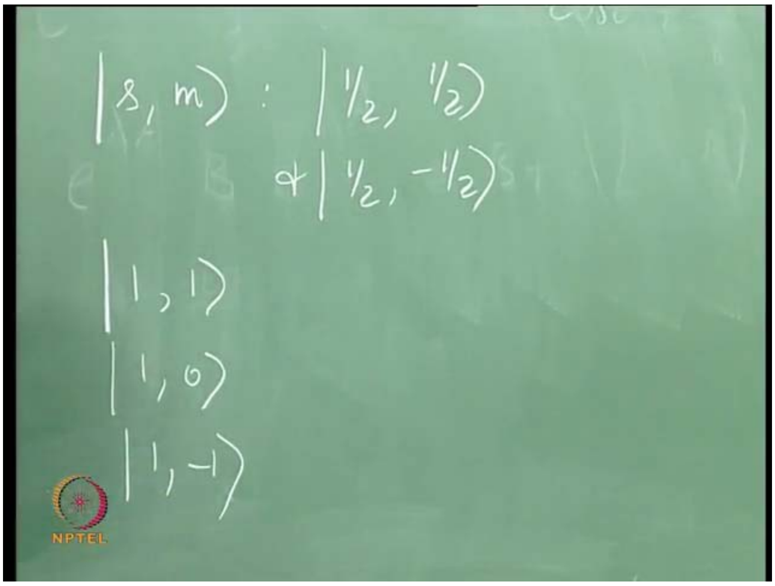

In the case of the spin doublet, we had half, half and half, minus half, so there was an s, m the same s with a minus m. This is what we had as the Eigen basis the spin Eigen basis, in the case of the spin doublet. And we simply noted the fact that m took values minus s to plus s in steps of 1. Similarly, when we did the 3 level atom problem and we had the same algebra as the angular momentum algebra or the spin algebra. We had 3 basis states and the values were 1, 1 1, 0 and 1, minus 1.

So when s was 1, again we realize that m took values minus s to plus s, in steps of 1. We would like to see if in general, this can be established. (Refer Slide Time: 14:30) In other words we want to find the value, the set of values that lambda can take and the set of values that m can take. It is clear, that lambda is simply a function of J; m has to be determined in terms of lambda.

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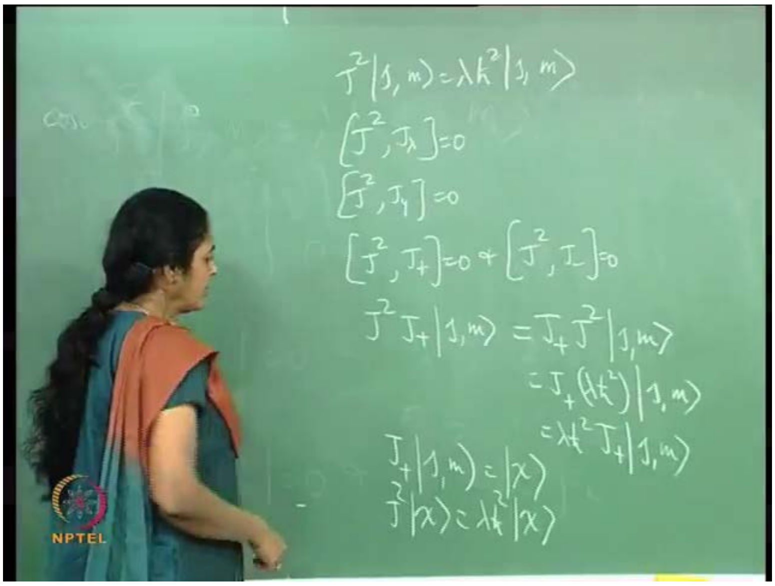

We proceed to do this as follows. Let us start with the fact that J squared commutes with J x and J square commutes with J y. Therefore, J squared commutes with J plus and J squared commutes with J minus. Recall that J plus is J x plus i J y and J minus is J x minus i J y. I therefore have, J squared J plus this operator acting on the state J, m to be equal to J plus J squared acting on the state J, m. I know that this object is J plus, times the number lambda h cross squared J, m. And since that is a number I can pull that out and I have J plus J m. Call the state J plus J m, as some ket chi it is now clear, that J squared chi is lambda h cross squared chi.

In other words we are saying that J plus acting on the state J, m, takes it to another state which I have represented by ket chi symbolically. That state is also an Eigen state of J squared with the same Eigen value lambda h cross squared, recall that J squared acting on the state J m, gave me Eigen value lambda h cross squared. Now, it seems that J plus J m this state, is also an Eigen state of J squared with the same Eigen value lambda h cross squared. And therefore, when J plus acts on the state J, m the ket J, m, it does not change the Eigen value corresponding to J squared.

