Quantum Mechanics - I
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Lecture - 11
The Displacement and Squeezing Operators

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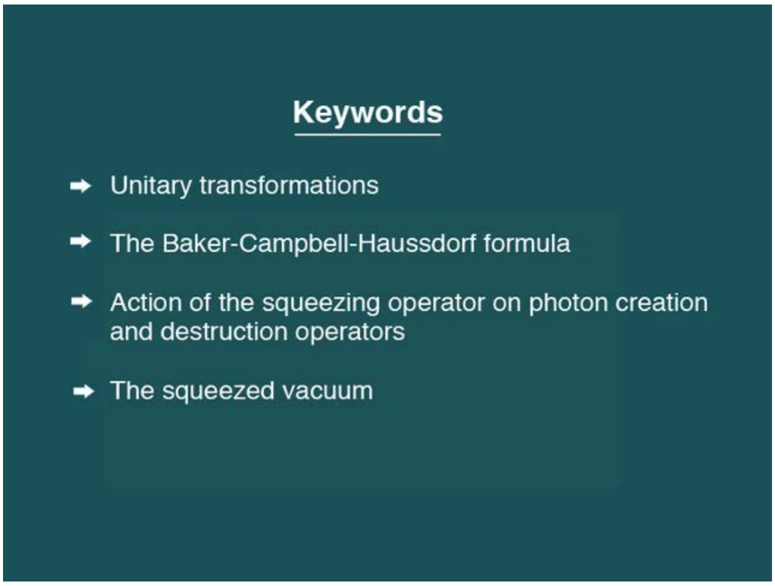

In the last lecture, I spoke about an interesting quantum superposition of the photon number states.

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The superposition is called the coherent state and by way of recapitulation the coherent state is represented by ket alpha. Alpha could be any complex number and this is e to the minus mod alpha squared by 2 summation n equals 0 to infinity, alpha to the n by root n factorial ket n, where the ket n's are the photon number state. So, you can have a superposition of the 0 photon state, the 1, 2, 3 and so on with a very specific weightage in front of the state. This state could be got by the action of a unitary operator D of alpha on ket 0. So, D of alpha on ket 0 gave me ket alpha. D of alpha is a unitary operator and you will recall the D of alpha was e to the alpha a dagger minus alpha star a.

While we prove this, I made a statement that D of alpha a D dagger of alpha was simply a shift in a. It just produced a minus alpha, this is to be read as a minus alpha times the identity operator. So, really there is an identity operator here, but I am just going to write this as a minus alpha. Similarly, D of alpha a dagger, D dagger of alpha was a dagger minus alpha star times identity. Once more I will write this as a dagger minus alpha star. So, to begin with I wish to establish these identities. I prove this using a variant of the Baker Campbell Hausdorff formula. There are a series of such formulae which we would establish later on.

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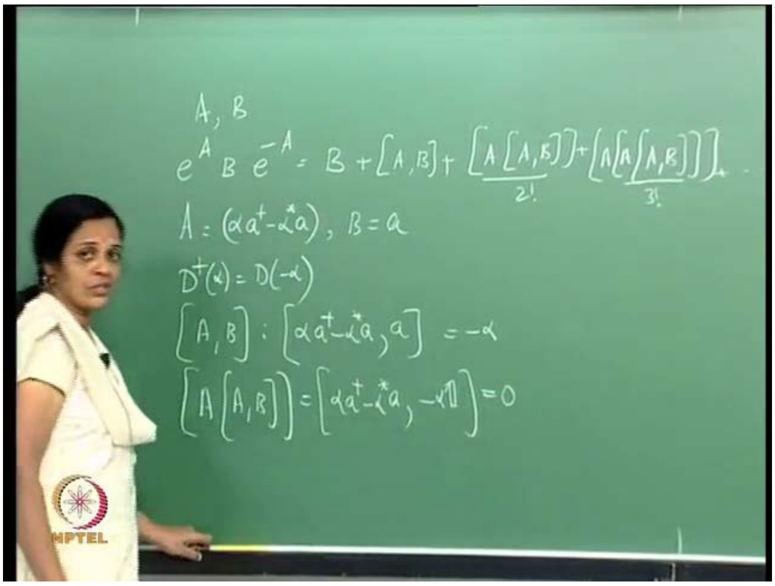

We will derive them in a tutorial session, but right now I will use the end result of that derivation and it is a variant of the Baker Campbell Hausdorff formula and it says: if you have an operator A and an operator B, then e to the A B e to the minus A is simply B plus the commutator of A with B plus the commutator of A with the commutator of A with B plus so on and there is a 2 factorial here. So, the next term to complete the series would be the commutator of A with the commutator of A with A with B by 3 factorial and so on, which is precisely what we will use here, we will use this identity.

In this problem A is simply alpha a dagger minus alpha star a. Recall that D dagger of alpha was simply D of minus alpha and therefore, on this side (Refer Slide Time: 00:27) I have D dagger of alpha in this identity, which is D of minus alpha which is minus A, because if I substitute minus alpha for alpha I simply get minus A. And therefore, if I wish to find the commutator of A with B in this problem that is the same as finding out the commutator of alpha a dagger minus alpha star A with B and B in this case is a. (Refer Slide Time: 00:27) Because I am trying to establish this identity.

The commutator of the 2nd term alpha star a with a drops out because a commutes with itself. That leaves behind the commutator of alpha a dagger with a. The commutator of a dagger with a is minus 1 and therefore, this just gives me minus alpha. The next term calls for the commutator of A with the commutator of A with B. Now, that is the same as writing commutator of alpha a dagger minus alpha star a. The commutator of A with B is minus alpha times identity if you wish which is what we have not put on here. But any operator commutes with the identity and therefore, that is 0. As a result all these terms vanish and do not make a contribution.

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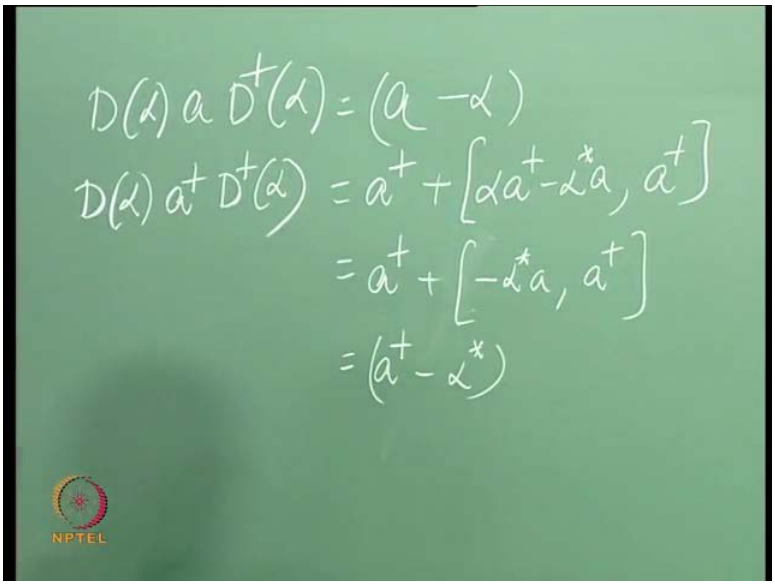

I therefore, can write D of alpha a D dagger of alpha is a, from the 1st term (Refer Slide Time: 03:05) plus commutator of A with B, which is minus alpha. Similarly, we can find D of alpha a dagger D dagger of alpha. That is a dagger plus the commutator of A with B in this case is alpha a dagger minus alpha star a, commutator with a dagger. All other commutators are 0. Minus alpha star a with a dagger in the commutator and that is a dagger minus alpha star. Since, this is just minus alpha star times identity all other higher commutators out here would vanish and therefore, I have D of alpha a dagger D dagger of alpha is a dagger minus alpha star. (Refer Slide Time: 00:27) That is how I establish these identities.

