Quantum Field Theory
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Module - 2
Interacting Quantum Field Theory
Lecture - 9
Interacting Field Theory - II

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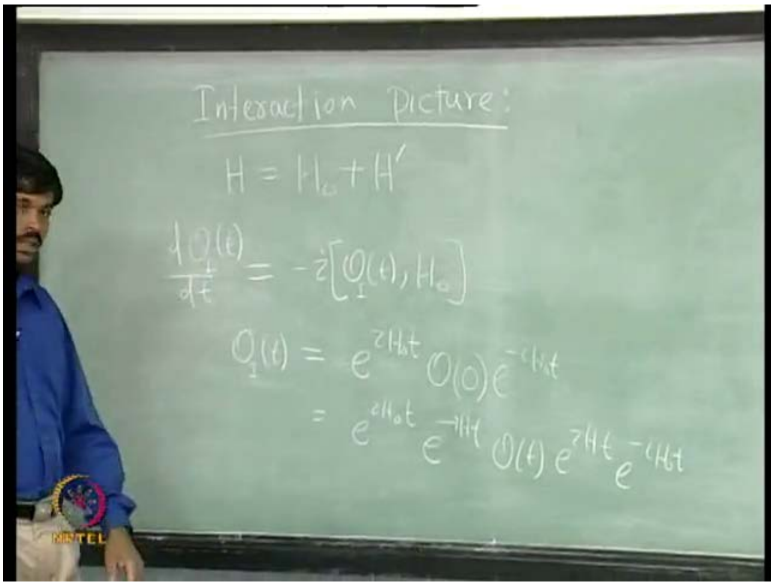

So, let us briefly discuss what we covered in the last lecture. We are discussing the interaction picture. In the interaction picture, both the operators as well as this state vectors are time dependent. However, the time evolution of the operators is covered by the free Hamiltonian, the full Hamiltonian as a free pattern in interaction part. The free part, we know how to quantize, we know the exact form. We do not know in general how to deal with the full Hamiltonian. So, what we did is we have introduced to the interaction pictures and then we will see how the interaction picture helps in finding this factrom and so on.

In the interaction picture, the operators if I call OI of t to be an operator, it has time dependents, which is given by the Heisenberg equation of motion where the Hamiltonian and as the free Hamiltonian minus i computation of O of t H 0 and is given by e to the power H 0 t. So, this is how the operators in the interaction picture are related to the operators in the Schrödinger picture. Note that here we took t equal to 0 as a reference time and then we found the operator in the interaction picture at time t; we did not do that we can take some time t 0 to be a reference time. In that case, here you should get t minus t 0 in both this places.

So, this is how the operator in the interaction picture is related to the operation operator in the Schrödinger picture. If this is so then this operator obeys the Heisenberg equation of motion with the free Hamiltonian in place of the full Hamiltonian. Now, we can also relate the interaction operators in the interaction picture with the operator. In the Heisenberg picture, they are given by e to the power minus i Ht.

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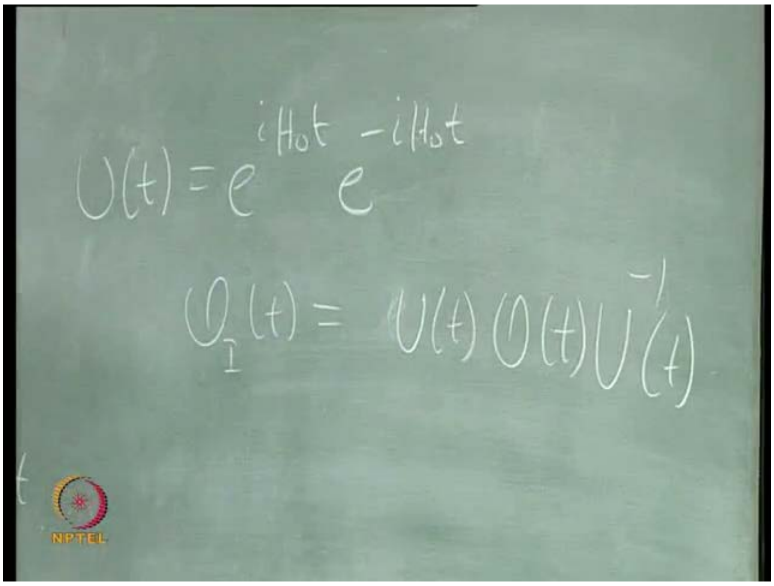

If we introduce an operator U, which is defined as e to the power i H0 t e to the power minus i H0 t, then this is how the operators in interaction picture is related to the operator at the Heisenberg picture. So, what is the use of this relation? The use is the pulling. Although we do not know how to solve the full Hamiltonian, we know how to solve and find this factor for the free Hamiltonian. The operators in the interaction picture obey the equation of motion as if the Hamiltonian is the free Hamiltonian T.

Therefore, we can solve, we can find the full solutions for the fields. Hence, any operator which is constructed out of these fields and this then we can from here, we can go to the interaction picture. We can find the operators in the both in the interaction as well as in the Heisenberg picture by using this relation. For example, let us consider the Klein Gordon effect.

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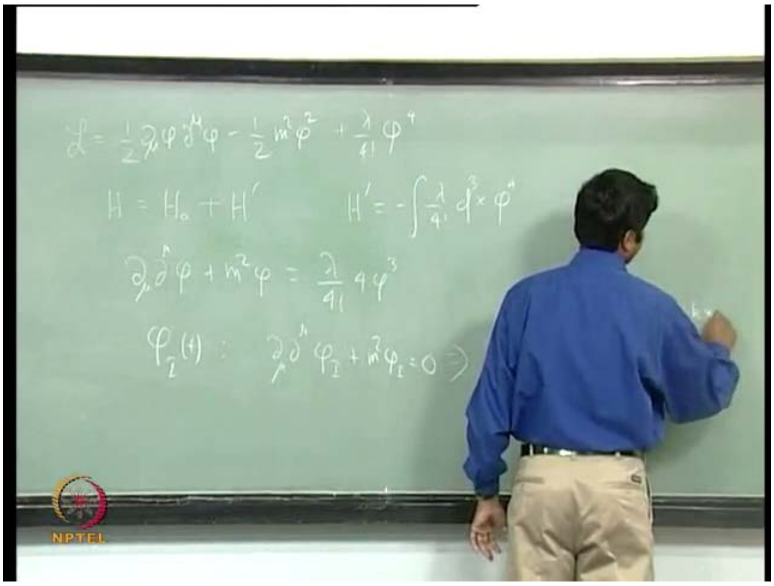

Suppose you are lagrangian density is given by half del mu phi del mu phi minus half m square phi square plus some interaction term, which I will call as lambda over 4 factorial phi to the power 4. This is the interaction term. Hence, the Hamiltonian has two parts. One is the free Hamiltonian and the other one is interaction Hamiltonian, which is the H prime is minus lambda over 4 factorial d cube x phi 4. This is the interaction Hamiltonian. The equation of motion is given by del mu del mu phi plus m square phi equal to lambda over 4 factorial into 4 phi cube.