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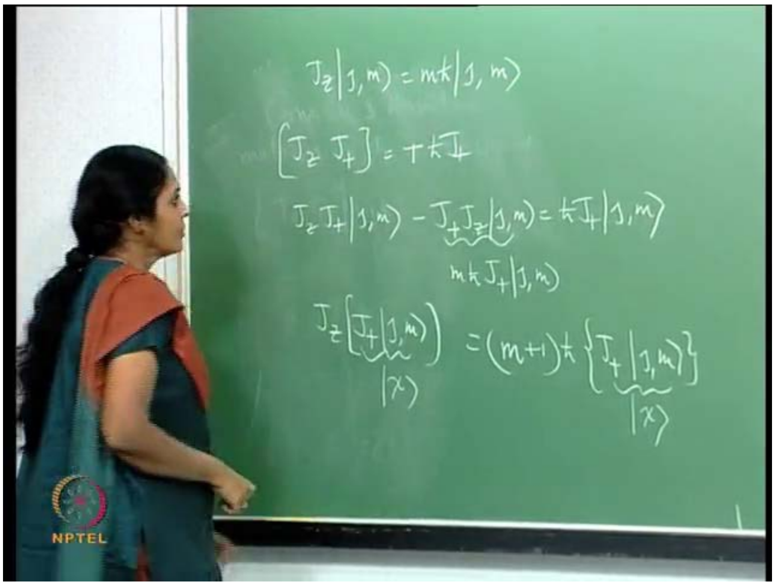

On the other hand, consider the commutator J z with J plus. Now, J z with J plus is plus h cross J plus, look at the state J z J plus J, m from this algebra I have that J z J plus J, m minus J plus J z J, m is h cross J plus J, m. Well certainly this object J z acting on J, m is m h cross, which is a number and therefore, I can pull that out and write the 2nd term in this manner. Therefore, I have J z J plus J, m, J z operating on this state which I labeled as ket chi symbolically.

This is equal to m plus 1 h cross J plus J, m this was ket chi. I have therefore, established that J plus, acting on the state J, m the resultant state here is an Eigen state of J z. But, with an Eigen value not m h cross but m plus 1 h cross. Recall, that J z acting on J m was m h cross J m. It is therefore, clear that the operation of J plus on the state J, m is as follows. It is not seem to change the J value but it does change m to m plus 1.

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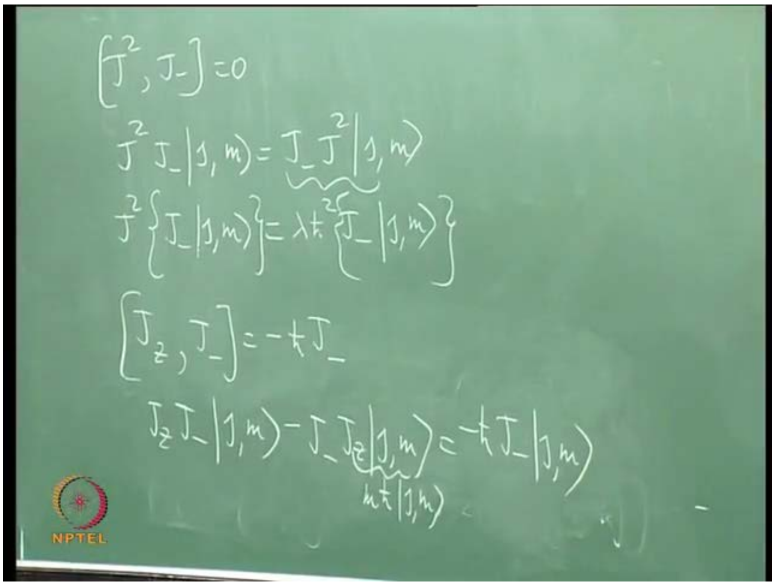

Now, let us look at what J minus does to the state J, m. I repeat this argument. I start off with the fact, that J square J minus is equal to 0. And therefore, J squared J minus J, m is equal to J minus J squared J, m. But, this is simply lambda h cross squared J minus J, m. Hence, I see that the state J minus J, m ket, is an Eigen state of J squared, with Eigen value lambda h cross squared. In other words, I have shown that J minus, acts on the state J, m without changing the Eigen value, that seems to be the same.

And since lambda is a function of J it looks like J is not changing when J minus acts on J, m. So whatever I said about J plus acting on J, m also holds for J minus acting on J, m in the sense that the J value does not change. On the other hand I know that the commutator, J z with J minus is minus h cross J minus and therefore, if I acted this commutator on the state J, m I get the following.