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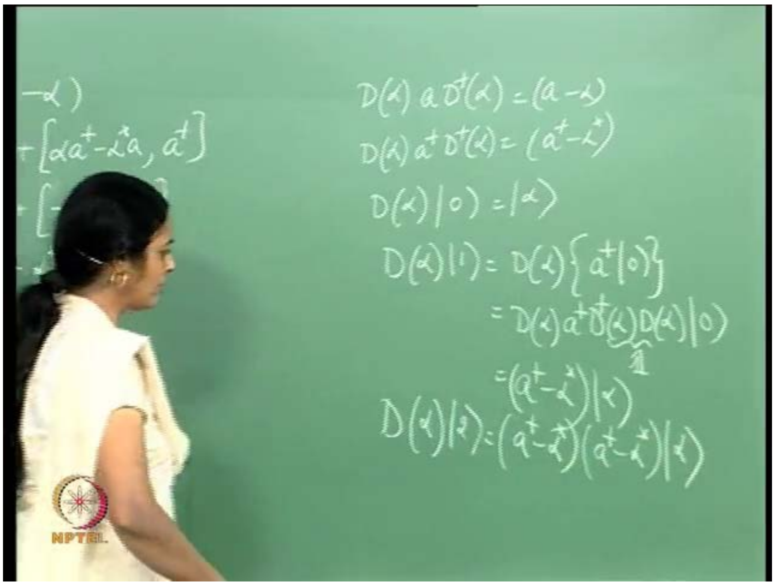

So, now we have the following picture. It is a dagger minus alpha star out here and D of alpha on ket 0, is ket alpha. What is D of alpha on ket 1? This is simply D of alpha, but ket 1 is a dagger ket 0 so I put that here. I can now put in a D dagger of alpha, D of alpha here, because that is identity. I use the fact that D of alpha a dagger D dagger of alpha is a dagger minus alpha star, an identity which I have just now proved. Then there is a D of alpha acting on ket 0 which gives me ket alpha.

So, you see this state D of alpha acting on ket 1 can be got from ket alpha by applying a dagger minus alpha star. So, basically D of alpha on ket 2 is a dagger minus alpha star. Again an a dagger minus alpha star on alpha. This gives me a very interesting picture.

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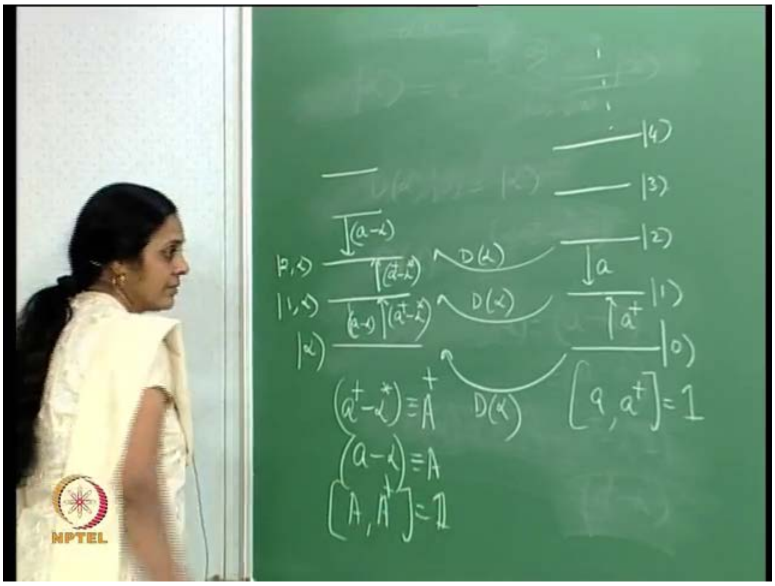

I can now have a diagram which says the natural basis, which I was considering had a set of Eigen states 0, 1, 2, 3, 4 etcetera. There was a raising operator a dagger and a lowering operator a. If we discuss the simple harmonic oscillator that is the language I will use. If I were talking about photon creation and destruction, a dagger would be the photon creation operator and a will be the photon destruction operator. You go from the 0 photon state to the 1 photon state and so on using a dagger repeatedly you come down from the 2 photon state to the 1, to the 0 by using a. And a and a dagger satisfies this algebra, by this I mean the identity operator.

Then if I act D of alpha on ket 0, I get the state ket alpha. If I work with D of alpha on ket 1, I get a state which I can reach from ket alpha by acting a dagger minus alpha star on ket alpha. (Refer Slide Time: 07:33) That is what I have proved here D of alpha on ket 1 is a dagger minus alpha star on ket alpha. Similarly, D of alpha on ket 2 would involve using a dagger minus alpha star on this state. Now, these states could well be labeled 1, alpha because D of alpha acted on ket 1 to produce this state. This state could be labeled 2, alpha. This is merely a notation, D of alpha acting on ket 2 give me this state and so on. The lowering operator is a minus alpha. I can go here using a minus alpha.

It is clear that if I represent by A dagger, a dagger minus alpha star, a dagger minus alpha star as A star and a minus alpha is A. It is evident that A, A dagger is 1. What is it that we have done? We have done a change of basis through a unitary transformation. The unitary operator that was used was the displacement operator D of alpha and this displacement operator took the basis set, the natural basis ket 0, ket 1, ket 2 ket 3 and so on to ket alpha, 1, alpha, 2, alpha and so on.

These states are all orthogonal to each other, as much as these states are all orthogonal to each other. The states have all been normalized to unity. And here therefore, is an example of a very specific unitary transformation which has taken us from one basis set in the Hilbert space spanned by ket 0, 1, 2, 3 etcetera to another basis set in the same Hilbert space. Any state in the Hilbert space could be written as a linear superposition of this infinite set or as a linear superposition of that infinite set. Since states changed by using d of alpha on ket n gives me ket n, alpha. Operators will change in the fashion that was mentioned earlier (Refer Slide Time: 07:33).

The operator gets sandwiched between D and D dagger. As a result of which the algebra does not change. Initially I had a, and a dagger and their commutator was 1. Now, I have A and A dagger and their commutator is 1. So, the algebra does not change. The new basis set is also mutually orthogonal and they are all normalized to 1, each state in the new basis set, so was the old basis set. And the whole transformation on the operators and on the basis set has been brought about by this particular unitary operator D of alpha. While we are at it, it is good to list out another property of D of alpha.