However, the equation of motion for the free Hamiltonian or the Heisenberg equation for the operators for the field if phi I of t is the field in Heisenberg picture in the interaction picture. Then phi I obey an equation of motion, which is given by del mu del mu phi I plus m square phi I equal to 0. You can consider the free Hamiltonian, which is the Hamiltonian derived from just this part. You can write down the Heisenberg equation of motion. You can show that that is equivalent to this equation. So, this is the equation obeyed by the field in the interaction picture. However, you know already how to solve this equation. What is the most general solution for this equation?

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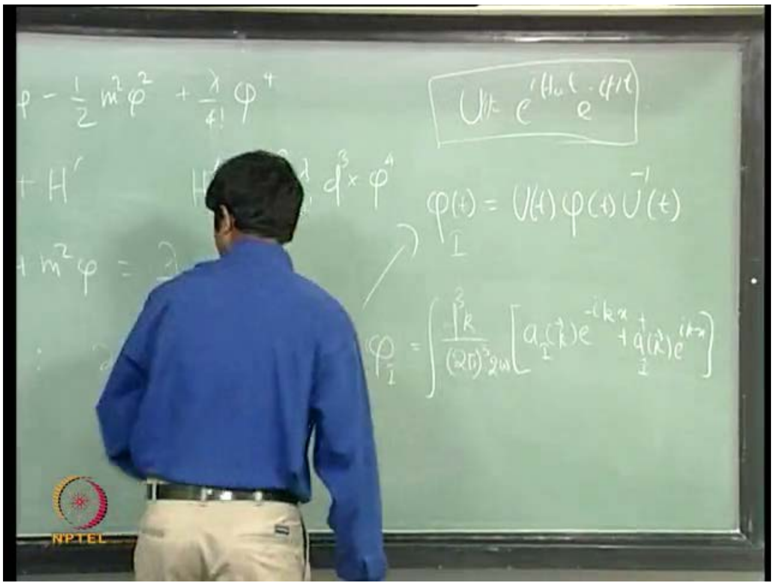

So, the most general solution for this equation is given by phi I equal to integration d cube k over 2 pi cube 2 omega times a i k e to the power minus i k dot x plus a i dagger k e to the power i k dot x. So, we already know phi I in the interaction picture and from here, I can construct these phi of t by using the relation phi. Phi I is equal to u of t phi of t U inverse of t. So, therefore, I know what the field is in the Heisenberg picture. If I know what the field is in the interaction picture provided, I know what U of t is. U of t as we have already seen is given by e to the power i H0 t e to the power minus i H t.

However, we know the fields in the interaction picture. Therefore, we would like to express this operator U as a functional of the fields in the interaction picture. So, to do that, we would like to find an equation, which is obeyed by this operator U. Then we will discuss how to solve that equation to find a general expression for this operator U in terms of the fields in the interaction picture. This is what we will do in the next part of this lecture.

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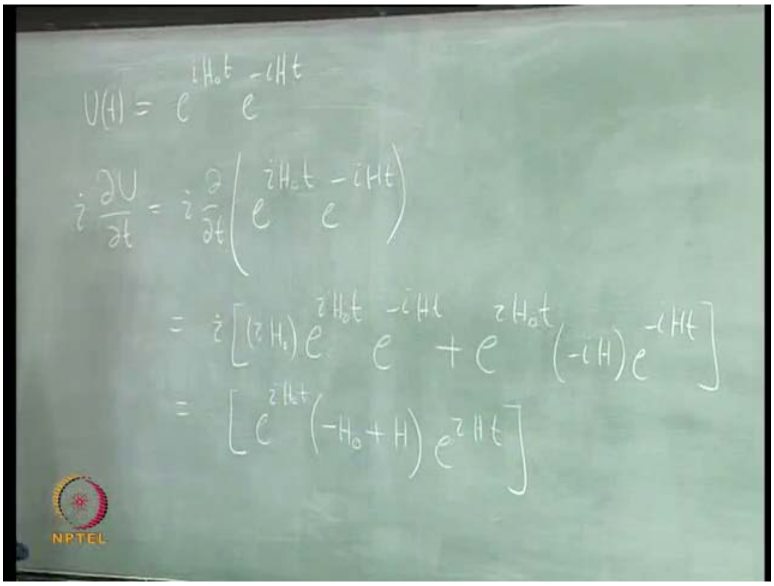

So, let us start with this expression for U of t. This is e to the power i H0 t e to the power minus i H t. Consider del u over del t with a factor of i. So, this is i del over del t times e to the power i H0 t e to the power minus i H t. It is quite straight forward to carry out this differentiation. This is i times i H0 e to the power i H0 t e to the power minus i H t plus e to the power i H0 t minus i H e to the power minus i H t. This computes with this. So, what I can do is this I can rewrite it as e to the power i H0 t minus H0 plus H e to the power i H t. Now, H minus H 0 is H prime the interaction Hamiltonian.

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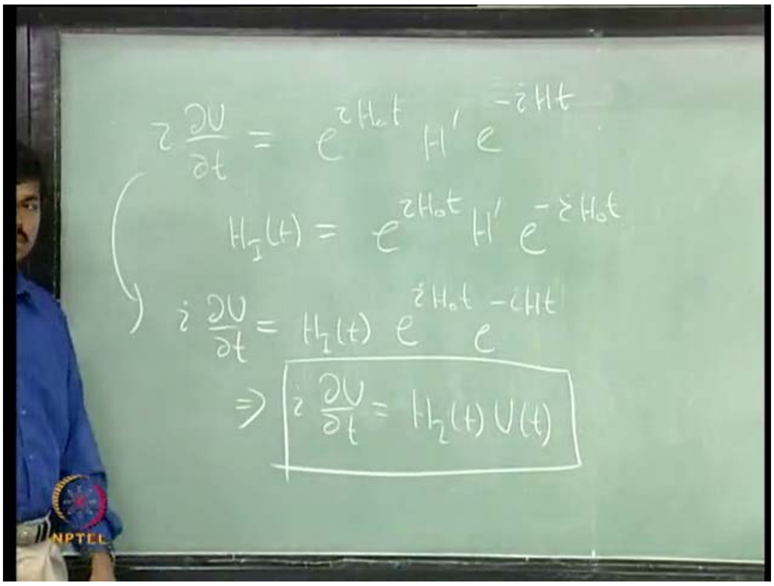

So, what I get is i del U over del t is e to the power H0 t H prime e to the power minus, there is a minus here minus i H t. However, we know HI of t the interaction Hamiltonian in the interaction picture is related to the H prime by this relation e to power i H0 t H prime e to the minus i H0 t. So, if we use this relation, what we can see is i del U over del t is here I can introduce a factor of e to the power i H0 t e to the power minus i H0 t with its identity. So, what I will get is HI of t and e to the power i H0 t e to the power minus i HT.