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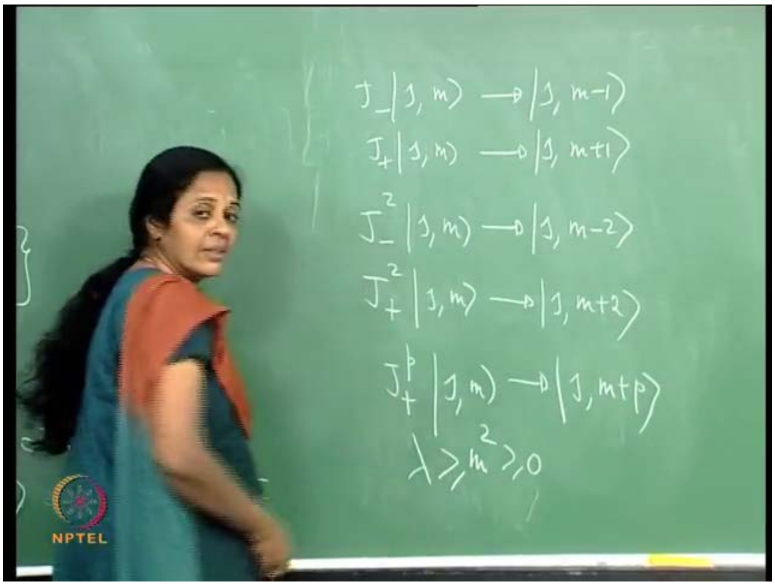

I know that this is m h cross J, m. It is therefore clear that, when J minus acts on the state J, m it reduces the m value by 1. Takes it to the state J, m minus 1 and J minus acts on J, m, and when J plus acts on J, m it takes it to a state where the m value increases by 1. So in this sense, that we say that J plus and J minus are the lowering and the raising operators respectively. They leave J untouched, but change the third component value m J minus decreases it by 1 and J plus increases it by 1, if it acts once.

It is clear that if J minus acts twice on J, m, this means it acts twice on J, m it takes it to the state J, m minus 2 and if J plus acts twice takes it to the state J, m plus 2. In this sense they are raising and lowering operators or ladder operators. However, this cannot go on indefinitely because if J plus acts p times say on J, m from whatever I have shown you till now, this takes it to a state J, m plus p but I have the constraint, that lambda corresponding to the Eigen value of J squared is greater than or equal to m squared and that of course, is greater than or equal to 0.

We began there so you see this cannot be increasing indefinitely, because there should come a stage where if I denote m plus p by m prime there should come a stage corresponding to some p, where m prime square cannot be greater than lambda. So there must be an upper value, beyond which J plus cannot increase the 3rd component value by 1.

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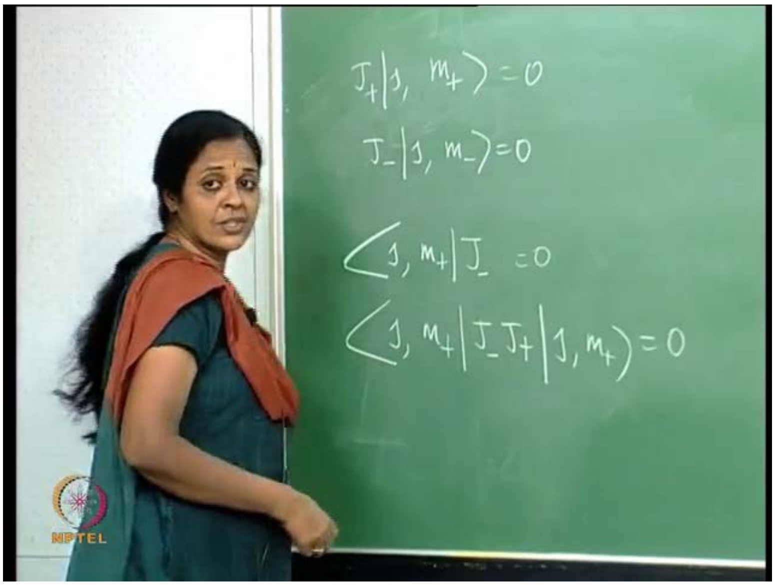

In other words I have the following condition now, I realize the following that there is an upper bound of m which I will call m plus such that when J plus acts on the state J, m plus, it gives me 0 because otherwise m plus squared could be greater than lambda and lambda is a function of J. Similarly, I know that there must be a lower bound for the same reasons, there is an m minus every time J minus acts on m it reduces the value of m by one but then there must be an m minus, which fixes the lower bound because J minus acting on J, m minus must be equal to 0.

Simply because (Refer Slide Time: 25:01) this constraint has to be satisfied. Look at the first relation I therefore, have J m plus J plus dagger which is J minus is equal to 0, consider the object J m plus J minus J plus J m plus, it is clear that this is 0 because J plus acting on J m plus is 0 similarly, the J minus acting on this side J m plus is 0.but, I can write J minus J plus in terms of J squared and J z in the following manner.