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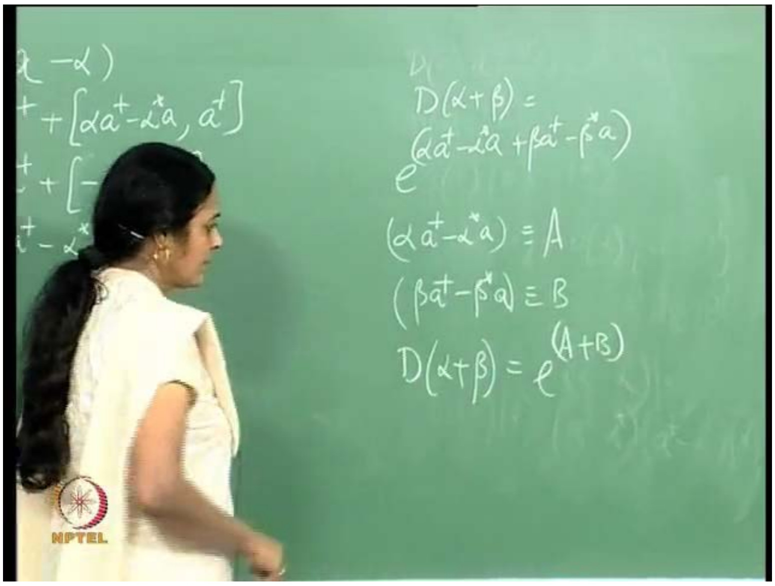

Consider D of alpha plus beta. Now, this object is e to the alpha a dagger minus alpha star a plus beta a dagger minus beta star a. Now, we could represent alpha a dagger minus alpha star a as the operator A and beta a dagger minus beta star a as the operator B. So, D of alpha plus beta has the structure e to the A plus B, by the Baker Campbell Hausdroff formula which is now familiar to you.

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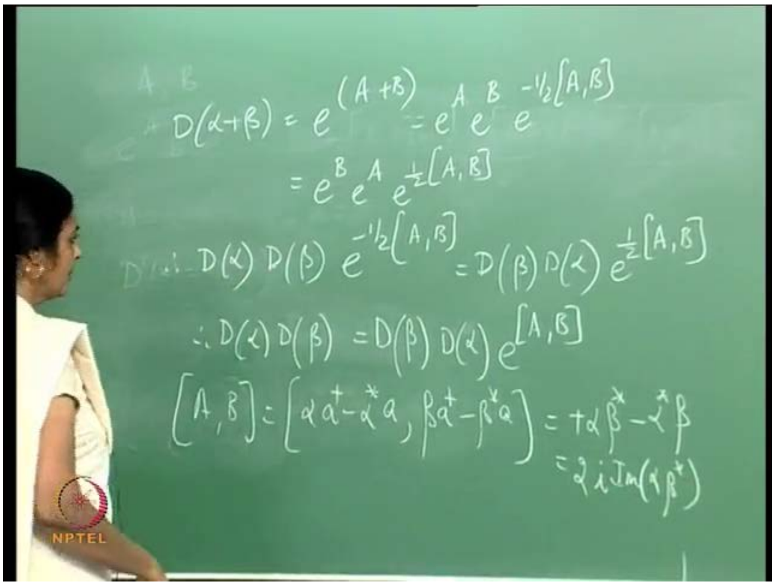

I can write D of alpha plus beta in the following manner. So, D of alpha plus beta is e to the A plus B and that is the same as e to the A e to the B e to the minus half commutator of A with B. But, it is also the same as e to the B, e to the A, e to the plus half commutator of A with B. I therefore, have now e to the A is simply D of alpha, e to the B is d of beta, e to the minus half commutator of A with B is e to the B which is D of beta e to the A which is D of alpha, e to the plus half commutator of A with B. Therefore, D of alpha and D of beta do not commute with each other. I have this object as an extra piece, they do not commute and D of alpha, D of beta is D of beta, D of alpha, e to the commutator of A with B.

Now, what is the commutator of A with B in this case? Alpha a dagger minus alpha star a, commutator with beta a dagger minus beta star a. Alpha a dagger commutes with beta a dagger, but alpha a dagger with minus beta star a gives me a contribution which is alpha times minus beta star commutator of a dagger with a, which is minus 1 therefore, giving me a plus sign there. Now, look at the 2nd term minus alpha star a commutator with beta a dagger gives me minus alpha star beta. The commutator of a with a dagger is 1. However, minus alpha star a commutes with minus beta star a and therefore, this is the only contribution. But this object is simply twice 2 i imaginary part of alpha beta star.

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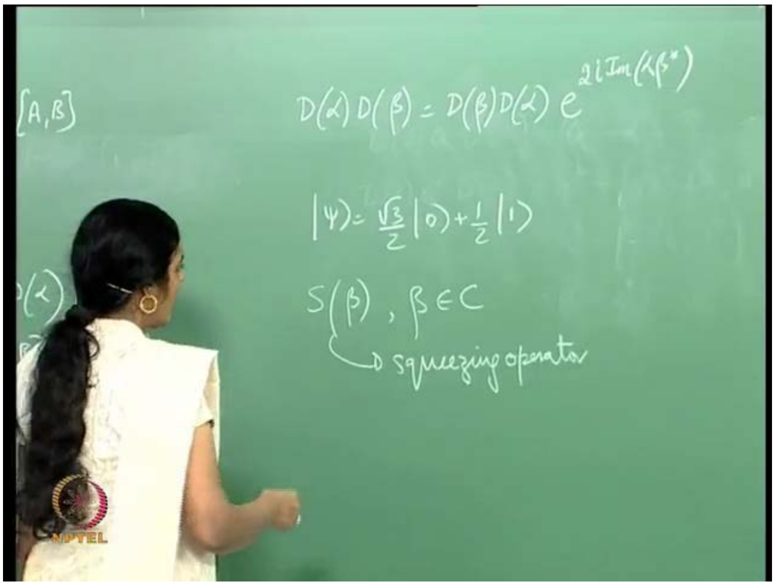

I therefore, have an interesting relation D of alpha D of beta is equal to D of beta D of alpha and then there is an extra piece, which is e to the 2 i, imaginary part of alpha beta star. So, it is important to realize that these two unitary operators, the displacement operator in question, with for an alpha and a beta as a argument. These two operators do not commute with each other. They certainly pickup an extra phase out here. There are several other interesting quantum superpositions of the photon number states.

Even as the coherent state is the ideal laser light and has been generated in the laboratory and is very useful in various branches of physics and has immense applications as well. There are other interesting quantum superpositions like the squeeze state of light which too has been realized in the laboratory. The squeezed state of light and example of squeezed light has already been given earlier. We had the state psi which is root 3 by 2 ket 0 plus half ket 1.

