Quantum Mechanics -I

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Lecture - 12
Exercises in Finite Dimensional Linear Vector Spaces

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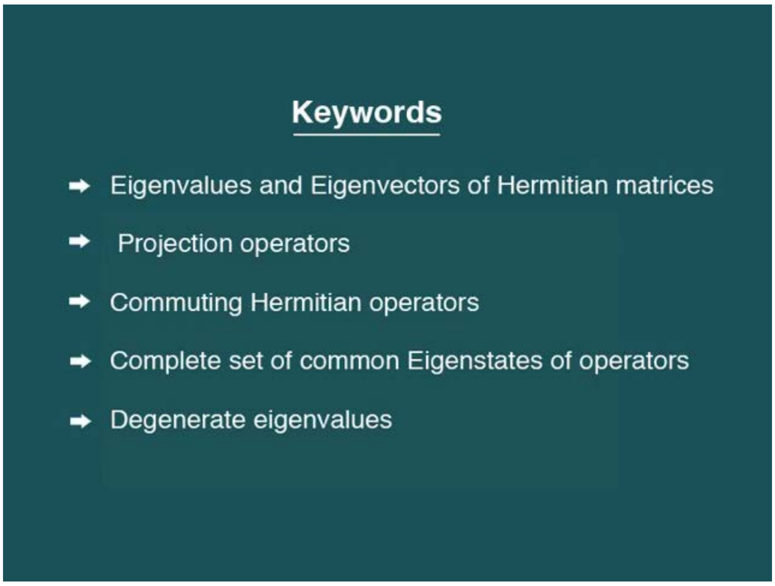

Until now, I have given you a set of lectures on finite dimensional linear vector spaces, emphasizing concepts like, operators, operator algebra, states, the role of matrices, matrix representation of operators, and so on. The importance of Hermitian operators, the importance of unitary operators, I have demonstrated many of these concepts, using the 2 level atom and the 3 level atom as examples. We have also looked at the spin half system or the spin doublet, the spin triplet and so on. The concepts that I have been taught till now would perhaps become clearer if we did a set of problems at this point. A set of exercises or tutorials and brought out the importance of some of these concepts better, by working out certain specific problems and certain aspects of finite dimension linear vector spaces, using specific examples.

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So, we would just have exercises in finite dimensional, linear vector spaces. So, today we will work out some exercises. The first one would be a recapitulation of what has already been done during the lectures. I told you that the Eigen values of a Hermitian matrix are real and Eigen vectors, corresponding to distinct Eigen values of a Hermitian matrix are mutually orthogonal. Let me quickly recapitulate the proof of that statement.

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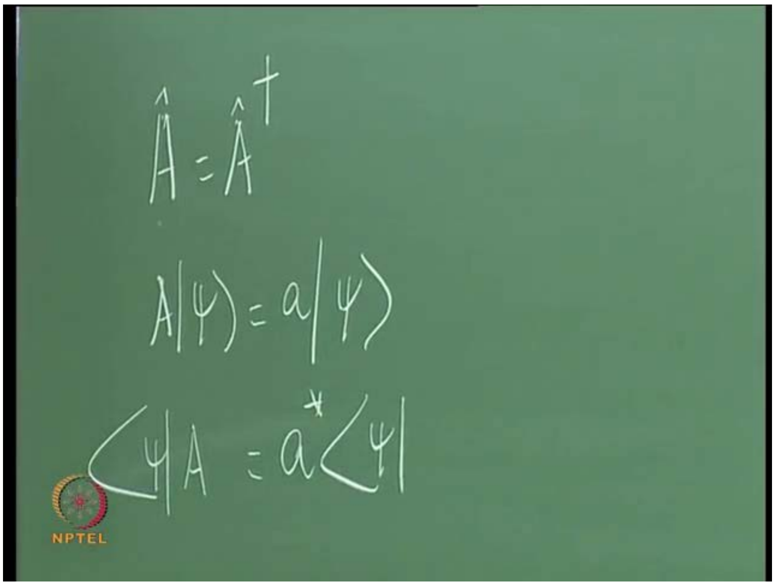

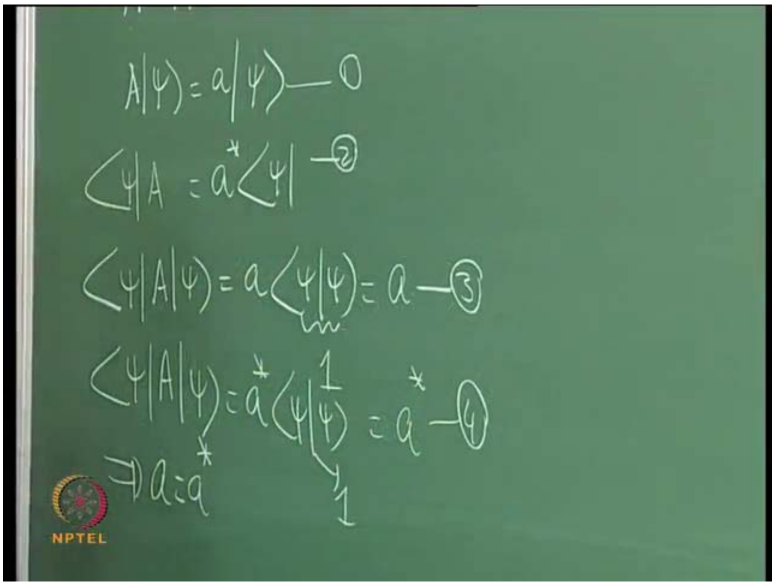

Let A, B, a Hermitian matrix, of course, in order to show that this is an operator which is represented by a matrix. We could put a hat on top. I would for the sake of simplicity, dispense off with the hat and we will remember, at the back of our mind that A is a matrix and a Hermitian matrix. The dagger would represent, transpose interchange the rows and the columns and take the complex conjugate of every element.

Suppose psi, were an Eigen state of A and the corresponding Eigen value is a. Then surely this is true. If I take the dagger, since A dagger is the same as A. I have this, the aim is to show that A is equal to a star and that can be simply done as follows. Start with equation one and do this. Let us imagine that psi is normalized to 1, then the expectation value of A in the state psi, is simply a, but I could have started with equation two and I could have done this. That is the same as a star psi, psi and this object is 1. From equations three and four, it is clear that psi A psi is equal to a and also equal to a star, which implies that a is equal to a star. So, Eigen values of Hermitian matrices are real. Let me also quickly recapitulate, how the Eigen vectors are mutually orthogonal to each other, if they correspond to different distinct Eigen values of a.

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So, for that purpose let us take, two Eigen vectors. Satisfying, this Eigen value equation, a 1 is not equal to a 2. That is given, they are distinct Eigen values, also a 1 and a 2 are real because a is Hermitian. Then from equation one I get this. I also have, so this object is there, I also have the bra of this, psi 1 A is equal to a 1 psi 1. Now, I could well do, psi 1 A psi 2, A psi 2 is a 2 psi 2. And A psi 1, is a 1 psi 1. So I have these two equations. On the one hand I worked, with A on psi 2, pulled out an a 2 and I had an inner product of psi 1 with psi 2. On the other hand I know that because A is Hermitian, A dagger is the same as A and here I have psi 1 A, should have normally been a 1 star bra psi 1. But, because a 1 is real, I have just written down a 1 there. And therefore, I have a 1 psi 1 psi 2.