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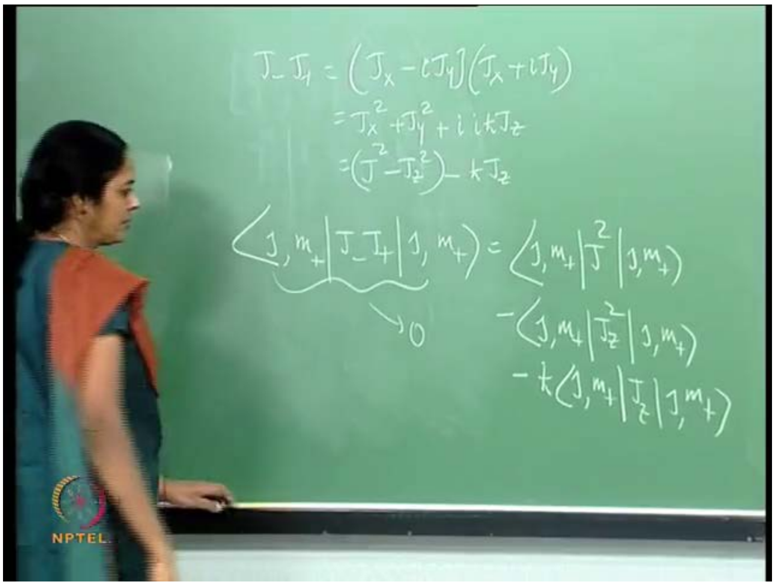

J minus J plus when written in terms of J x and J y, is J x minus i J y times J x plus i J y in this object is J x squared plus J y squared plus i commutator of J x with J y which is i h cross J z. But, that the same as J squared minus J z squared minus h cross J z. I therefore, feed that in there and I have, J minus J plus sandwiched between these states is the same as J m plus J squared J m plus plus minus J m plus J z J m plus minus h cross J m plus that is a J z squared this is a J z J m plus I have this. And this object I know is 0. That these J m plus is an Eigen state of J squared J z squared and J z. So I might as well pull out the Eigen values.

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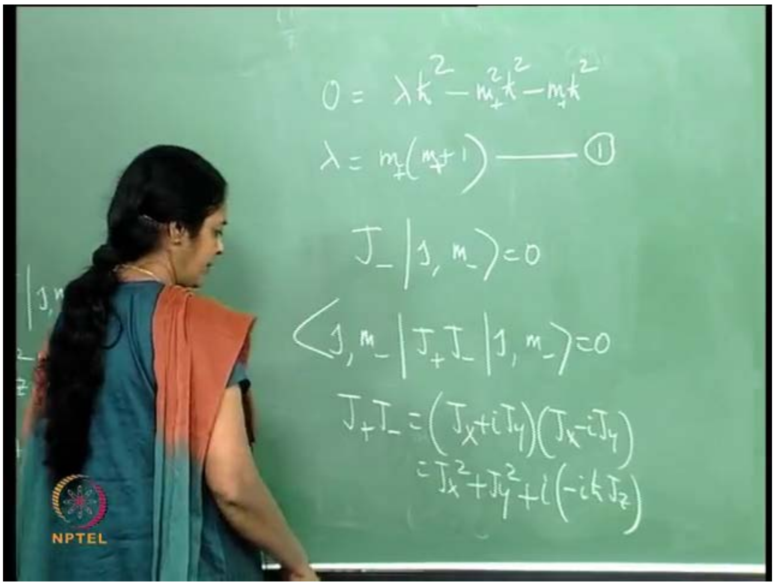

And then I have the following I have 0 is equal to, (Refer Slide Time: 29:19) the first term gives me a lambda h cross squared these are normalized states. So the inner product of J, m plus bra with J, m plus ket is 1 and therefore, I Just get lambda h cross squared from the first term, minus m squared h cross squared from the second term, (Refer Slide Time: 29:19) minus m h cross squared from the third term, which tells me that lambda is equal to m into m plus 1 except that in all these cases it is m plus, it is m plus times m plus 1.

So, this is the first relation I get which relates lambda to the upper bound on m which I have denoted by m plus, I can repeat this calculation starting with the fact, that J minus acting on the state J, m minus is 0. Therefore, I have this object to be 0 but once more I can write this in terms of J squared J z squared and J z because J plus J minus product is J x plus i J y times J x minus i J y, that is J x squared plus J y squared plus i commutator of J y with J x, the commutator of J y with J x is minus i h cross J z, this therefore, tells me the following.