So, this was a specific squeezed state which was the superposition of the 0 photon state, with the 1 photon state. There are other interesting superpositions and even as the coherent state is the superposition of an infinite set of photon number states. I define a squeezing operator which could act on the vacuum and produce a very specific interesting infinite superposition of photon number states, which also display squeezing properties. So, similar to the displacement operator one introduces the squeezing operator, S of beta where beta is any complex number. This is the squeezing operator. S of beta is defined in the following manner.

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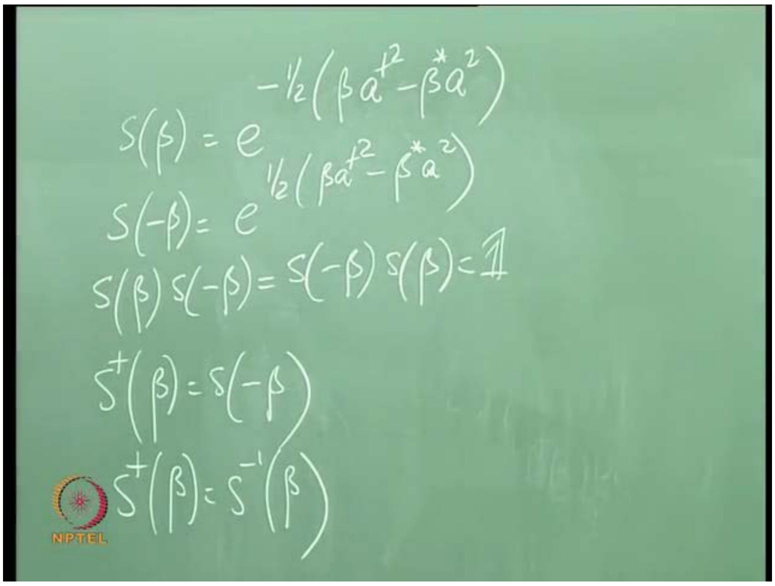

S of beta is e to the minus half beta a dagger squared minus beta star a squared. It is clear therefore, that only even photon number states will appear when S of beta acts on the vacuum for instance. And when S of beta acts on any photon number state, it increases or decreases the number of photons in the infinite superposition by 2 at a time, because of a dagger squared and a squared being present there.

Now, S of minus beta is e to the half beta a dagger squared minus beta star a squared. It is clear that S of beta S of minus beta is equal to 1 and that is the same as S of minus beta S of beta. By 1 I mean the identity operator. Therefore, S dagger of beta is S of minus beta. This is the unitary operator and therefore, S dagger of beta is S inverse of beta. Even as the displacement operator is a unitary operator, the squeezing operator is another unitary operator. Now, let us find out the action of the squeezing operator on a and a dagger.

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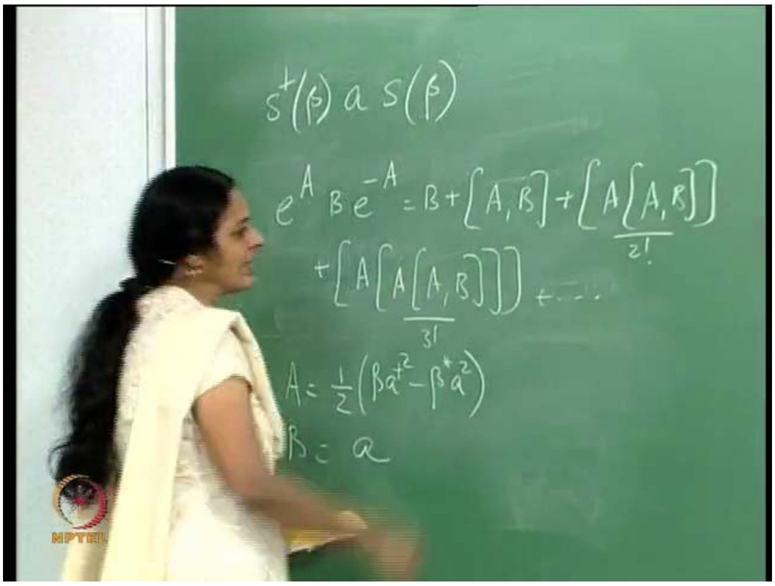

I wish to find the following: S dagger of beta a S of beta where S dagger and S are defined there. Once more, I recall the identity e to the A B e to the minus A is B plus commutator of A with B plus commutator of A with A with B by 2 factorial plus commutator of A with A with A with B by 3 factorial and so on. In this case A is simply half beta a dagger squared minus beta star a squared. That is what we have here. (Refer Slide Time: 20:29) S dagger of beta is S of minus beta and that is here, e to the half beta a dagger squared minus beta star a squared. B itself is the operator a, and we need to find these commutators and simplify.

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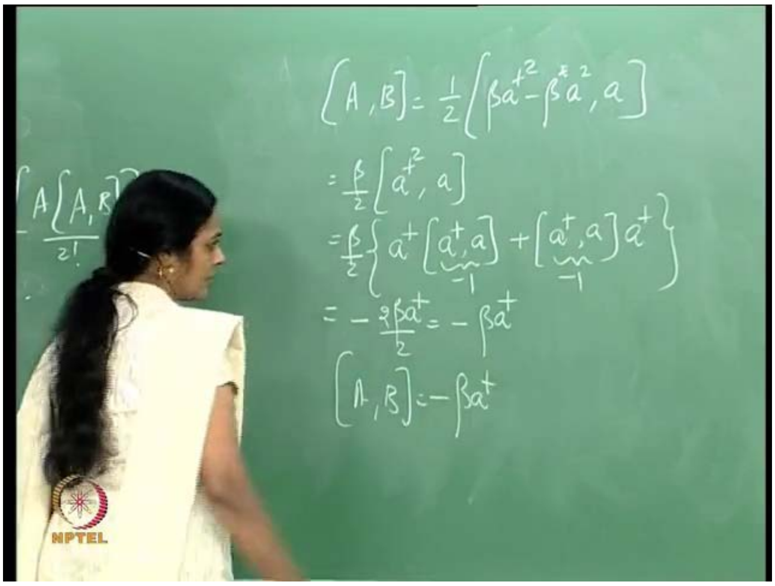

So, we have commutator of A with B is half beta a dagger squared minus beta star a squared with a certainly, the 1st term contributes. I pull out a beta and I have commutator a dagger squared with a. The 2nd term here commutes with a because there is an a squared here and there is an a there and any object commutes with itself or its powers. I use the a b c rule here so that is beta by 2 a dagger commutator of a dagger with a plus a dagger a commutator with a dagger.

Now, a dagger a commutator is minus 1 and therefore, this is minus 2 beta a dagger by 2 which is equal to minus beta a dagger. So, this is what I have for the commutator of A with B. (Refer Slide Time: 20:29) Now, I need to find the next term which is the commutator of A with A with B by 2 factorial.