However, this is nothing but U of t. So, what I get is i del U of del t is equal to HI of t acting on U of t. So, what we did is we started with the definition of this operator U, and then we have shown that this operator U obeys this equation. Now, what I can do is I can solve this equation.

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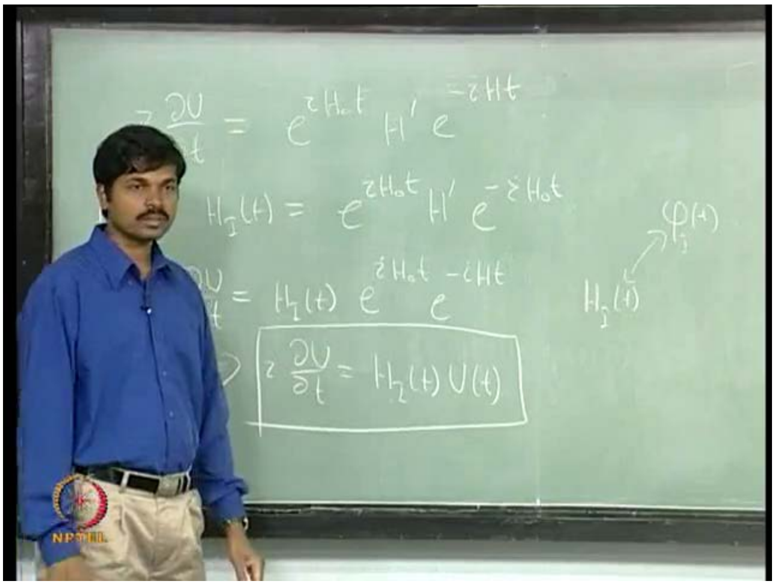

HI of t is the interaction Hamiltonian in the interaction picture. So, I can express HI of t as a functional of the fields in the interaction picture. So, what I will do is I will solve this equation. Then I will express this operator U as a functional of the interaction Hamiltonian in the interaction picture and hence the operator U as a functional of the fields in the interaction picture. So, let us solve this equation.

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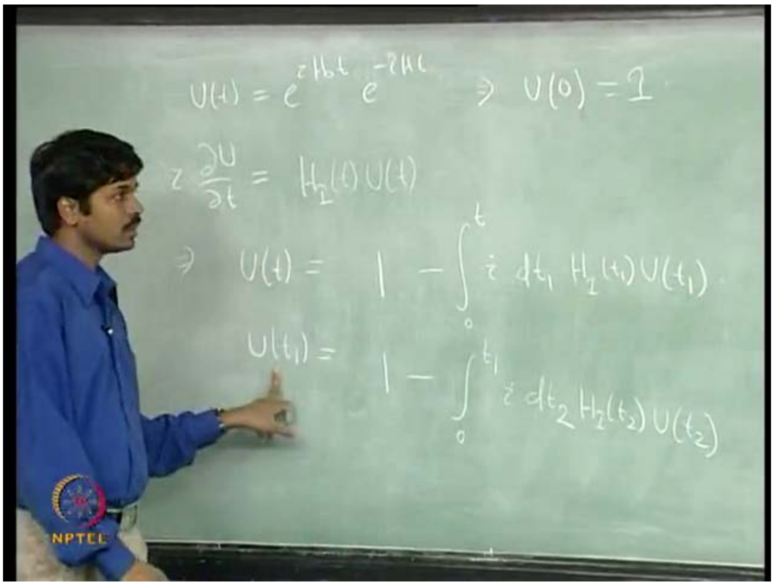

Before solving, note that since we have defined U to be e to the power i H0 t e to the power minus i HT. This implies U of 0 is the identity operator. So, at t equal to 0, this is identity. Using this condition, we can integrate this equation i del U over del t equal to HI. This implies U of t equal to U of 0 minus integration 0 to sometime t i d t 1 HI of t 1 U of t 1. Since, U of 0 is identity, this is 1 minus this. You can recover this equation by differentiating this with respect to t. As you can see, if you differentiate this respect to t, you will get del u over del t is minus i HI t acting on U of t, which is this, the same equation.

So, I have got this equivalently U of t 1. Since, this t is some arbitrary time, 1 minus 0 to sometime t 2 i d t 2 HI of d 2 U t 2.

Student: 0 to t 1.

Thank you. So, what I will do? Now, I will substitute this expression for U of t 1 here.

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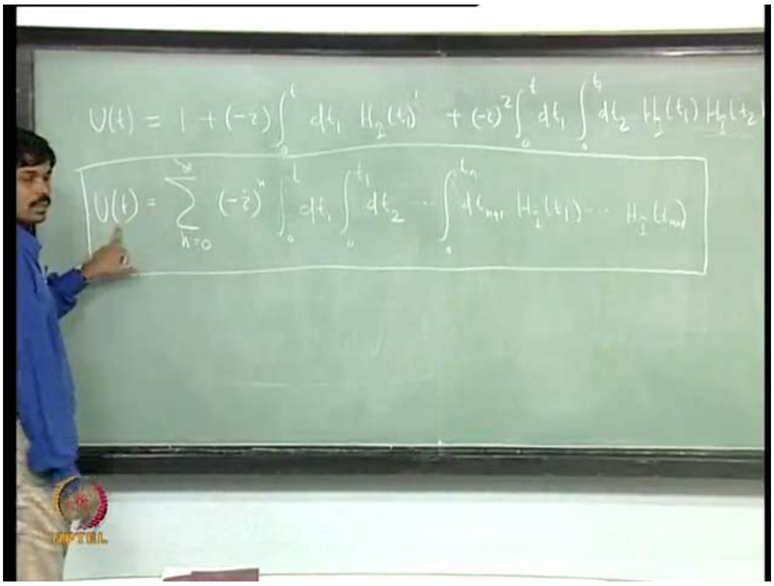

If I substitute what I will get is U of t is 1 plus minus i, 0 to t d t 1 HI of t 1 plus minus i square 0 to t d t 1, 0 to t 1 d t 2 HI of t 1 HI t 2 U t 2, but now again, I substitute for U of t 2. I can keep doing this. So, when I do that I will get a series, which in general can variant sum over n equal to 0 to infinity minus i to the power n 0 to t d t 1 0 to t 1 d t 2 and so on up to 0 to t n d t n minus 1 HI t 1 up to HI t n. This is what is the expression for U of t d t n plus 1 here, HI of t n plus 1. So, what we will do is we will start with this expression for U of t. Then we will rewrite it in a nice form. To do that, let us look at this second term in this expression.