This clearly means, that a 2, inner product of psi 1 with psi 2, should be the same as a 1 inner product of psi 1 with psi 2. But, a 1 and a 2, are not equal to each other, that is given to us implies, psi 1 psi 2 is 0. And therefore, Eigen vectors corresponding to distinct Eigen values of a Hermitian matrix, are mutually orthogonal. So, this is a quick recapitulation, of what was already done, during one of the lectures. I want to work more with Hermitian matrices. The next thing I want to show is the following.

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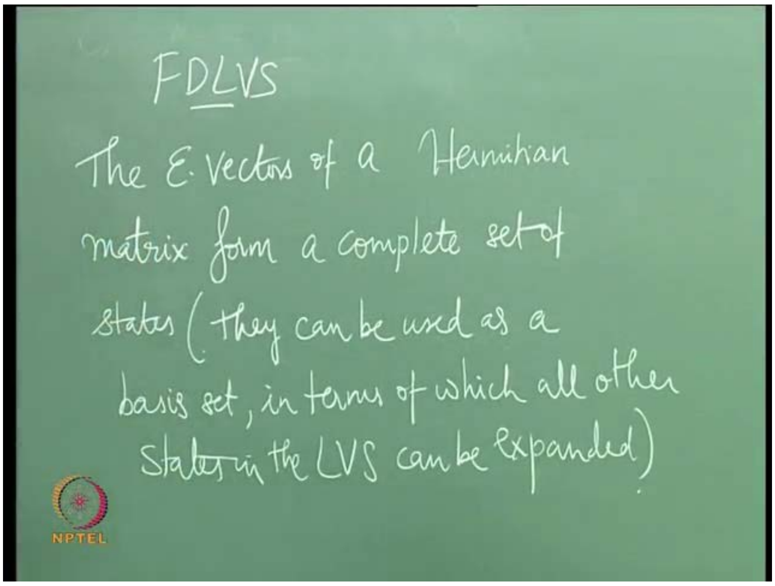

We are working with finite dimensional linear vector spaces. And I want to show, that the Eigen vectors of a Hermitian matrix, form a complete set of states. In other words, they can be used, as a basis set, in terms of which all other states in the linear vector space can be expanded. So, basically I need to show that the Eigen vectors of a Hermitian matrix span the entire linear vector space. They form a complete set. To prove this I do the following.

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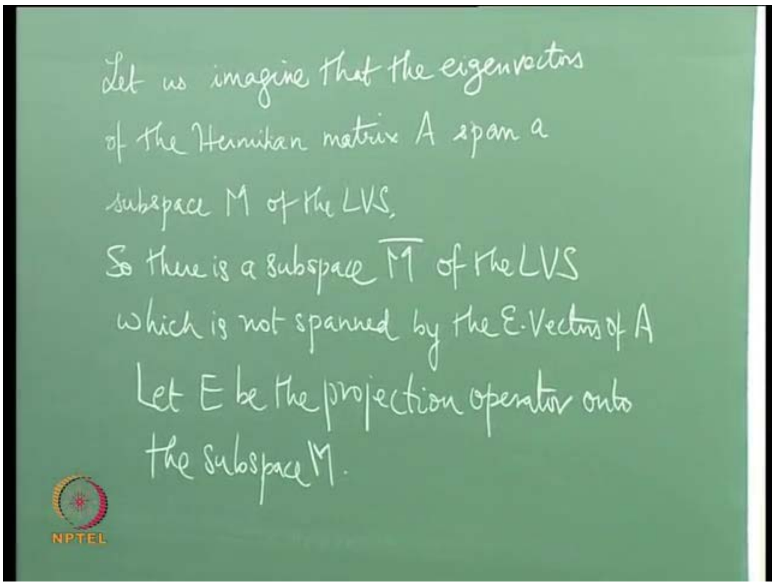

Let us imagine that the Eigen vectors of the Hermitian matrix A, span a subspace M of the linear vector space. In other words we are now saying that, let us imagine that the Eigen vectors do not span the entire linear vector space. All states in the linear vector space, cannot be expanded as a superposition of the Eigen vectors of the Hermitian matrix. The idea is to show that there is a contradiction, if we imagine so. So, there is a subspace M bar of the L V S, which is not spanned, by the Eigen vectors of A. The total linear vector space is composed of M and M bar. Now, consider the projection operator, let E be the projection operator onto the space, subspace M. Let me quickly recapitulate, what is meant by a projection operator? We have seen projection operators both in the context of the 2 level system and the 3 level atom.

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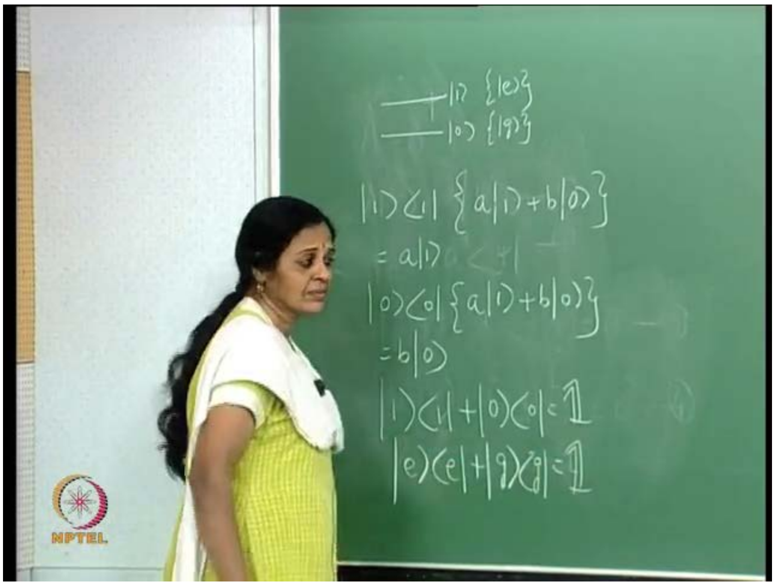

So. if you recall in the case of the 2 level atom the projection operator 1, 1 for instance, would act on any arbitrary state, a times ket 1 plus b times ket 0, for instance. So, these are the 2 levels 0 and 1.Iif you wish you can call this g and you can call this e. So, by 1, 1 in this notation I mean e, e that projection operator, certainly gives you a times ket 1. So, this is the projection operator on to the subspace of the linear vector space, which is spanned by the basis state 1. Similarly, 0 0 acting on any arbitrary state, gives me b 0. And this is therefore, the projection operator on to the subspace span by ket 0. It is evident that 1 1 plus 0 0 is the identity. In the language of ket e and ket g, we showed that e e plus g g was the identity. (Refer Slide Time: 08:48) So, here we have a situation, where we do not have just 2 basis states. But, in general we have many states spanning the space. Now, let E be the projection operator on to the subspace M.

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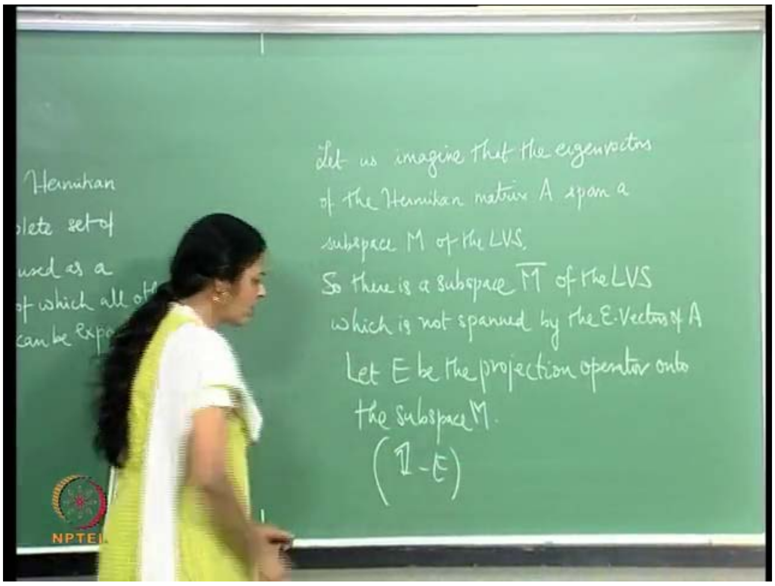

Then surely identity minus E, is the projection operator on to the subspace M bar. (Refer Slide Time: 11:17) That is like saying, that if e e is the projection operator on to this space, spanned by ket e. Identity minus e e is the projection operator on to the space spanned by ket g. In the 2 level atom problem, that would be the corresponding statement.