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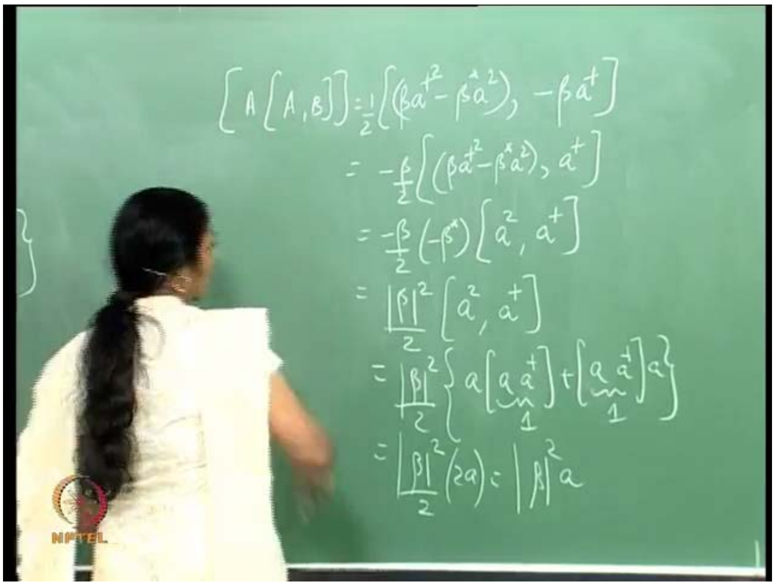

So, let us work out that commutator now. (Refer Slide Time: 23:41) Commutator of A with A with B is a half beta a dagger squared minus beta star a squared. Its commutator with minus beta a dagger. So, that is minus beta by 2, I have pulled out the beta and the minus sign. This is the commutator that I need to find, a dagger squared commutes with a dagger. Therefore, this commutator the 1st term commutes with a dagger leaving behind a minus beta star commutator of a squared with a dagger, which is the same as mod beta squared by 2, commutator of a squared with a dagger.

And this commutator again can be evaluated using the a b c rule. The commutator of a with a dagger is plus 1 and therefore, this gives me mod beta squared by 2 times 2 a, which is mod beta squared a. (Refer Slide Time: 22:22) Of course, I need to take into account the fact that there is a 1 by 2 factorial multiplying that term which I can put down. Now, I need to find the commutator of A with A, with A, with B by 3 factorial. (Refer Slide Time: 25:27) So, this object here is simply the commutator of A with A with B.

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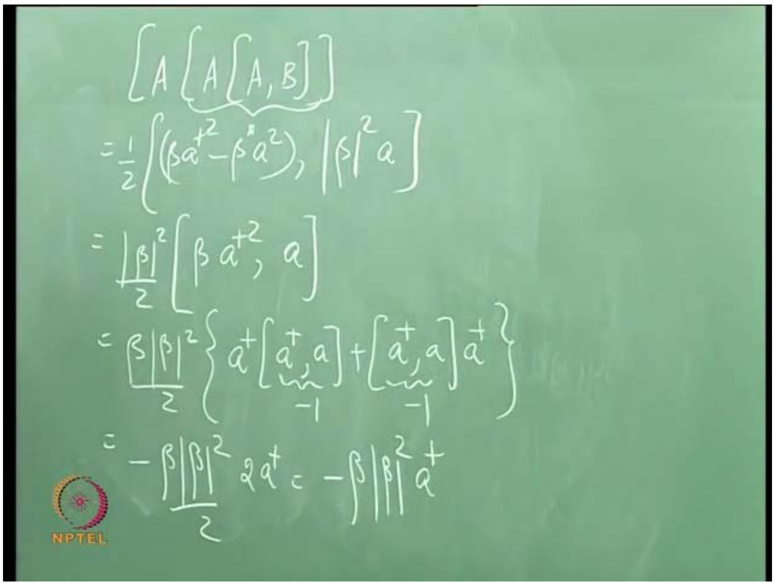

So, let us just do one more term and that is the commutator of A with this object. That is the same as half commutator of beta a dagger squared minus beta star a squared. (Refer Slide Time: 25:26) With this commutator, which is what I have put down here mod beta squared a I can pull out the mod beta squared. The 1st term is the commutator of beta a dagger squared with a, because I have pulled this one out here. The 2nd term does not contribute because a squared commutes with a. This gives me a beta mod beta squared by 2 and again I use the a b c rule. It is a dagger commutator of a dagger with a plus commutator of a dagger with a times a dagger.

And, this as we know by now is minus 1 and therefore, I have minus beta, mod beta squared by 2, 2 a dagger which is minus beta mod beta squared a dagger. Of course, I will have a 1 by 3 factorial multiplying this term. So, let us put down all the terms together.

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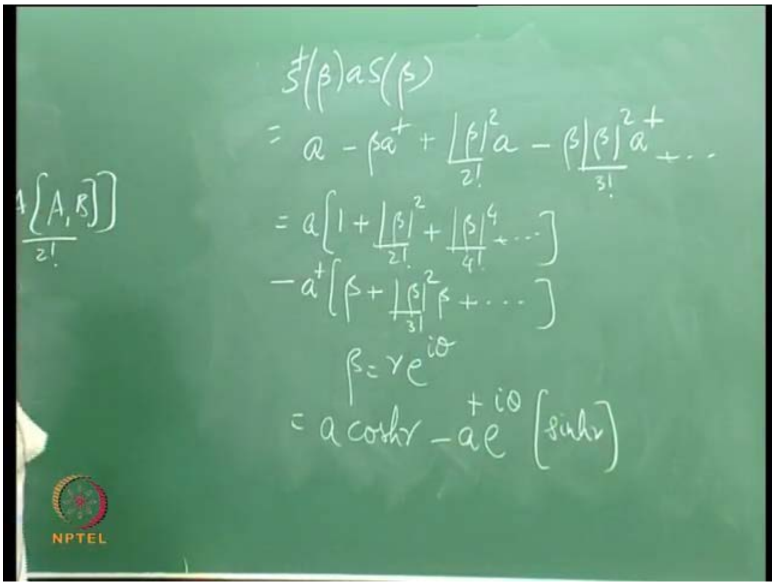

And therefore, I have S dagger of beta a S of beta. It is simply a as a 1st term, minus beta a dagger that is the 2nd term. Then I have (Refer Slide Time: 22:22) plus commutator of A with A with B by 2 factorial and that is out there plus mod beta squared by 2 factorial a. And then I have that (Refer Slide Time: 27:34) commutator of A with A with A with B that gives me a minus beta mod beta squared a dagger. That comes with the 3 factorial and so on.

So, clearly there are terms multiplying a and there are terms multiplying a dagger. And it should be possible now to do the following: you can write this as a times 1 plus mod beta squared by 2 factorial plus mod beta to the power of 4 by 4 factorial plus so on. Minus a dagger beta plus mod beta squared by 3 factorial times beta plus so on. That is the same as a and let me define beta in the polar form as r e to the i theta. So, in the mod squared I just have an r squared and an r to the power of 4 and so on. So, this quantity is a cos h r.