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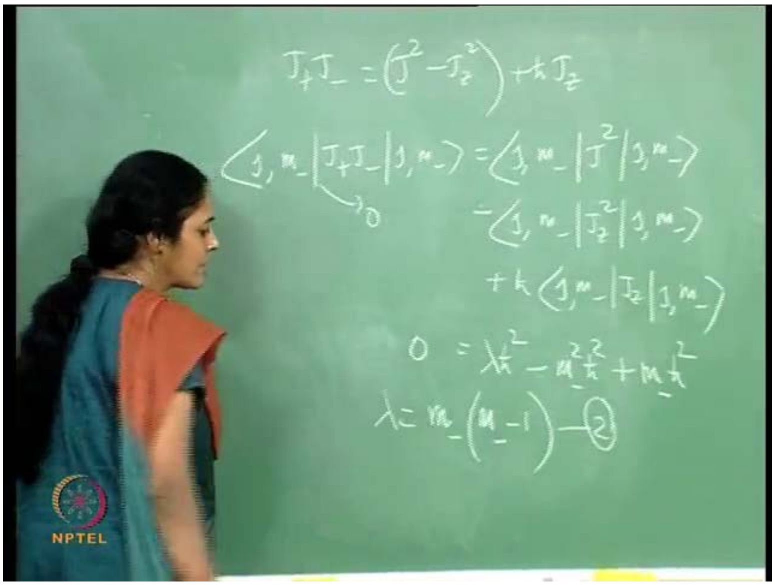

I have J plus J minus equals J squared minus J z squared (Refer Slide Time: 31:03) because I have written J x squared plus J y squared in that fashion, plus h cross J z, taking the expectation value in this manner. This object is 0 remember is simply the following, as before I have this minus this object plus h cross. Well that gives me, a J squared a lambda h cross squared it pulls out lambda h cross squared this pulls out plus m squared h cross squared except that m is now m minus, comes with a negative sign because of the negative sign here plus m minus h cross squared. Since, this object is equal to 0.

I have lambda is equal to m minus times m minus minus 1 and that is a second relation that I have. (Refer Slide Time: 31:03) So I have two descriptions of lambda lambda is m plus times m plus plus 1 is also equal to m minus times m minus minus 1. I emphasize that m plus is the higher bound on the value that m can take for a given J and m minus is the lowest value that m can take for the same value of J. (Refer Slide Time: 25:01) Since these two are equal I have the following.

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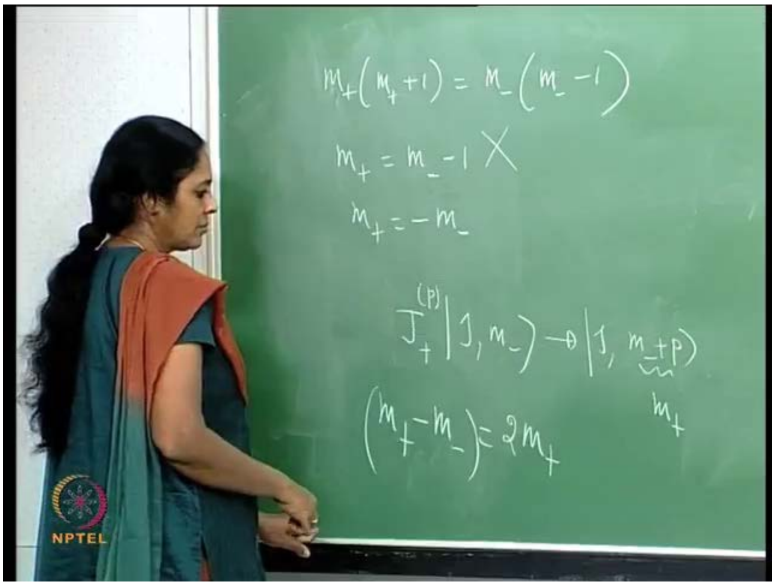

I have m plus times m plus plus 1 is equal to m minus times m minus minus 1. Well this tells me the following, I can have one of two solutions. I can have m plus is m minus minus 1, certainly that is satisfied or m plus is equal to minus m minus this is another possibility. We need to understand that this is not possible, this is not possible because if I had started with the state J, m minus and repeatedly applied, J plus on it to the power of p perhaps I would have landed in a state m minus plus p and m plus was the highest value that I it could have taken.

As a result of which m plus is greater than m minus the difference between m plus and m minus has to be an integer, because every time J plus acts on this state it increases the m value by 1. And therefore, I know that the difference m plus minus m minus is an integer. I also know that m plus is the upper bound and m minus is the lower bound on m and therefore, this relation is not true whereas, this is more acceptable. And therefore, if m plus is equal to minus m minus m plus minus m minus is 2 m plus. The solution is this, since lambda is simply a function of J. (Refer Slide Time: 35:24) I now have m plus minus m minus is 2 m plus