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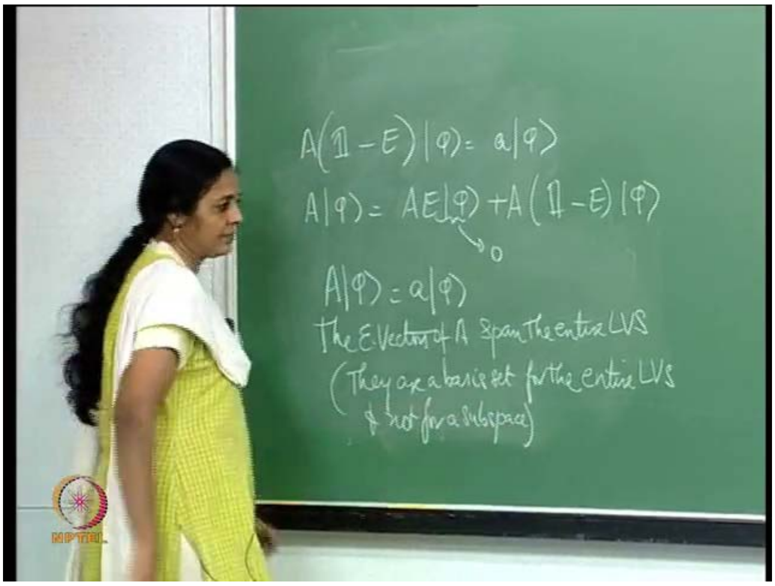

Now consider the operator A times, identity minus E. Since, identity minus E is the projection operator which acts on any state, in the linear vector space and projects it on to the subspace M bar. A, M bar is an operator, which surely has at least one Eigen vector with the corresponding Eigen value. And let us imagine that that Eigen vector is phi, with corresponding Eigen value a. This is a matrix, an operator with the matrix representation I would therefore expect it to have, it is a Hermitian matrix. I would expect it to have 1 Eigen value at least, with the corresponding Eigen vector. But, this object therefore phi, is an Eigen vector, corresponding to this operator. Now consider A phi. A phi can well be written, as A, E phi, plus A times identity minus E phi. But, phi is a vector in M bar. Therefore, E phi is 0, because E is the projection operator on to the state M.

So, there is nothing to project, into the subspace M. And therefore, A phi, is the same as A times identity minus E phi, which is A phi. In other words, what we have seen is that a state which belongs to M bar it is an Eigen state of A with Eigen value A. But, all Eigen states of A, we imagined would span only the subspace M. (Refer Slide Time: 12:56) Therefore, there is a contradiction. Here we have a state phi, which initially we said belong to the subspace M bar but we have now seen that it is one of the Eigen vectors of A. And therefore, the Eigen vectors of A, span the complete linear vector space, the full linear vector space. In other words, they are a basis set, for the entire linear vector space and not for a subspace. This is an important point, because we have just established that the Eigen vectors, of a Hermitian operator span the entire linear vector space. In other words they act as the basis set, in terms of which every state in the linear vector space, can be expanded.

Now, we move on to the next problem, which is a very important problem and that is to show. That if two Hermitian operators commute you are guaranteed to find a complete set of common Eigen states, of these two Hermitian operators. So, that is the next problem and this is something which should be proved systematically.

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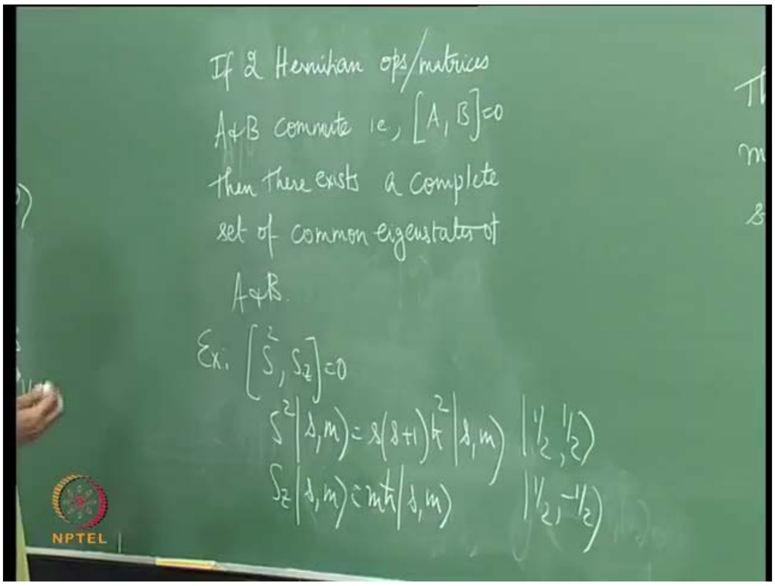

So, the statement is this. If 2 Hermitian operators could well be matrices A and B, commute with each other, that is commutator of A with B is equal to 0. Then there exists a complete set, of common Eigen states of A and B. You will recall that this 2 has been established in the context of the spin system, where we had two Hermitian operators s squared and s z, which commuted with each other. So, we had S squared S z commutator equal to 0. And we found a complete set of common Eigen states, labeled by the two quantum numbers s and m and we certainly demonstrated in the case of the 2 level atom, that this could be written as s into s plus 1 h cross squared s comma m, and S z acting on s m is m h cross s m.

So, this is a complete set, of common Eigen states. Where S takes a certain value and M takes values, from minus s to plus s in steps of one. While we did not establish, that M indeed takes values in general, between minus s and plus s. We certainly demonstrated that is what happened in the case of the 2 level atom, where we had two states half half and half minus half. So, that for s is equal to half m took values, plus half and minus half. That is plus s to minus s in steps of one. We now have to establish this statement. That in general, if two Hermitian operators A and B commute, then you are guaranteed, that there exist a complete set of common Eigen states of A and B. So, that if you measure A and B, in the system simultaneously, the system will collapse to one of this complete set of common Eigen states after the measurement.