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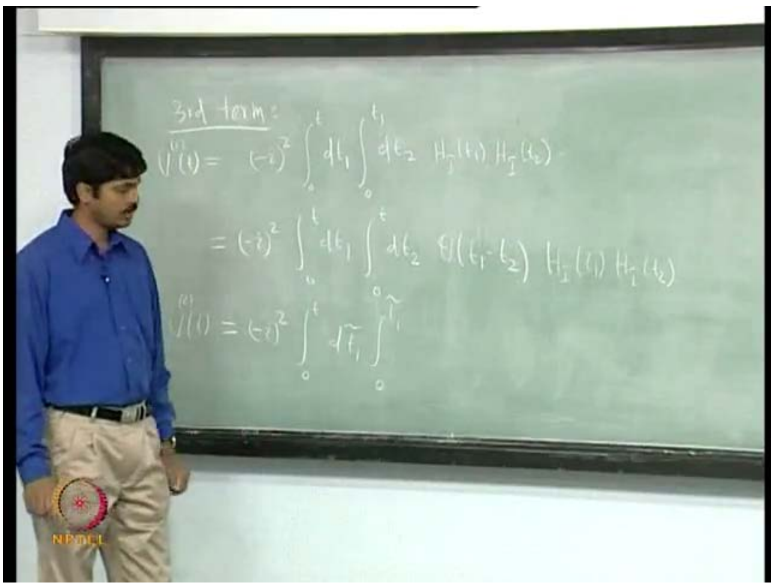

The second term in this expression is given by minus i square. So, this is the third term minus i square integration 0 to t d t 1 0 to t 1 d t 2 HI of t 1 HI of t 2. So, this is some i will give this term as some name which is U 2 t. Look at this, this t 1 and t 2 are dummy variables. So, what I can do is I can, so first of all, I can do two things. First thing is this integration limit this t, t 1 varies from 0 to t, whereas t 2 varies from 0 to t 1. There is no U of t 2. So, what we did is we substituted it for U of t 2 again. Then this gives us an infinite series here. We look at this infinite series and then term by term, this series term by term.

The second term in this series is given by this. So, because t 2 varies from 0 to t 1, I can rewrite this as minus i square 0 2 t d t 1 0 2 t d t 2 times theta of t 1 minus t 2 HI of t 1 HI of t 2. Is this clear? This step function makes you that although here I have taken the range of this variable t 2 to be from 0 to t, this step function will make you that it is non zero, only in the range 0 to t 1 that is number 1. Number 2 is because this t 1 and t 2 are dummy variables. I can rewrite this U 2 of t to be something as minus i square 0 to t d t 1 tilde. I will call this to be t 1 tilde because it is a dummy variable. However, the integration here will range from 0 to t 1 tilde and here this is d t 2.

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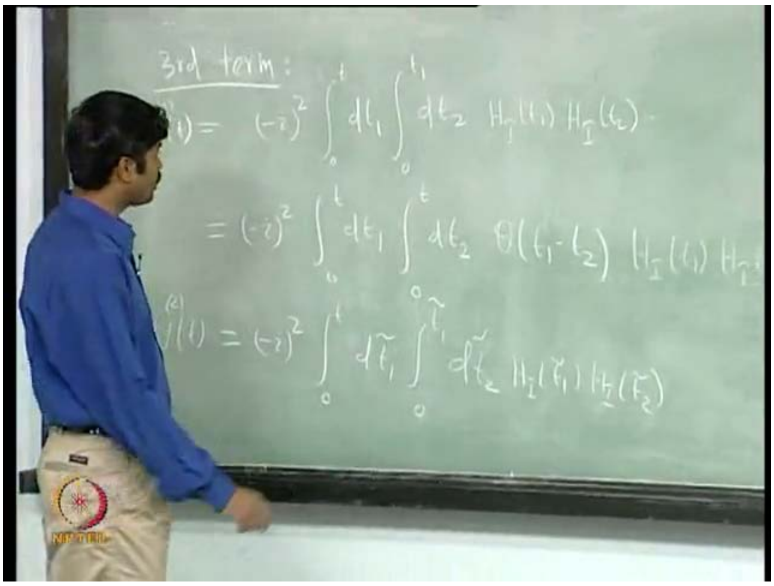

Again since t 2 is a dummy variable, I will call this to be t 2 tilde and HI of t 1 tilde HI of t 2 tilde.

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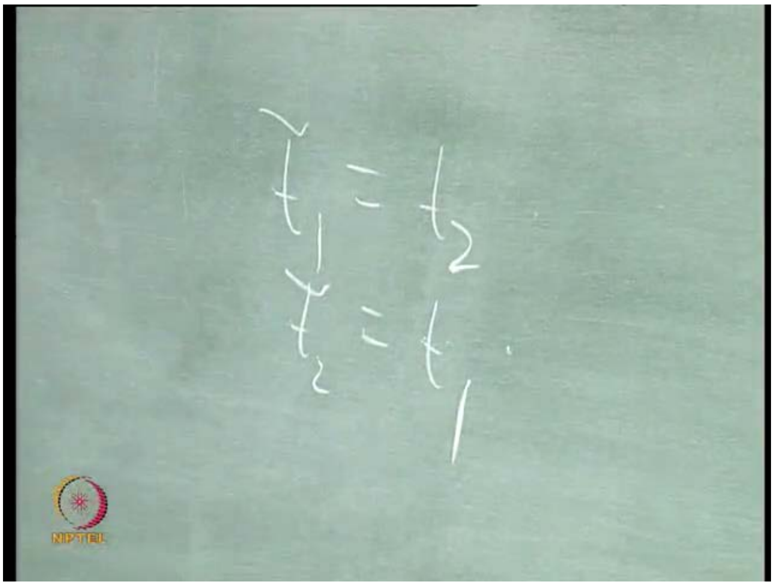

But, now again because t 1 tilde and t 2 tilde are dummy variables, let us label them as t 1 tilde equal to t 2 and t 2 tilde equal to t 1.