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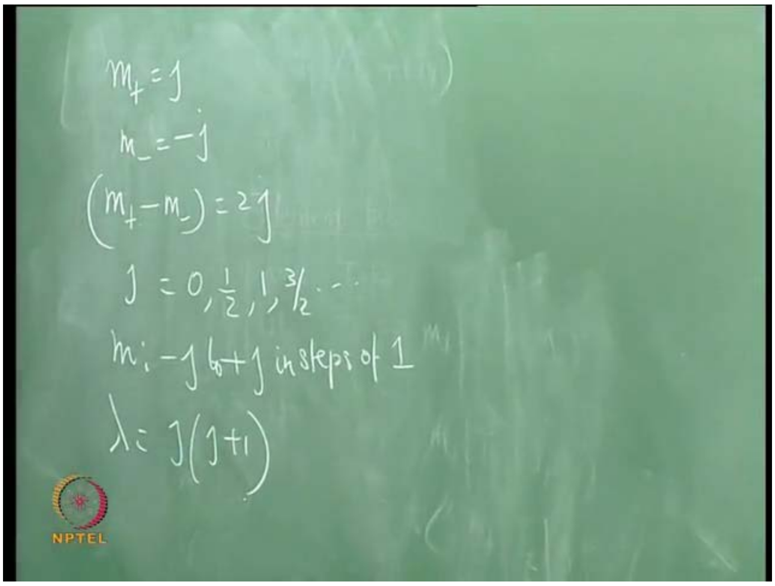

And m plus is J and therefore, m minus is minus J. And therefore, m plus minus m minus is 2 J. But, this is an integer. Therefore J takes values 0 half 1 3 by 2 and so on. If this is 0 of course, it is 0 if this 1 it is half, if it is 2 it is 1 and so on. And m can take values minus J to plus J in steps of 1 and lambda itself is therefore, J into J plus 1 (Refer Slide Time: 35:24) it is J into J plus 1. So what we have established is the following, simply on the basis of the s u 2 algebra

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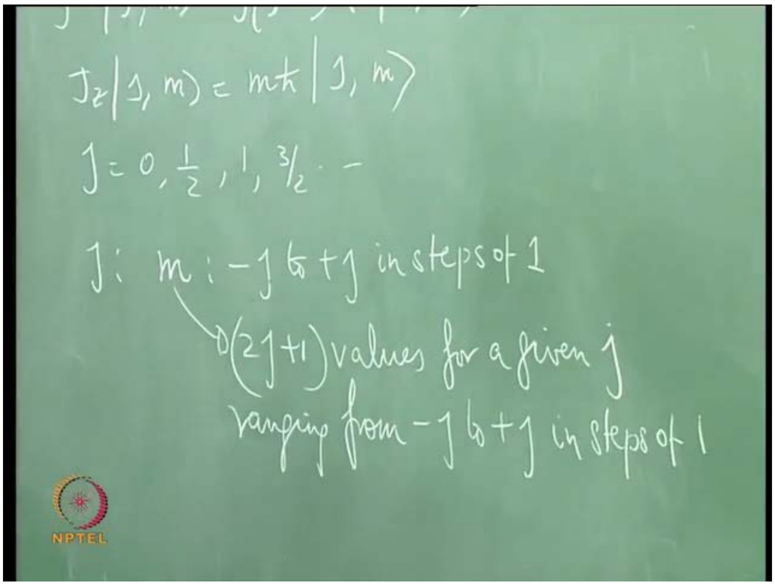

we have established that J squared acting on J m is J into J plus 1 h cross squared J, m, J z acting on J m is m h cross J, m J takes values 0 half 1 3 by 2 and so on. It takes integer or half integer values, for a given value of J m takes values minus J to plus J in steps of one. You will recall that this is m plus and that was m minus the upper and lower bounds on m. And every time J minus acted on a state it reduced the value of m by 1 if J plus acted on the state it increased the value by 1. And therefore, m takes 2 J plus 1 values over the range minus J to plus J, for a given J ranging from minus J to plus J in steps of 1.

And indeed this is what we had verified, when we did the spin doublet problem equivalently the two level atom problem. When we had the two basis states half, half if J was half call it s then because it was spin. The second entry was m, so we had two states half, half and half, minus half. So m took values minus half to plus half in steps of 1. When we did the three level atom problem, we had the states 1, 1 1, 0 and 1, minus 1 once more for given value of J in this case it was 1, m took values minus 1 0 and plus 1, ranging from minus J to plus J in steps of 1. We can now proceed, and find out what exactly is the action of J plus on, m. We have realized that J plus is raising operator it takes J, m to J, m plus 1 but what are the coefficients what is the coefficient outside.

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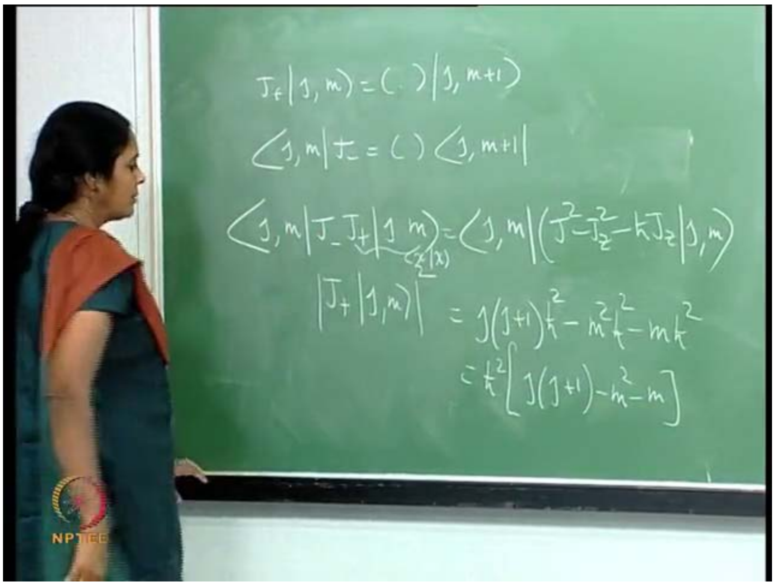