There is always a beta here. So I can pull out an e to the i theta and what is left behind is an r plus there is an r squared here and an r here so an r cube by 3 factorial and so on. And this is what I have. So I have S dagger of beta a S of beta is simply a cos h r minus a dagger e to the i theta sin h r.

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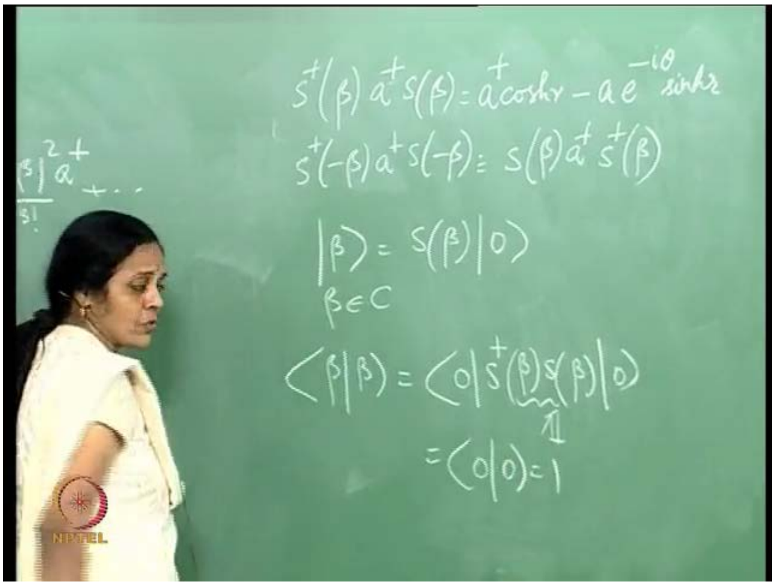

We can proceed in the same way and show the following that S dagger of beta a dagger S of beta is a dagger cos h r minus a e to the minus i theta sin h r. Now, let us consider S dagger of minus beta a dagger S of minus beta. Simply means replace all betas by minus betas, but that would just be identical to S of beta a dagger S dagger of beta, because S dagger of minus beta is merely S of beta. These relations are good to know and remember because right away we will be looking at a very interesting quantum superposition, which is the squeezed vacuum. The squeezed vacuum, which I will denote by ket beta is simply S of beta acting on ket 0.

So, if we are talking about the simple harmonic oscillator, it is S of beta acting on the ground state of the oscillator or the vacuum state. And if it is quantum optics, then this is a 0 photon state or the vacuum state and S of beta acts on the vacuum. To produce ket beta we can check that ket beta is a squeezed state and that is why it is called the squeezed vacuum. Remember that beta is any complex number. It is very clear that ket beta is normalized to unity because this is the same as S dagger of beta S of beta ket 0 and this object is identity and therefore, this is the same as the inner product of 0 with ket 0 with bra 0 and that is 1. So, this is the normalized state. What we need to show is that, it is the squeezed state.

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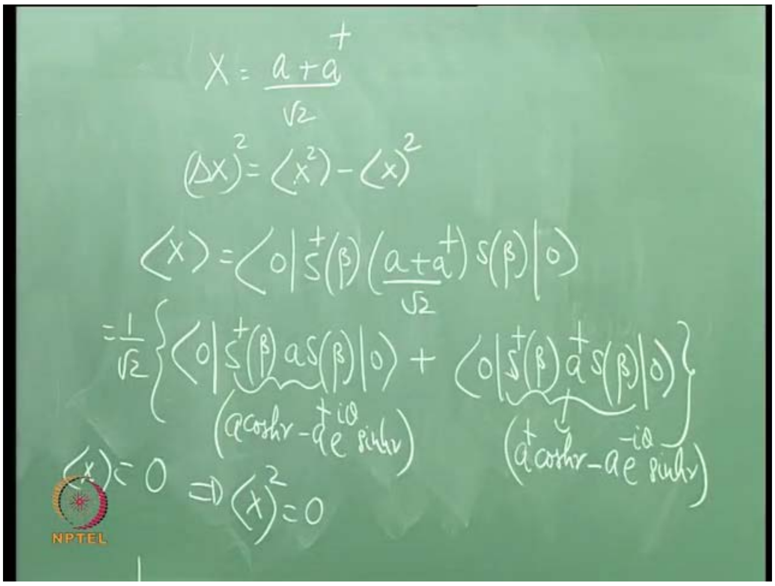

So we recall that we had X which was a plus a dagger by root 2 and the variance in X was expectation X squared minus expectation X the whole squared. We need to compute this and find out if it is less than half or greater than half. If it is less than half, it is a squeezed state with squeezing in the X quadrature. To calculate expectation X the whole squared in the case of the squeezed vacuum, we need to first find expectation X and that is this object S dagger of beta plus a dagger S of beta 0 divided by root 2. But we know this.

This quantity is just 1 by root 2. The 1st term is 0, S dagger of beta a S of beta 0. The 2nd term is 0 S dagger of beta a dagger S of beta 0. This is expectation X in the state S of beta ket 0. (Refer Slide Time: 32:01) This object is known to us S dagger of beta a S of beta is out here. (Refer Slide Time: 29:09) It is a cos h r minus a dagger e to the i theta sin h r.

So, this is a cos h r minus a dagger e to the i theta sin h r and S dagger of beta a dagger S of beta. This object is a dagger cos h r minus a e to the minus i theta sin h r. What would this give us? These are numbers cos h r, sin h r, e to the i theta are merely numbers. So, here I have the expectation value of a in the state ket 0 and that is 0 because a annihilates ket 0. Now, a dagger acts on ket 0 to give me ket 1, but bra 0 is orthogonal to ket 1 and therefore, the contribution from here is 0. Similarly, a dagger acts on ket 0 to give me ket 1. The inner product of bra 0 with ket 1 is 0.

So the 1st term does not contribute. As to the 2nd term a simply annihilates ket 0 and therefore, none of the terms contribute and I have expectation value X 0 which implies expectation value X the whole squared is also 0. The non-trivial contribution therefore, comes from expectation X squared which is what we will proceed to find now.

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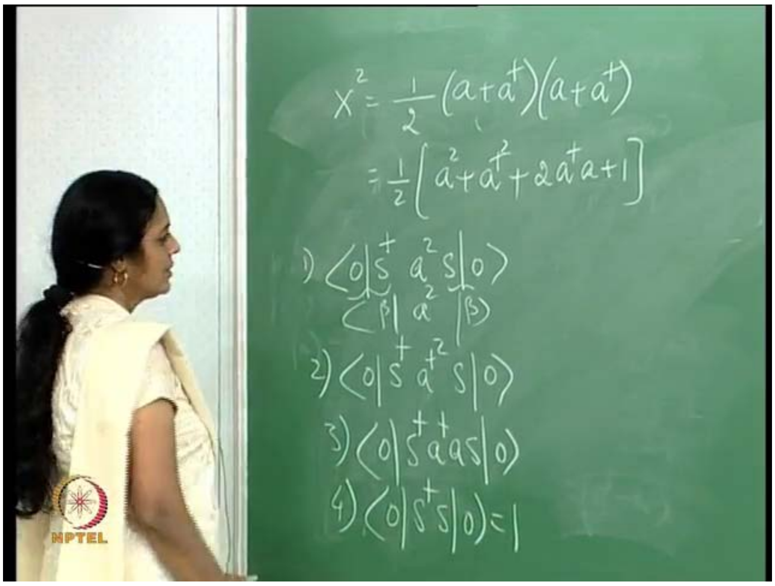

Now, X squared is simply 1 by root 2 a plus a dagger the whole squared and therefore, that gives me a half a plus a dagger times a plus a dagger. We have done this in the earlier lecture and this gives us a squared plus a dagger squared plus a dagger a plus a a dagger, that is 2 a dagger a plus 1. So, this is X squared therefore, have to find various expectation values. First of all, we need to figure out this object. We need to find out expectation value of a squared when it is sandwiched between ket beta and bra beta. This is something we need to find out.