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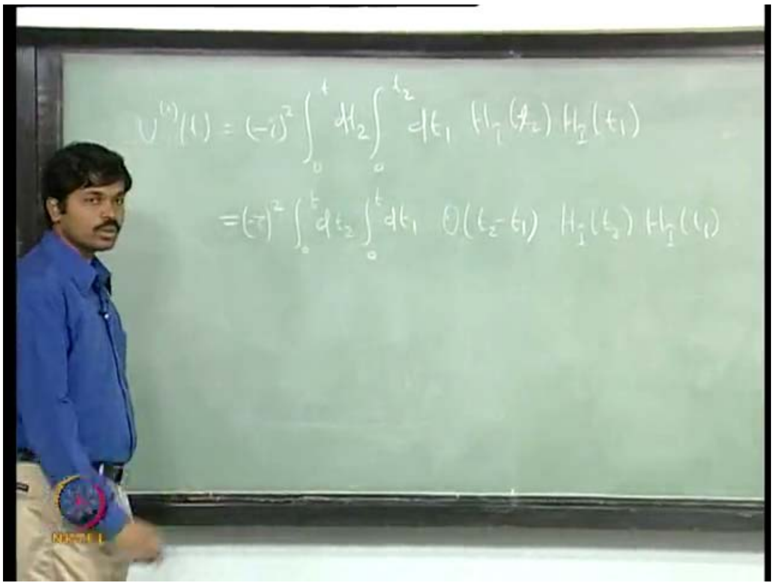

If you do that, then what you will get is U 2 of t will be minus i square 0 to t d t 2 0 to 2 d t 1 HI of t 2 HI of t 1. What I have done? What I have done is I merely exchange t 1 and t 2 because they are just dummy variables. Again, I can introduce a step function here. I can take the range of this variable t 1 to be from 0 to t. So, when I do that, this is minus i square 0 to t d t 2 0 to t d t 1. However, now it will be theta of t 2 minus t 1 HI of t 2 HI of t 1. So, I can take this and this expression. I can add them and divide by 2.

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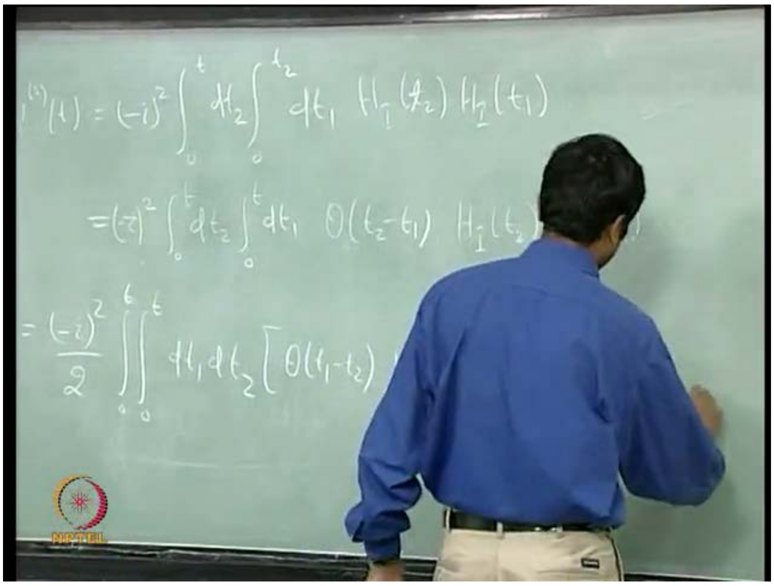

So, as a result, what I get is the same expression as minus i square over 2. Now, d t 1 d t 2 times theta of t 1 minus t 2 HI t 1 HI t 2 plus theta of t 2 minus t 1 HI t 2 HI t 1.

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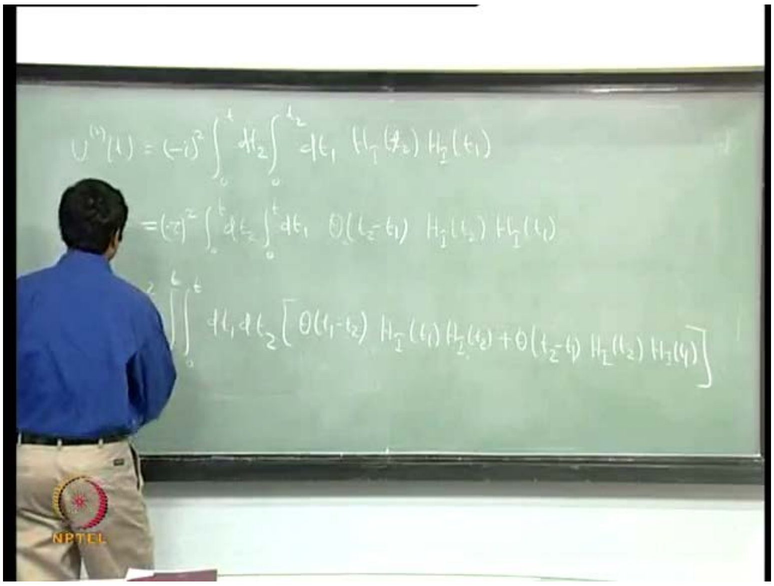

However, there is a notation that we are using for this particular expression. What is that? This is nothing but the time order product of HI t 1 and HI t 2. If t 1 is greater than t 2, then this whole expression is HI t 1 HI t 2, whereas if t 2 is greater than t 1, this is simply HI t 2 HI t 1. Therefore, this is nothing but the time order product of HI t 1 and HI t.

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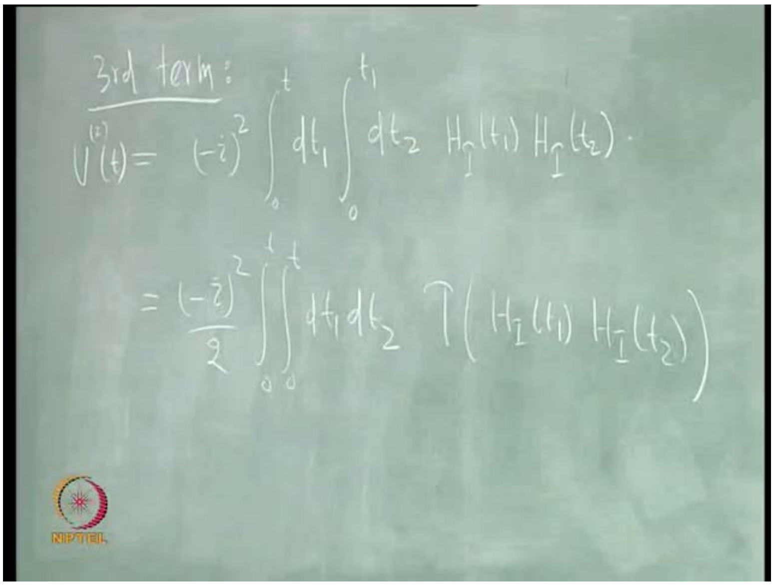

So, this expression here is given by minus i square over 2 0 to t d t 1 d t 2 time order product of HI t 1 HI t 2. You can use the same argument and you can look at the n plus 1th term in the expression for U. What you will get is the following.