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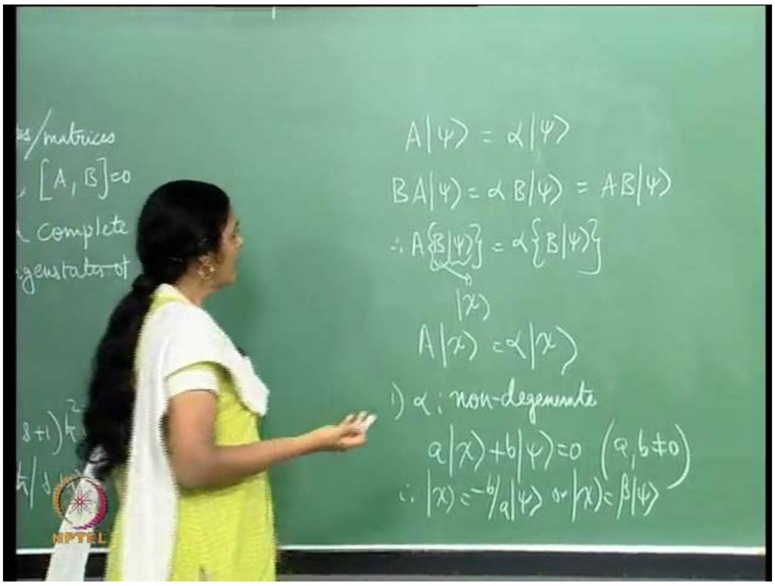

(Refer Slide Time: 17:24) We prove this as follows. Let, psi be an Eigen state of A, with Eigen value alpha. Then B A psi, is alpha B psi but B A psi, is also equal to A B psi because A and B commute with each other. Therefore, A, B psi, is alpha B psi, where alpha is the Eigen value of A, corresponding to the Eigen vector psi. We have then shown, that B psi this is another state which I will call chi, because it is an operator acting on a state that gives me another state. We have established therefore, that B psi is also an Eigen state of A, with Eigen value alpha. This means the following; there are many cases here. The 1st case is alpha is non degenerate, non degenerate Eigen value means, that there is exactly one Eigen vector corresponding to that Eigen value.

So, the 1st case is alpha non degenerate. There is no more than one Eigen vector of A corresponding to that Eigen value alpha. Degenerate would mean that there is more than one Eigen vector, may be two may be three, may be many more. All these Eigen vectors are Eigen vectors of A corresponding to the same Eigen value alpha. So there are cases to be considered the 1st case is alpha is non degenerate. Which means, that if you have A psi is alpha psi and A chi is also alpha chi and alpha is non degenerate, chi is clearly linearly dependent on psi. In other words some a chi plus b psi equal to 0, a b not equal to 0. Therefore, chi is minus b by a psi or chi is equal to some beta psi, that is the only way by which this could have happened.

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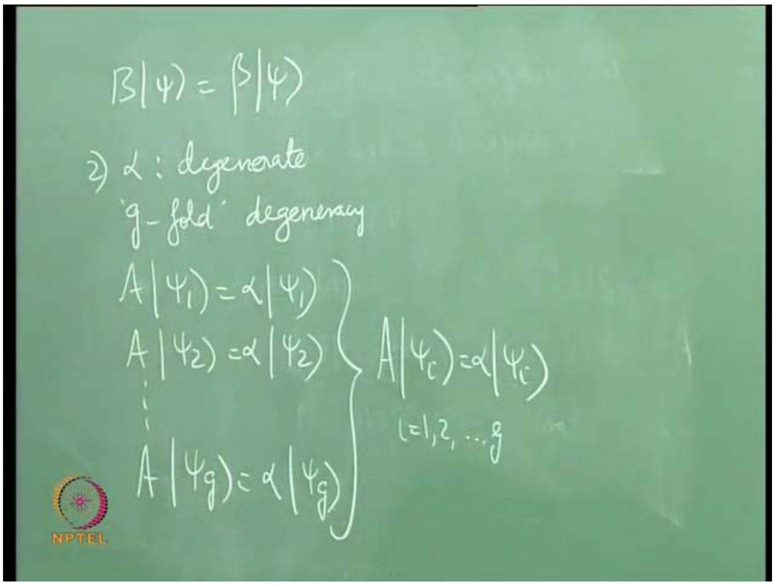

Clearly this means, writing chi in terms of B psi, B psi is equal to beta psi. What is it that we have established? (Refer Slide Time: 20:31) If psi is an Eigen vector of the Hermitian operator A, we are only concerned with Hermitian operators, if psi is an Eigen vector of the Hermitian operator A with Eigen value alpha. And alpha is non degenerate and if B is a Hermitian operator which commutes with A, psi is also an Eigen vector of B. In general with the different Eigen value beta.

We have already shown earlier that the Eigen vectors of A form a complete set, which spans the entire relevant linear vector space. Since the set of psi's form a complete set of Eigen vectors of A and since these are also Eigen vectors of B. It is clear that in this case, A and B, have a complete set of common Eigen states. So, the 1st part was proved earlier that the Eigen vectors of a Hermitian operator, in a finite dimensional linear vector space form a complete set.

In the event that alpha is non degenerate it is clear that this complete set of Eigen vectors of A, is also the complete set of Eigen vectors of B and therefore, there is a complete set of common Eigen states. I have worked with just psi, you can look at the complete set of Eigen vectors of A and the argument would go through as such. The more non trivial case, is where alpha is degenerate. So, let us look at the 2nd case. Alpha degenerate, let us say g fold degeneracy. What does that mean? There are a set of g Eigen vectors of A, all of them corresponding to Eigen value alpha. In other words, A psi i is alpha psi i, i takes values 1, 2 all the way to g. So, that is a g fold degeneracy, the Eigen value is g fold degenerate. To illustrate whatever follows, we will look at the simplest case, where g is 2. So, we will have 2 fold degeneracy. Whatever I say for g is equal to 2, can be easily extended to n fold degeneracy. So, let us look at the case when g is 2.

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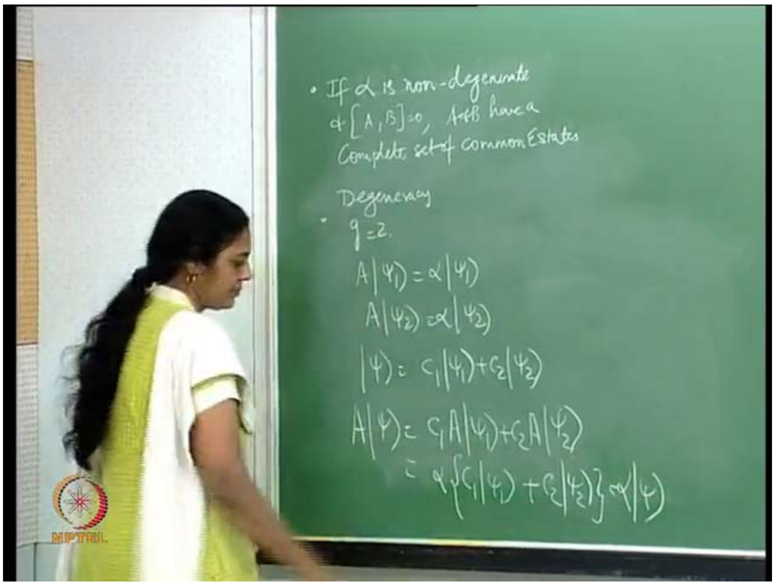

Before that we recall that we have already established, if alpha is non degenerate and A, B equal to 0, A and B have a complete set of common Eigen states. So, now alpha is we are looking at the 2nd case of degeneracy and there, we look at the simple example g is equal to 2. In other words there are 2 Eigen states psi 1 and psi 2, with Eigen value alpha. Consider a linear combination psi, written this way where C 1 and C 2 are for the moment, arbitrary constants no conditions on them. Then A psi is C 1, A psi 1, plus C 2, A psi 2. That is the same as alpha C 1 psi 1, because A psi 1 is alpha psi 1 and similarly, A psi 2 is alpha psi 2. Therefore, I have C 2 psi 2, which is alpha psi. So, any arbitrary linear superposition, of psi 1 and psi 2, is also an Eigen state of A with the same Eigen value alpha.