So for doing that we need to find out this object, we need to find out this object. What is it that multiplies this ket J, m plus 1 this can be handled in the following fashion; this is what I have when I take the Hermitian conjugate. So I consider the object J m J minus J plus J m. It is pretty clear that J plus will increase the m value to m plus 1. And therefore, this would go to a state J, m plus 1. But, J minus will act on that state and decrease the m value by 1 again, bringing it back to J, m and then the orthogonality relation, the fact that J, m inner product with J, m is 1, that will give us a number.

That is precisely what is happening, and as we did earlier we can substitute for J minus J plus. So J minus J plus was J squared minus J z squared minus h cross J z this is what I have. It's also clear that this object can be written as J plus J m squared, not squared because J plus J m is like some ket chi this is like the bra. And therefore, this is like mode square of that object. The coefficients are in general the coefficients here are in general taken to be real and therefore, if I evaluate the number here that will be the square of the coefficient arrival need here. So, this object is J into J plus 1 h cross squared minus m squared h cross squared minus m h cross squared. This is the same as h cross squared J into J plus 1 minus m squared minus m; this is the square of the number that comes here.

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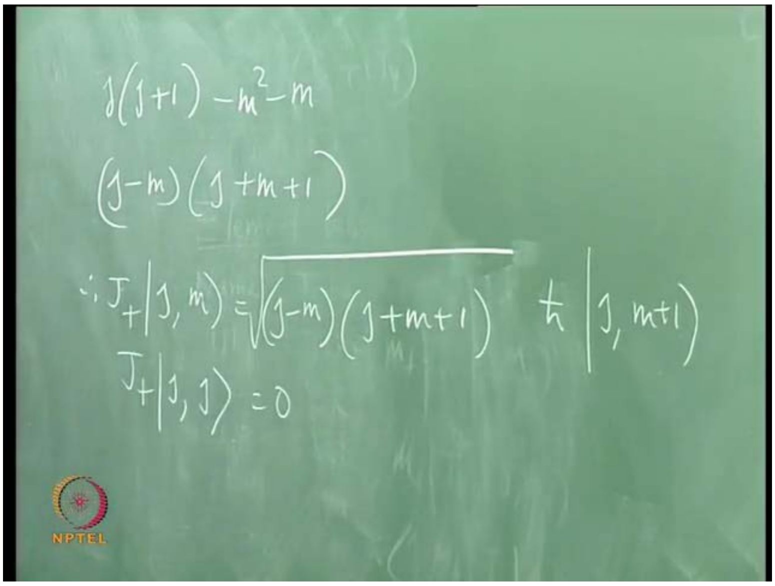

And therefore, the number that comes here can be easily evaluated. I would like to write J into J plus 1 minus m squared minus m as J minus m into J plus m plus 1 that is a way of writing it. And therefore, J plus acting on J m is the square root of J minus m into J plus m plus 1, there is an h cross J m plus 1 the h cross was because I needed to take the square root of this object and therefore, I got the h cross. The h cross is outside the square root. It is clear therefore, that when J plus acts on the state J, J which is the maximum value that m can take that gives me 0, which is consistent with what I wanted that J, J was the upper most state that I can have and J plus cannot raise the m value beyond that.

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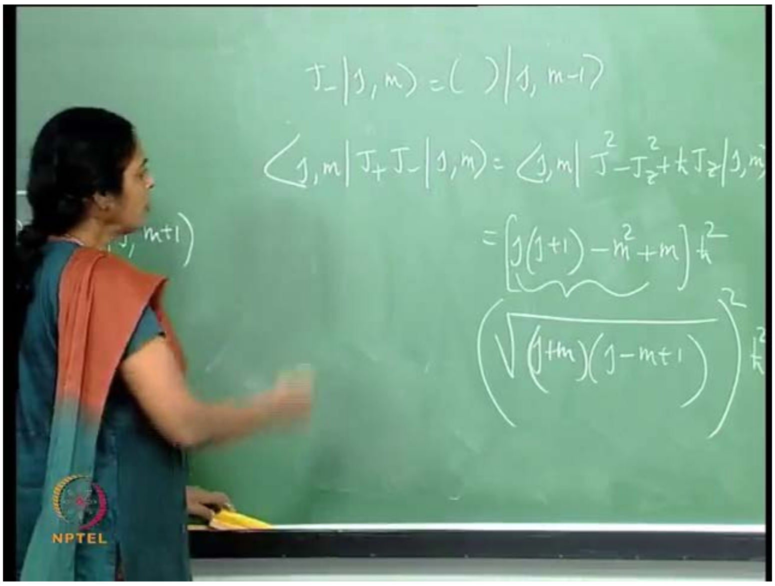