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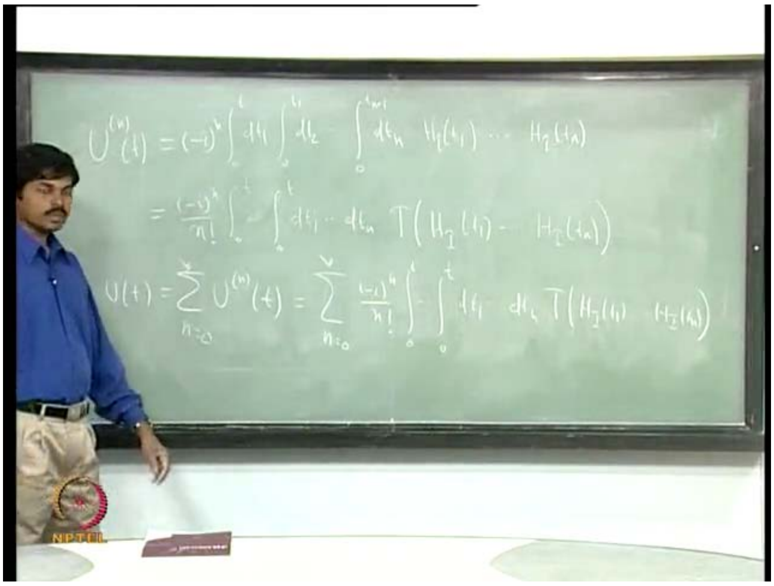

So, in a similar way, it can be shown that U n t, which is minus i to the power n 0 to t d t 1 0 to t 1 d t 2 and so on up to 0 to t n minus 1 d t n HI t 1 HI t n. This expression here can be written as minus i to the power n divided by n factorial 0 to t d t 1 up to d t n time order product of HI t 1 of t n. So, this operator U is nothing but the sum over m equal to 0 to infinity U n t and this is sum over n equal to 0 to infinity, this term minus i to the power n over n factorial. If there is no time order product, this would look like exponential of these terms.

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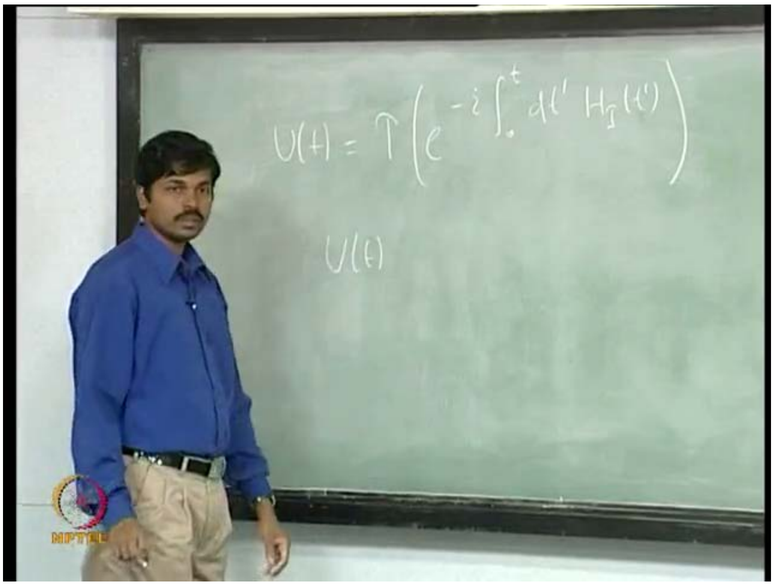

So, what I will do is I will write this term here as U of t is time order product of e to the power minus i 0 to t d t prime HI of t prime. But, time order what I mean? I expand this exponential in terms of the power series of this integration. In each term, I take the time order product of these operators. That is what I mean by time order product of this whole thing all. Remember when I define this U of t, actually what I did is e to the power i H0 t e to the power minus I H t. This was the case when t equal to 0 was the reference time. However, there is no reason for us to take t equal to 0 to be the reference time.

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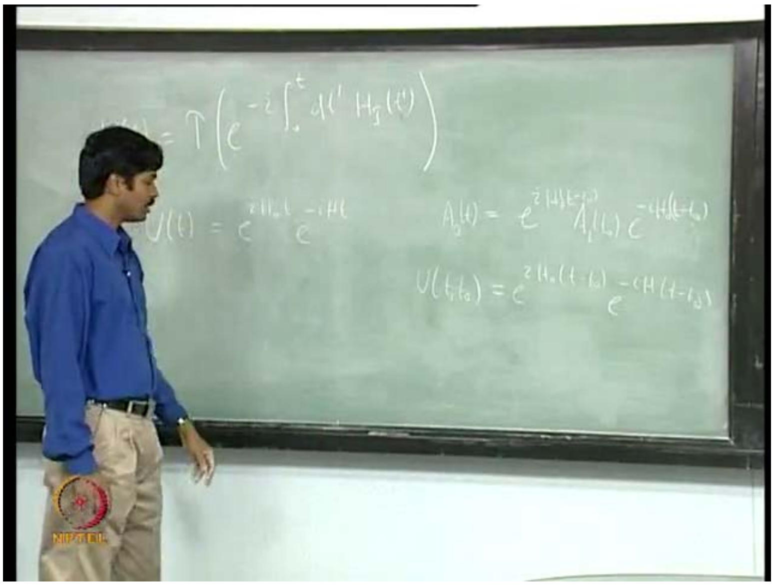

In other words, in this expression AI t which relates this operator in the interaction picture with the operator in the Schrödinger picture by this relation AI of 0 e to the power minus i H0 t, I took t equal to 0 to be the reference time which is arbitrary. So, there is no reason for us to take t equal to 0 to be reference time. Instead, I can t 0 to the some reference time and then define the operator with the interaction picture.

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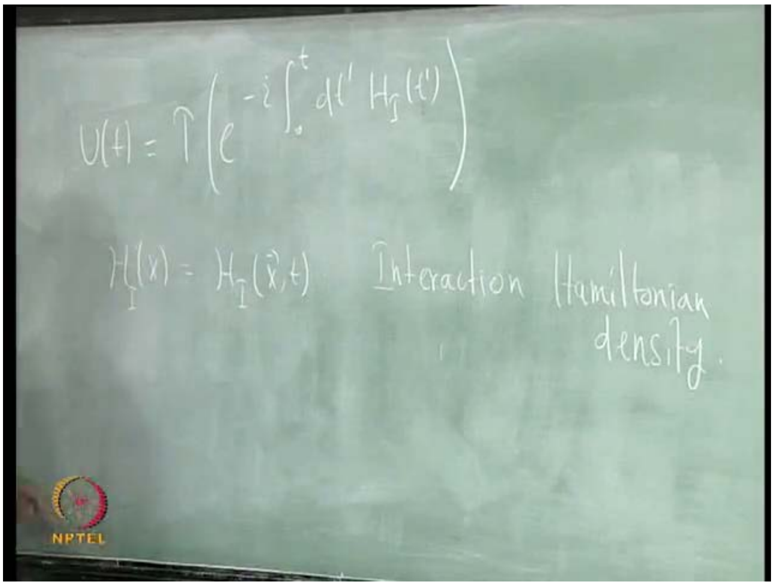

Then, what I have to do is I have to merely substitute here this as U t minus t 0. Accordingly, the operator U will have both t as the time on which the operators in the interaction picture is defined and t 0 to be the reference time. So, this will be e to the power i h 0 t minus t 0 e to the power minus i h t minus t 0. So, in this case, everything will through except that the integration range will be from t 0 to t. This is the only change that is there. In field theory, we can define the interaction Hamiltonian and density HI of x, I will denote which is also HI of x t.