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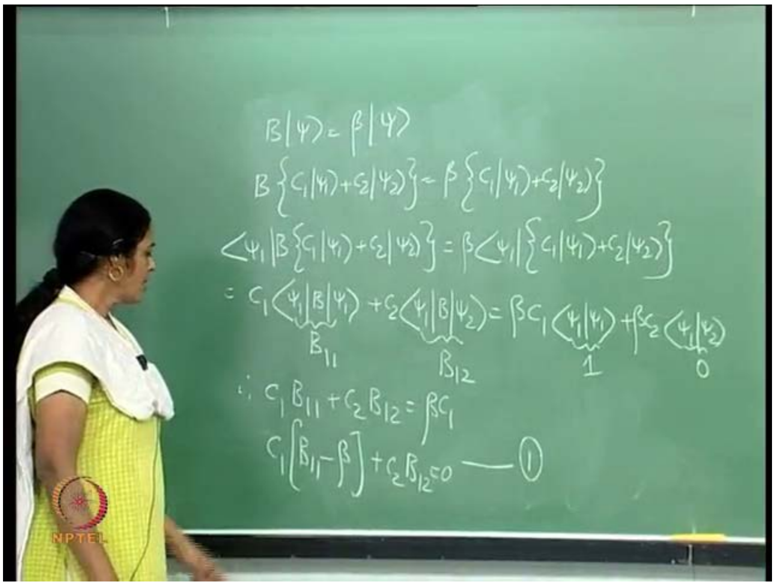

Now, let us look at B. We wish to know, if it is possible to find some set of values for C 1 and C 2, such that B psi is beta psi. In other words would all linear superpositions of psi 1 and psi 2, given in general by C 1 psi 1 plus C 2 psi 2. Would all such linear superpositions be also Eigen states of B, of course in general with the different Eigen value beta or are there conditions on C 1 and C 2. Is it at all possible, to find a complete set of common Eigen states of A and B, if alpha is degenerate. That is the problem that is being addressed. So, this is the same as saying, expanding psi in terms of psi 1 and psi 2. We have this.

Now, let us go ahead and do the following operation, psi 1 B, C 1, psi 1 plus C 2 psi 2. That is clearly beta psi 1, C 1 psi 1 plus C 2 psi 2. C 1 is a number which can be pulled out, psi 1 B psi 1, C 2 is a number. So the 2nd term gives me psi 1 B psi 2 and that is what I have in the left hand side. That is beta C 1 psi 1 psi 1 plus beta C 2 psi 1 psi 2. I have already assumed that psi 1 and psi 2 are orthogonal to each other and therefore, I have a short hand notation B 1 1 for this B 1 2 for that, this is normalized 1. So, I have C 1, B 1 1 plus C 2 B 1 2 is beta C 1. In other words C 1 times B 1 1 minus beta plus C 2 times B 1 2 equal 0, let me call that equation 1.

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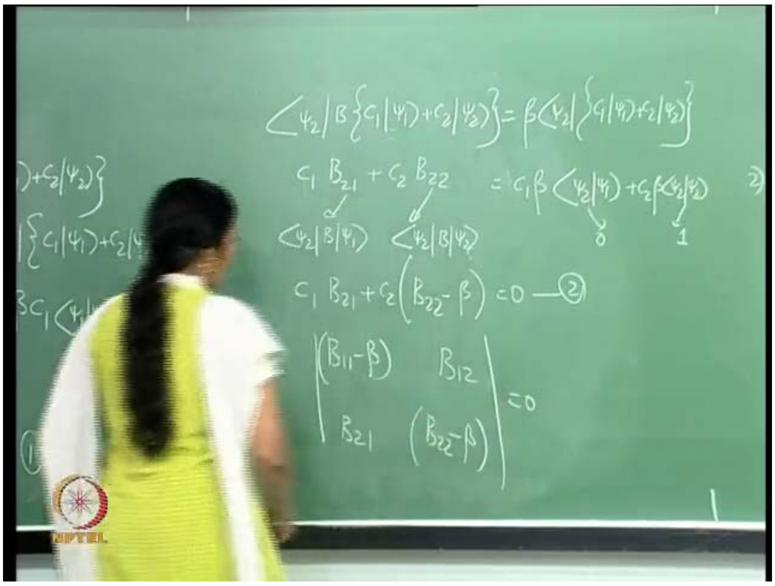

Similarly I can take, I can do the same operation except that, I can use psi 2 there instead of psi 1 and then what do I get? psi 2 B C 1 psi 1 plus C 2 psi 2. I am using psi 2 ok, is equal to beta psi 2, C 1 psi 1 plus C 2 psi 2. Once more the 1st term gives me C 1 B 2 1, where B 2 1 is simply psi 2 B psi 1. In my notation plus C 2 B 2 2 and B 2 2 is psi 2 B psi 2. This object is clearly equal to C 1 beta psi 2 psi 1, but that term is 0, plus C 2 beta psi 2 psi 2, but that term is 1. Therefore, I have C 1 B 2 1 plus C 2 B 2 2 minus beta is equal to 0 and I call this equation 2. (Refer Slide Time: 29:05) So, I have these two equations C 1 B 1 1 minus beta plus C 2 B 1 2 is 0 and remember that B 1 1 and B 1 2 are simply numbers. Similarly, C 1 B 2 1 plus C 2 B 2 2 minus beta is equal to 0.

Now clearly, if indeed it is possible to have this super position to be an Eigen state of B, with some Eigen value beta. The determinant of the coefficients given this way should be 0. This determinant should be 0, because I have two homogeneous equations, in C 1 and C 2 and in order to have a solution, I need to have this determinant equal to 0. And that is going to place conditions on beta and the beta is the Eigen value corresponding to the Eigen vector of B and therefore, that should tell me what the values of beta are. In other words I have reduced this to finding out the possible values of beta.

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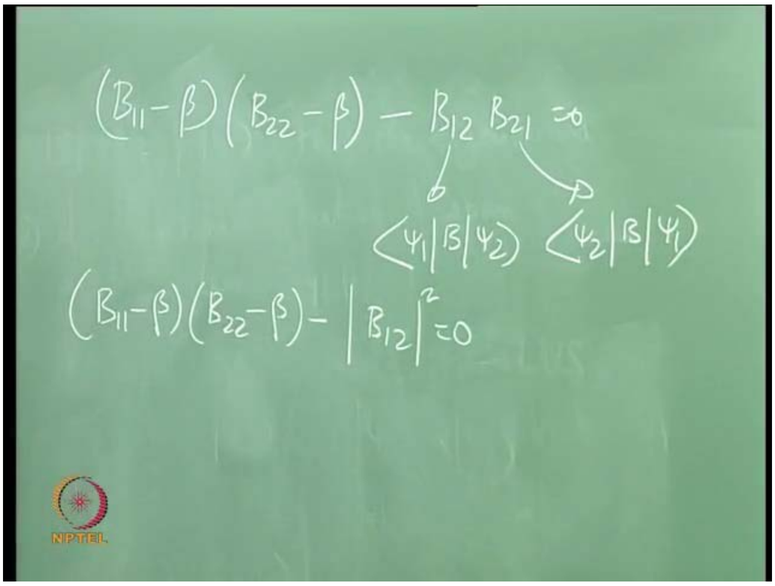

So this determinant when expanded gives me, B 1 1 minus beta, times B 2 2 minus beta. (Refer Slide Time: 32:02) Minus recall that B 1 2 and B 2 1, are complex conjugates of each other. Because, B 1 2 is psi 1 B psi 2. Which is a number and B 2 1 is psi 2 B psi 1, which is the complex conjugate of that. And therefore, I can write this as B 1 1 minus beta, times B 2 2 minus beta, minus modulus of B 1 2 the whole squared is equal to 0.