Similarly, we need to find out 0 S dagger a dagger squared S 0. The third thing we need to find out 0 S dagger a dagger a S ket 0. The last term is clear and is simply 0 S dagger. The identity operator and therefore, this is 1.Tthere is an overall factor half multiplying these terms.

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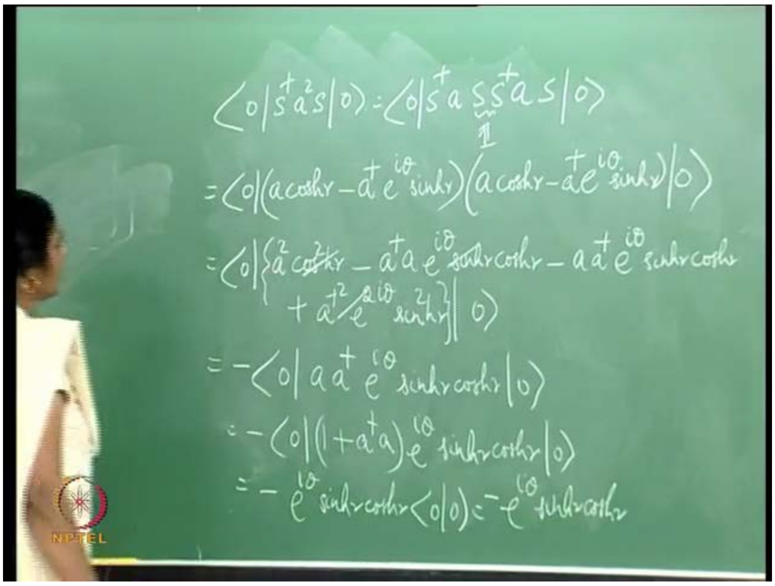

So we begin with the first term. Now, 0 S dagger a squared S acting on ket 0 can be written as 0 S dagger a and I can insert an S S dagger, because this is an identity out there. The a squared was written as a times a, S ket 0. But I know what is S dagger a S? And therefore, this is 0. S dagger a S is a cos h r minus a dagger e to the i theta sin h r. That repeats because there is another S dagger a S here. So, that is an a cos h r minus a dagger e to the i theta sin h r acting on ket 0.

So, I multiply this term by term. The first term is a squared cos squared h r minus a dagger a e to the i theta sin h r cos h r minus a a dagger e to the i theta sin h r cos h r and then the last term is plus a dagger squared e to the 2 i theta sin squared h r. And this I need to find the expectation value of this object in the state ket 0. It is evident that the first term does not contribute because a squared on ket 0 is 0. Now a dagger a on ket 0 is also 0, because ket 0 is an Eigen state of a dagger a with Eigenvalue 0. So, these terms do not contribute, a dagger squared acting on ket 0 gets me ket 2 and by the orthonormality property, that ket 0 is orthogonal to ket 2. This 2 will not contribute and then I have minus a, a dagger left.

So, this is the same as minus ket 0 a a dagger e to the i theta sin h r cos h r ket 0. I know that a a dagger is 1 plus a dagger a and there is an e to the i theta sin h r cos h r ket 0. Again a dagger a does not contribute because a dagger a acting on ket 0 gives me 0. So this is just minus e to the i theta sin h r cos h r ket 0 with bra 0. That is the same as minus e to the i theta sin h r cos h r. This is how I find this object when a squared is sandwiched between ket beta and bra beta. This is the manner in which I proceed and figure out what the answer is and this is what I get. I get minus e to the i theta sin h r cos h r.

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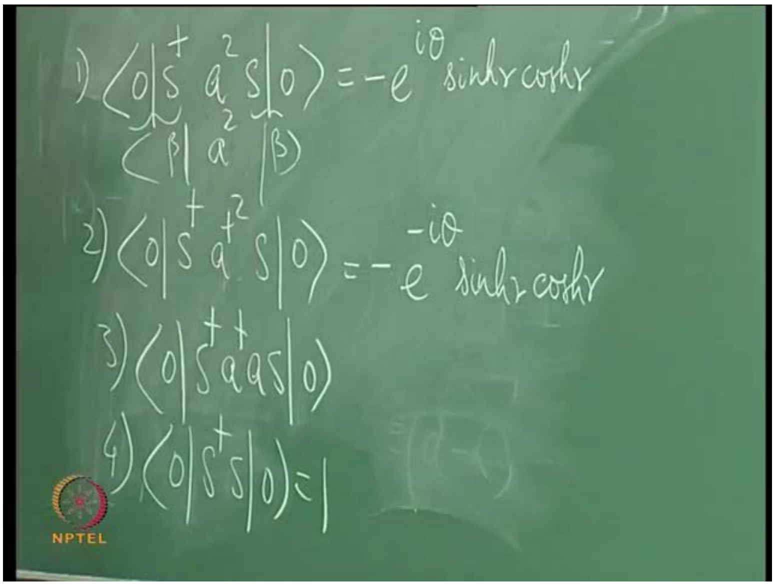

So, returning to this the first item here has already been simplified and I get minus e to the i theta sin h r cos h r. I need to now find out a dagger squared S 0. Now suppose, I took the dagger of this object, ket beta would become bra beta, bra beta would become ket beta and a squared would become a dagger squared. So, I simply need to take the dagger of this object in order to find out what is ket 0 S dagger a dagger squared S ket 0 and therefore, I get the answer in this case to be minus e to the minus i theta sin h r cos h r. Now, the difference is out here, where I had e to the i theta in the first case, I have e to the minus i theta in the 2nd case. The fourth is already clear, it is 1. All I need to do is find out 0 S dagger a dagger a S ket 0 and I will proceed to do that now.