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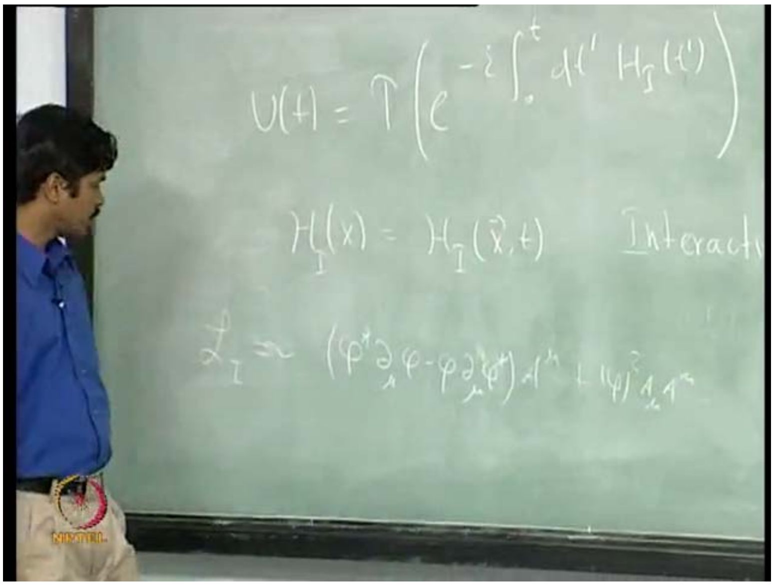

So, when you write down the lagrangian, the full lagranzian, it will have free parts as well as it will parts like in the case of electromagnetic field interacting with a charged scalar field. We have seen the interaction term in the lagrangian to be something like del mu phi minus phi del mu phi star times A mu plus phi square A mu, A mu something like that the terms where non linear in the field, but they were all functional of these fields.

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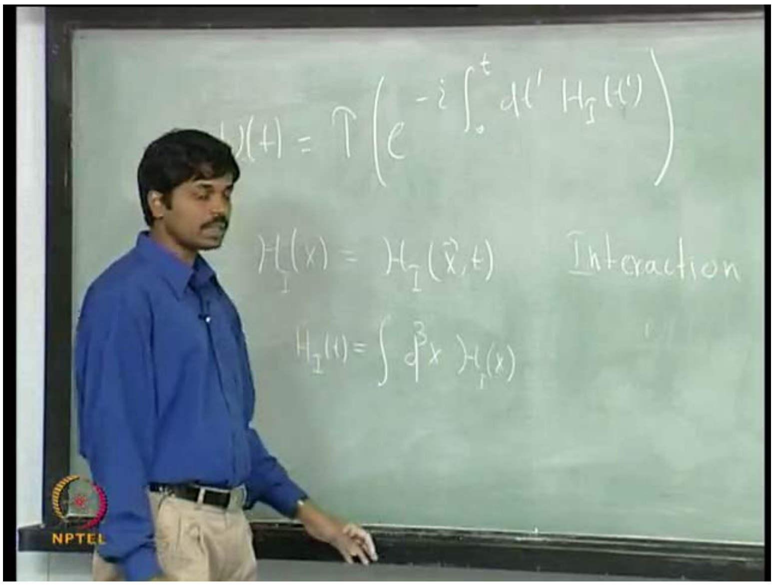

So, when you write down the interaction Hamiltonian, this will also be given by integration over d cube x of some Hamiltonian density HI of x. So, the interaction Hamiltonian in the interaction picture typically is given like this In terms of an interaction Hamiltonian density. So, I can write this operator U of t in terms of HI, the interaction Hamiltonian density.

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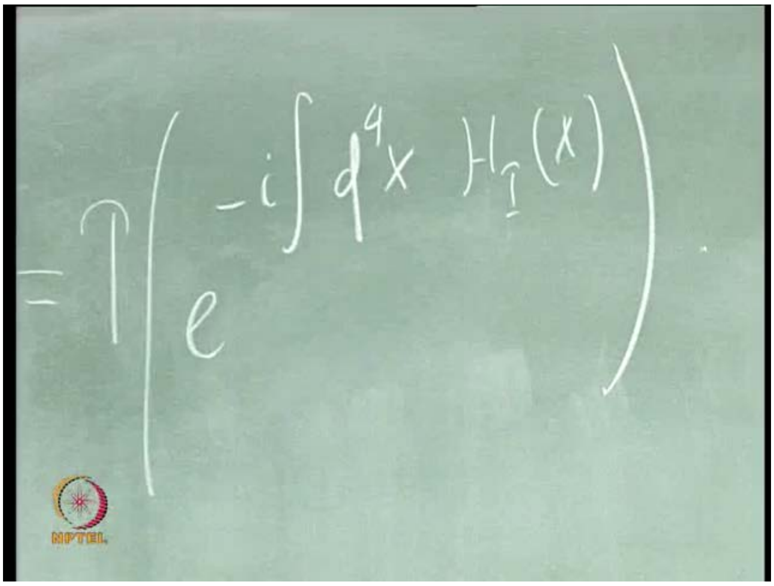

When I do that, this expression will look like time order product of e to the power minus i d 4 x HI of x. So, this looks like a Lawrence covariant expression. Instead of considering the operators, you can also consider the state vectors. We know these state vectors in the interaction picture obey a Schrödinger equation where the Hamiltonian is the interaction Hamiltonian.