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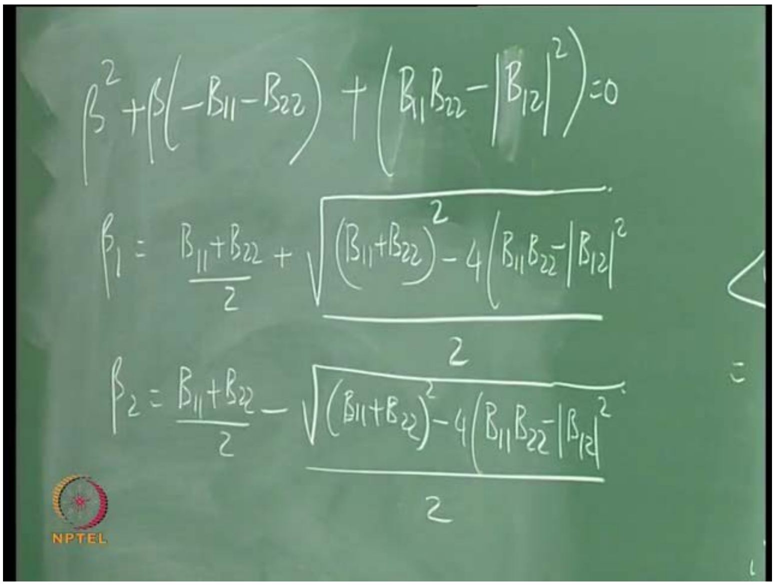

This is the equation I have to solve and beta will it is a quadratic in beta. So, I will have two roots, beta squared plus beta times minus (Refer Slide Time: 32:02) B 1 1 minus B 2

2 plus B 1 1 B 2 2, minus modulus of B 1 2 the whole square. So beta 1 is this. That is beta 1 and beta 2, again is the solution, minus root of the same quantity, B 1 1 plus B 2 2. The whole square minus 4 B 1 1, B 2 2 minus modulus of B 1 2 the whole squared by 2. I, now have various cases to discuss. Recall that alpha was 2 fold degenerate and we had A psi, is alpha psi, where psi was C 1 psi 1 plus C 2 psi 2. Now, I have a situation, where I have 2 roots beta 1 and beta 2 and I demand, that B psi is beta psi. That psi is a common Eigen state of A and B. Is it possible to have these 2 roots equal? Yes, it is possible.

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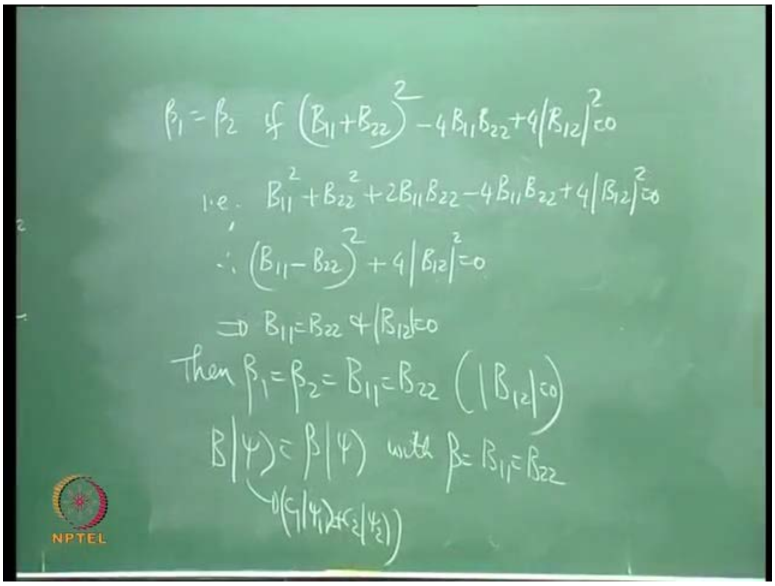

Beta 1 equals beta 2, if this quantity within the square root is 0. So, if B 1 1 plus B 2 2 the whole squared, minus 4 B 1 1 B 2 2 plus 4 mod B 1 2 whole squared is 0. (Refer Slide Time: 35:41) So, certainly if this vanishes then beta 1 is B 1 1 plus B 2 2 by 2 and so is beta 2. So, let us expand this, that is B 1 1 squared plus B 2 2 squared plus 2 B 1 1 B 2 2 minus 4 B 1 1 B 2 2 plus 4 mod B 1 2 squared, equal to 0. Now, that is like saying, B 1 1 minus B 2 2, the whole squared. Because this gives me a minus 2 B 1 1 B 2 2, between these two terms, plus 4 mod B 1 2 the whole squares equals 0. If this were true then beta 1 is equal to beta 2, no conditions on C 1 and C 2.

Since both of them are positive quantities this implies, that B 1 1 is equal to B 2 2 and B 1 2 is equal to 0. I could write modulus of B 1 2 is equal to 0. Then beta 1 is equal to beta 2, (Refer Slide Time: 35:41) from here this is just B 1 1 this term drops out and so beta 1 is twice B 1 1 by 2, beta 2 is also twice B 1 1 by 2. So, that is B 1 1 but that is the same as B 2 2, remember mod B 1 2 is equal to 0. And then we have shown therefore, that B psi, this psi was a linear super position C 1 psi 1 plus C 2 psi 2, is equal to beta psi, with beta equal to B 1 1, which is the same as B 2 2. It clearly follows, that psi 1 is an Eigen state of B, with Eigen value beta and psi 2 is an Eigen state of B with Eigen value beta.

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I therefore, have C 1 psi 1 plus C 2 psi 2, is B 1 1, C 1 psi 1 plus C 2 psi 2. This is true for any C 1and C 2. Therefore, I have B psi 1 is B 1 1 psi 1, B psi 2 is B 1 1 psi 2. Remember the beta was B 1 1 and any linear combination B psi is equal to B 1 1 psi. In other words we have shown, that with no conditions on C 1 and C 2. In other words any superposition of psi 1 and psi 2, would also be an Eigen state of B. The Eigen value is degenerate, again 2 fold degeneracy. Because, there are two states psi 1 and psi 2, corresponding to the Eigen value beta, which turns out to be B 1 1 in this case. So, what is it that we have established? If alpha is 2 fold degenerate, that is A psi 1 is alpha psi 1 and A psi 2 is alpha psi 2.

It is possible that B psi 1 is beta psi 1 and B psi 2 is beta psi 2. The degeneracy is not lifted and the set of Eigen states of A are also Eigen states of B. We have found that set, so we have a complete set of common Eigen states of A and B, the degeneracy has not been lifted. On the other hand it is possible, that the degeneracy gets lifted and that is the case that we will consider now. So, returning to the solutions beta 1 and beta 2. (Refer

Slide Time: 35:41) Let me recall that we have considered the case, where this object within the square root is 0 and therefore, beta 1 was equal to beta 2. But, in general beta 1 need not be equal to beta 2. So, that is the case that we will consider now.