Similarly, I can find out the effect of J minus acting on J, m. Quite apart from taking this state to the state J, m minus 1 what is it that multiplies it what is the constant that multiplies it. Some number so, I need to find this number as before I try to find this, that is going to be J, m J plus J minus is J squared minus J z squared plus h cross J z J, m as before, I have this to be J into J plus 1 minus m squared plus m h cross squared. And this can be written as root of J plus m into J minus m plus 1 it is simply a way of writing it, the whole squared, from which it is clear.

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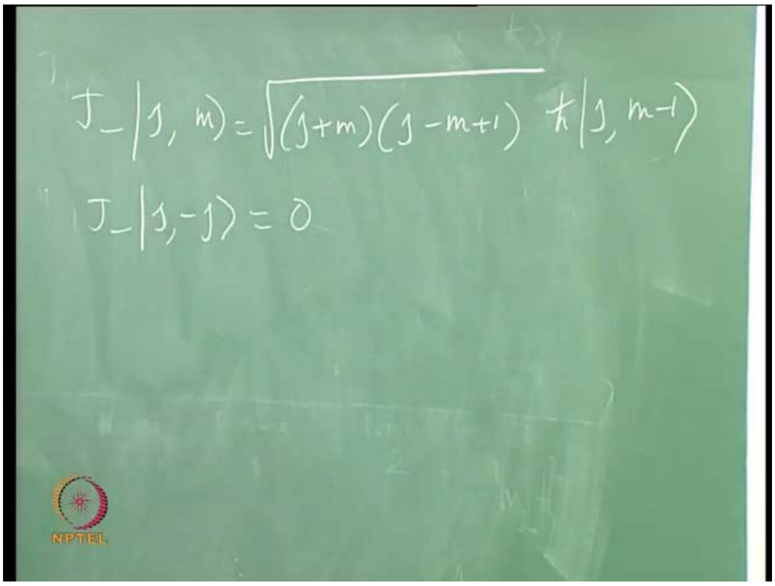

That J minus acting on the state J, m gives me root of J plus m into J minus m plus 1 h cross J, m minus 1. Once more it is clear, that when J minus acts on the state J, minus J, that is the lowest value that m can take I refer to this as m subscript minus earlier on when I did the derivation, this object is automatically 0 because J minus J is 0 and that is what I want. There is a lower bound and J minus acting on that state gives me 0.

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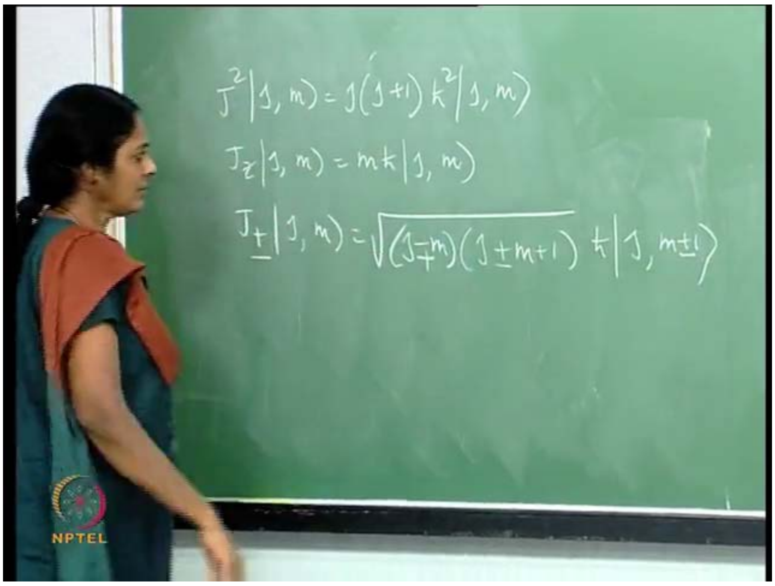

To summarize therefore, I have gotten for you the angular momentum algebra. And the manner in which these states act the operators acts on the states, I have J squared acting on J m is J into J plus 1 h cross squared J m J z acting on the state J m is m h cross J m, m taking 2 J plus 1 values ranging from minus J to plus J in steps of 1. I have J plus J m is root of J minus m into J plus m plus 1 h cross J m plus 1 and J minus acting on J m gives me this. So this is the manner in which, the angular momentum operators act on the basis states J, m, what I said holds for spin matrices where J is replaced simply by the label s and orbital angular momentum, where J x J y J z are denoted in this context as l x l y and l z, being x y and z components of the orbital angular momentum operator.