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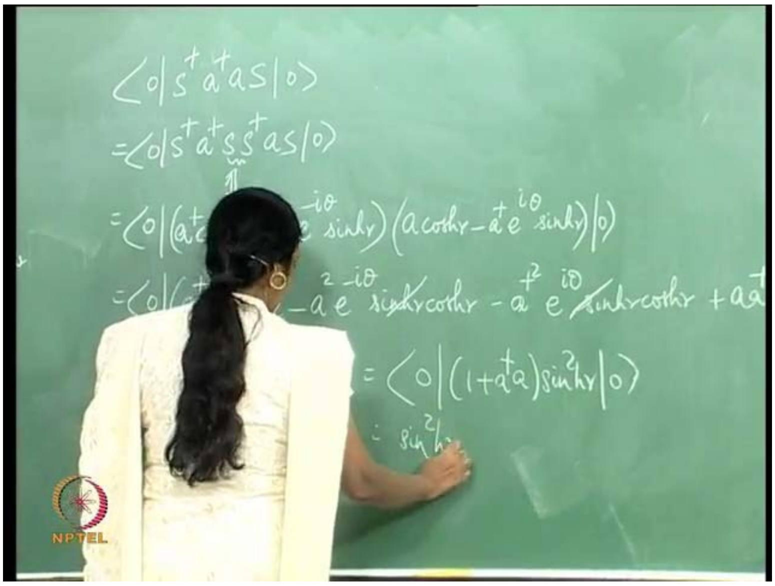

So here we go, we need to find 0 S dagger a dagger a S ket 0, that is item 3 here. Once more I use the old trick and I introduce an S S dagger in between here on ket 0. Now S dagger a dagger S is a dagger cos h r minus a e to the minus i theta sin h r and S dagger a S is a cos h r minus a dagger e to the i theta sin h r and of course, there is a ket 0. Once more I can expand the terms. The first term is an a dagger a cos squared h r. Then I have a minus a squared e to the minus i theta sin h r cos h r. Then I have a minus a dagger squared e to the i theta sin h r cos h r and the last term is plus a a dagger sin h r sin squared h r and this whole thing acts on ket 0. This can be simplified as before.

Look at the first term a dagger a acting on ket 0 gives me 0 so that does not contribute, a squared simply annihilates ket 0, a dagger squared acting on ket 0 takes it to ket 2 and by orthogonality this is 0 and I am left with 0 a a dagger sin squared h r 0. As before I can work it out and this object is simply 0, 1 plus a dagger a sin squared h r ket 0 and while a dagger a does not contribute the one does and I get a sin squared h r. So, this is all that I have here.

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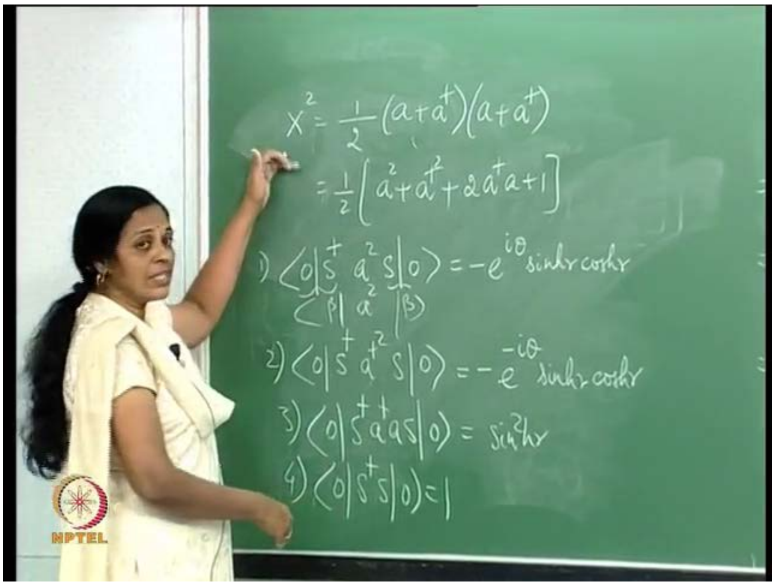

So, I have various terms which I have put down. This object is simply sin squared h r and all we need to do is re-substitute these in X squared and find out the expectation value.

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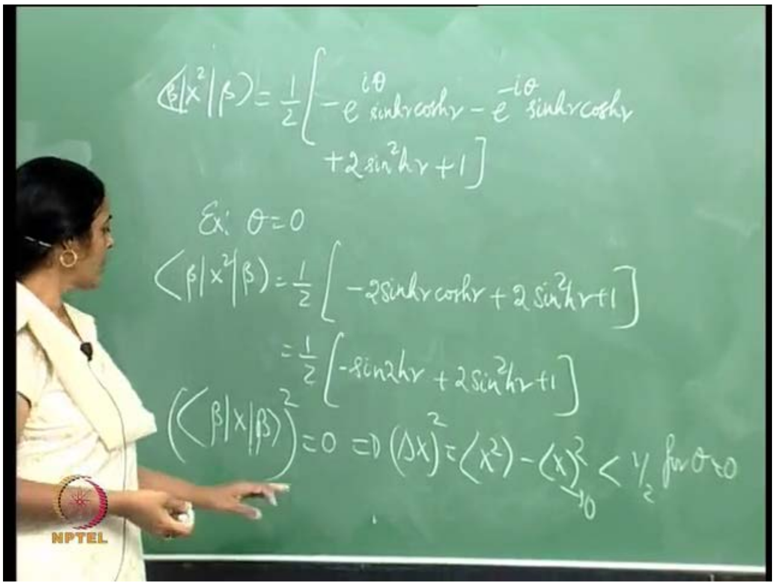

So let us look at what is expectation X squared. Expectation X squared in the state beta is what we are out to find. (Refer Slide Time: 47:21) And that is half expectation a squared which is minus e to the i theta sin h r cos h r minus e to the minus i theta sin h r cos h r. So, that takes care of a squared plus a dagger squared plus twice a dagger a. (Refer Slide

Time: 47:21).That is 2 sin squared h r plus 1 and that is just a 1 there and the whole thing multiplies half. Let set theta equal to 0. Example: theta is a phase it can take various values. If we say theta is equal to 0 that is minus 2 sin h r cos h r plus 2 sin squared h r plus 1. That is half minus sin 2 h r plus 2 sin squared h r plus 1.

This object is always less than half. Beta X squared, beta is always less than half for r greater than 0, r has to be greater than 0 because you will recall that beta is r e to the i theta and for all values of r, this object is less than half. Since expectation X the whole squared was 0 because expectation X was 0 and expectation X squared is less than half for this example when theta is equal to 0. This implies delta x the whole squared is expectation X squared minus expectation X the whole squared. Since, this object is less than half. It is less than half for theta equal to 0.

So, with this value of theta at least, we have seen that there is squeezing in the X quadrature. We can calculate delta p, where p is a minus a dagger by root 2 i as defined in an earlier lecture. We will find that for theta equal to phi, there is squeezing in the p quadrature. The squeezed vacuum is very interesting for the following reason that for various values of the parameter theta, you could either get squeezing or no squeezing.

And therefore, I can change the parameter and change the various squeezing properties of this particular state, the squeezed vacuum. Such interesting quantum superpositions of light have been realized in the laboratory. The coherent state, superpositions of the coherent state, ket alpha and say 1, alpha 2, alpha etcetera call the photon added coherent states, the squeezed vacuum and so on have been produced in the laboratory and used for various purposes.