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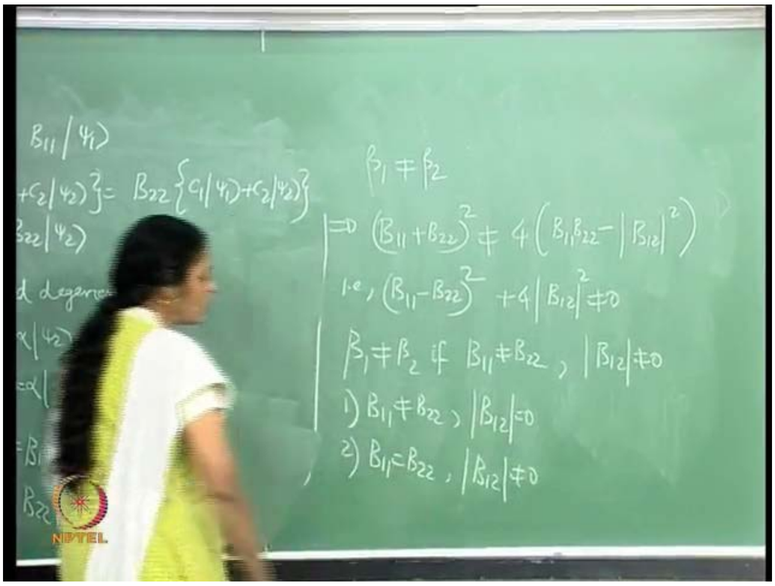

What does that mean? That means that square root is not 0. So, B 1 1 plus B 2 2, the whole square, is not equal to 4 times, B 1, B 2, B 1 1, B 2 2 minus modulus of B 1 2 the whole square. So, we are looking at a situation like this. As you can see this simplified if this were equal, it is simplified to B 1 1 minus B 2 2 the whole squared plus 4 modulus of B 1 2 the whole squared equal to 0. So we are essentially saying, B 1 1 minus B 2 2 the whole squared. (Refer Slide Time: 37:53) Plus 4 modulus of B 1 2 the whole squared is not equal to 0. In general beta 1 would not be equal to beta 2, if any of these quantities is non vanishing. If B 1 1 not equal to B 2 2, or mod B 1 2 not equal to 0. So, we can consider the case, the 1st case, B 1 1 not equal to B 2 2 and for simplicity, mod B 1 2 equal to 0, or B 1 1 equal to B 2 2, but mod B 1 2 not equal to 0. In either case, beta 1 is not equal to beta 2 and we have 2 different roots corresponding to the quadratic in beta.

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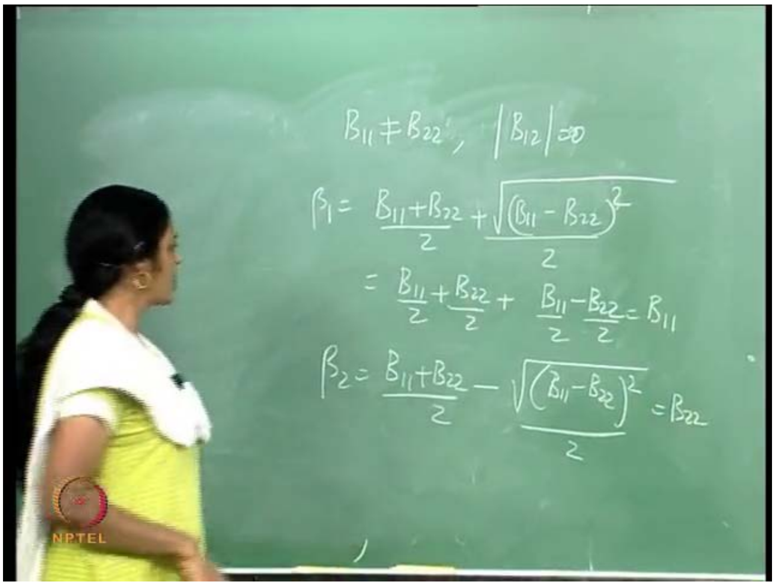

So, let us look at this situation. So, let us look at the 1st situation, B 1 1 not equal to B 2 2, B 1 2 equal to 0. So, that is the situation that I intend considering now. So, what is beta 1? Beta 1 is B 1 1 plus B 2 2 by 2 plus square root of B 1 1 out there minus B 2 2 the whole square, divided by 2. So, this object is simply B 1 1 by 2 plus B 2 2 by 2 plus B 1 1 by 2 minus B 2 2 by 2, which is just B 1 1. Similarly, beta 2, is the other object B 1 1 plus B 2 2 by 2, minus square root of B 1 1 minus B 2 2 the whole square the whole by 2, which is the same as B 2 2. So, I have 2 roots beta 1 is equal to B 1 1 and beta 2 is equal to B 2 2.

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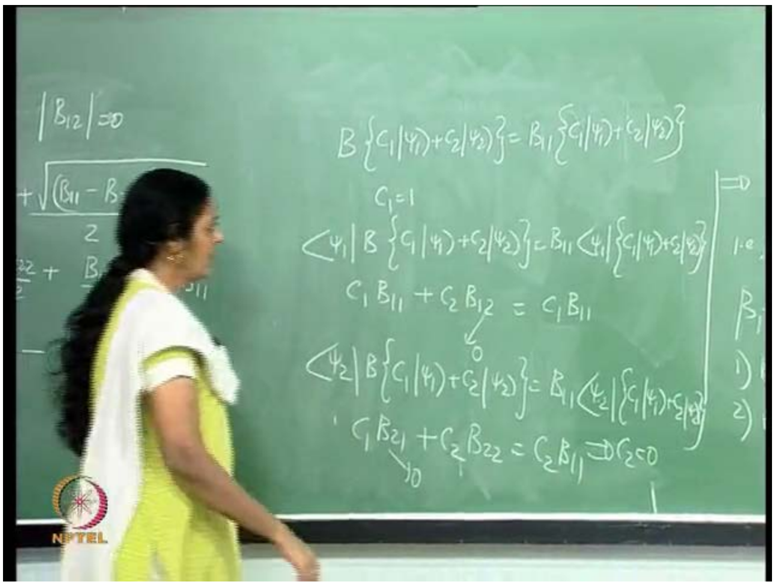

What does that mean? It means the following. That there are 2 Eigen values. C 1 psi 1 plus C 2 psi 2, is equal to B 1 1,C 1 psi 1 plus C 2 psi 2. So, let me consider this similarly, I have another Eigen value B 2 2 and I have some superposition. So, let me consider this and let me look at C 1 equals 1 and fix C 2. So, if I took C 1 equals 1 and if I work with psi 1 on this side as before, B 1 1 is a number. So, I can pull that out, this gives me C 1 B 1 1 plus C 2 B 1 2. But, that object we said was 0, is equal to C 1 B 1 1, which is just a consistency condition. I have put C 1 equals 1, if I now do psi 1 B times C 1 psi 1 plus C 2 psi 2, that object is B 1 1 psi 2 times C 1 psi 1, C 2 psi 2, by this I mean the inner product.

The 1st term gives me C 1 B 2 1 but that is 0. If B 1 2 is 0, B 2 1 is 0, plus C 2 B 2 2. The 1st term is 0, because psi 2 psi 1 inner product is 0, but the 2nd term survives and that is C 2 B 1 1. If C 2 B 2 2 should be equal to C 2 B 1 1 and B 1 1 is not equal to B 2 2. It implies that C 2 is equal to 0. In other words, I have established the following.

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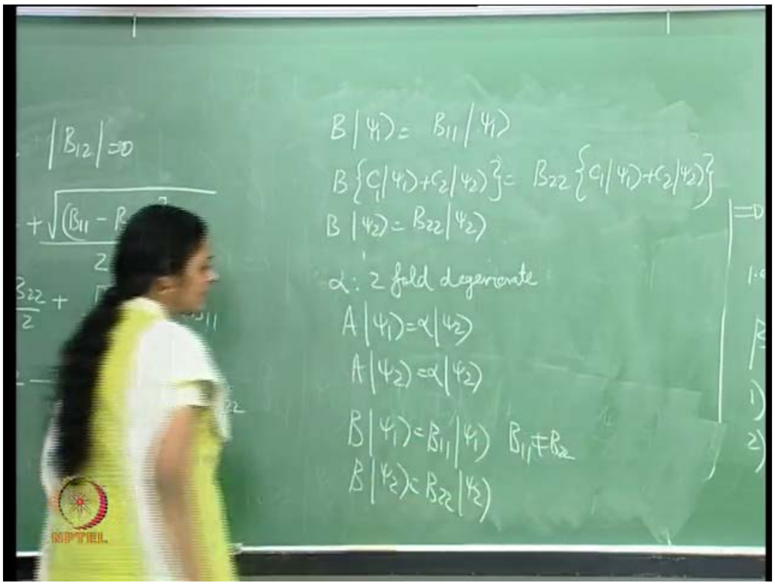

I have established that B psi 1, (Refer Slide Time: 47:05) because C 1 is equal to 1 and C 2 is equal to 0. I have established that B psi 1 is equal to B 1 1 psi 1. Similarly, I can start with B times C 1 psi 1 plus C 2 psi 2 equals B 2 2, C 1 psi 1 plus C 2 psi 2. Proceeding on the same lines as before, I can now show that in this case if C 2 is 1 and C 1 is 0, it follows then that B psi 2 is B 2 2 psi 2. What is it that I have established? I have looked at a very specific case, where B 1 1 is not equal to B 2 2 and B 1 2 is equal to 0. And I have shown that there are 2 Eigen vectors psi 1 and psi 2, which are Eigen vectors of B, except that the degeneracy is now lifted. One of them comes with Eigen value B 1 1 and the other comes with Eigen value B 2 2. So, this situation corresponds to alpha is 2 fold degenerate, A psi 1 is alpha psi 1, A psi 2 is alpha psi 2. However B psi 1 is B 1 1 psi 1 and B psi 2 is B 2 2 psi 2, B 1 1 not equal to B 2 2. The degeneracy has been lifted in this case, by B, there was a 2 fold degeneracy. Those Eigen vectors continue to be Eigen vectors of B, they are Eigen vectors of A, corresponding to a 2 fold degenerate Eigen value. They continue to be Eigen vectors of B, but the degeneracy has been lifted. So, this is one case that we have. Let us look at the last case, which is this. (Refer Slide Time: 43:14) Where B 1 1 equals B 2 2. But, modulus of B 1 2 is not equal to 0. So, let us look at that case now. What is it that we get?

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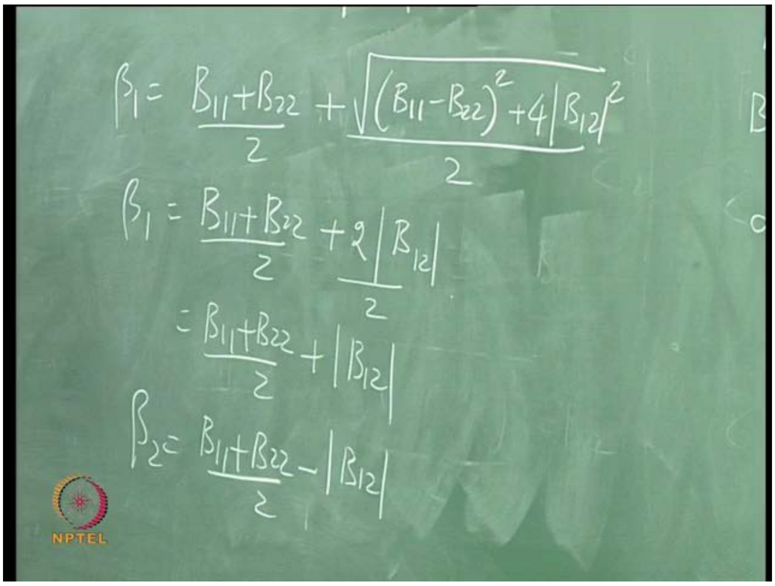

B 1 1 equals B 2 2 and modulus of B 1 2, not equal to 0. So, what is it that we have here? (Refer Slide Time: 35:41) The root beta 1 would be B 1 1 plus B 2 2 by 2 plus, now this object this whole exponent, was simply plus square root of B 1 1 minus B 2 2 the whole squared, 4 modulus of B 1 2 the whole squared. This was beta to begin with, this exercise simply tells us that, beta 1 in this case, is B 1 1 plus B 2 2 by 2. This quantity vanishes, plus twice modulus of B 1 2 by 2 and therefore, I have B 1 1 plus B 2 2 by 2 plus mod B 1 2.

In the other case beta 2, would correspondingly be B 1 1 plus B 2 2 by 2 minus modulus of B 1 2. Once more I have 2 distinctly different Eigen values. And since B 1 1 equals B 2 2, we have beta 1 equals B 1 1 plus modulus of B 1 2 and beta 2 equals B 1 1 minus modulus of B 1 2. You can now see that you could repeat this argument and you will get conditions, on C 1 and C 2.

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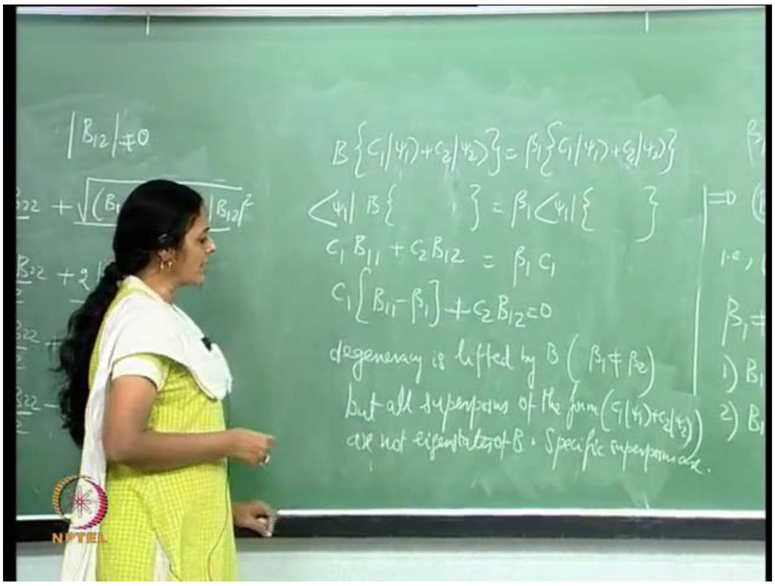

In other words you started by saying, is beta 1, C 1 psi 1 plus C 2 psi 2. What is it that I have? I have if I work with the inner product, of psi 1 with this object. If I did this, the 1st term is C 1 B 1 1 plus C 2 B 1 2 is equal to beta 1, C 1 the 2nd term there drops out because psi 1 is orthogonal to psi 2, B 1 1 is not 0, B 1 2 is not 0.

Therefore I have C 1 B 1 1 minus beta 1, is equal to plus C 2 B 1 2 is equal to 0. Neither B 1 1 or B 1 2 are 0, this would therefore, lay conditions on C 1 and C 2. Similarly, if I started with beta 2 here it would lay a different set of conditions, on C 1 and C 2. In other words this is the situation where degeneracy, is lifted by B. That is beta 1 is not equal to beta 2, but all superpositions of the form C 1 psi 1 plus C 2 psi 2, are not Eigen states of B. Specific superpositions are, the degeneracy is lifted.

Once more we have found a complete set of common Eigen states, of A and B in this case and B lifts the degeneracy. There was a 2 fold degeneracy which was lifted, in this case as well. Whatever I have said for a 2 fold degeneracy can be extended to a g fold degeneracy. So, the problem has a following outcome, that if you have 2 Hermitian operators that commute, you can find a complete set of common Eigen states of these 2

Hermitian operators.