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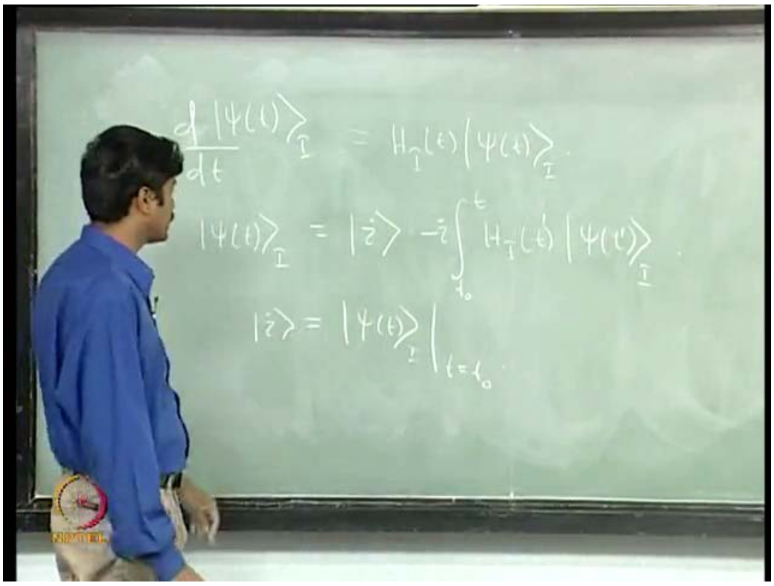

So, if psi of t I is the state in the interaction picture, then this obeys the equation i s psi over d t equal to HI of t acting on psi t. So, you can consider this equation instead and then we can solve it for psi I of t. When we do that, we can again get some expression like psi t I equal to, let us say there is some initial state and then minus integration, some t 0 to t minus i HI of t prime psi I of t prime exactly. The same way we can proceed and then we can solve this equation for the state vector. Then we can get something like that where this state i is defined as psi I of t at time t equal to t 0. Then you can do iteration. Ultimately what you will get is again you can substitute for psi I of t prime and so on.

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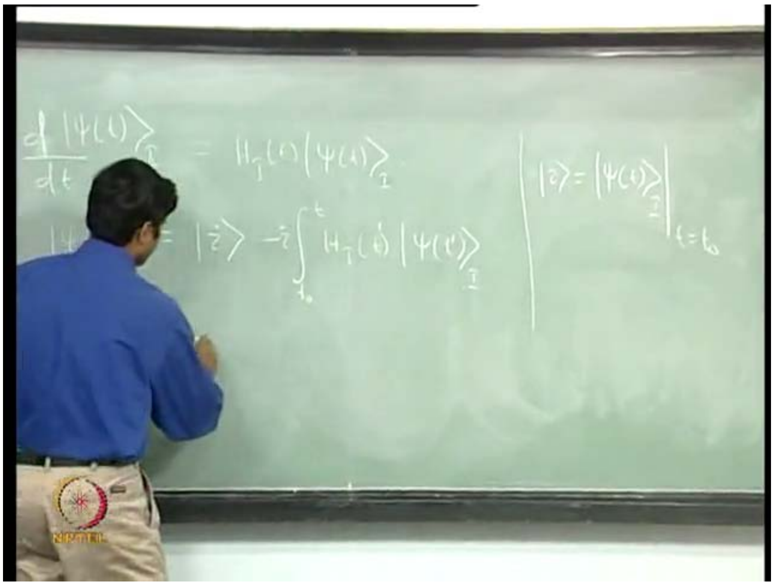

When you do that, you will see that again you get an expression where this becomes, let us write i is psi t at t equal to t 0.

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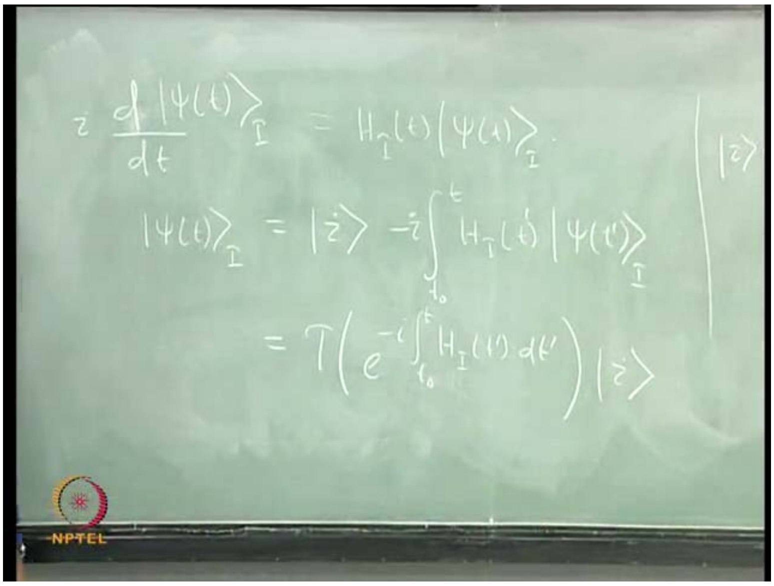

This is the time order product of e to the power minus i HI of t prime d t prime where the integration range is from 0 to t acting on this state i.

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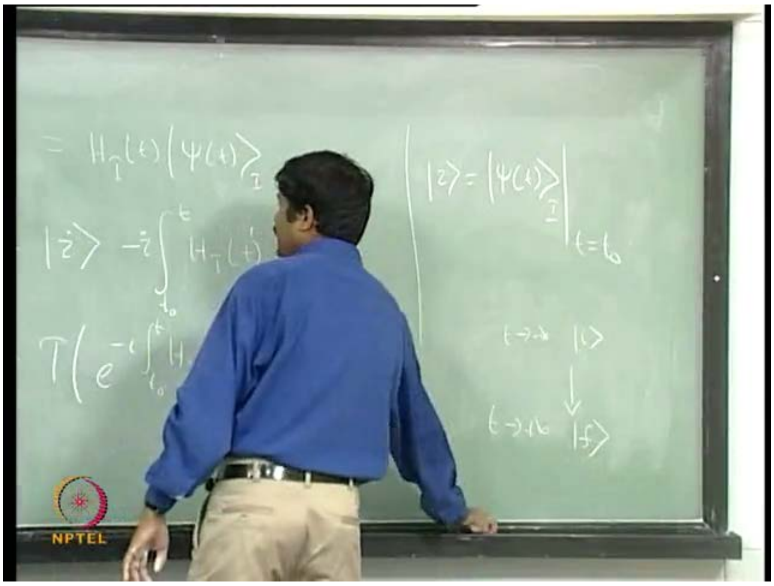

Now, when you are scattering, what typically you do is you prepare the initial state at some time, which is t goes to minus infinity where the particles were essentially non interacting. So, the interaction part of the Hamiltonian at as time t goes to infinity H0, these particles come together, they propagate, they come close. At some finite time, they interact. Then after the interaction take place again, they just essentially, they scatter or they interact some physical interaction takes place.

Then, finally, they go away and again HT goes to plus infinity, you get some final state. You would like to ask is what is the probability of suppose you started with some state i H time at time t equal to minus infinity; in presence of this interaction, what is the probability of this state i going to some final state f at time t goes to plus infinity? So, these are the questions you would like to ask.

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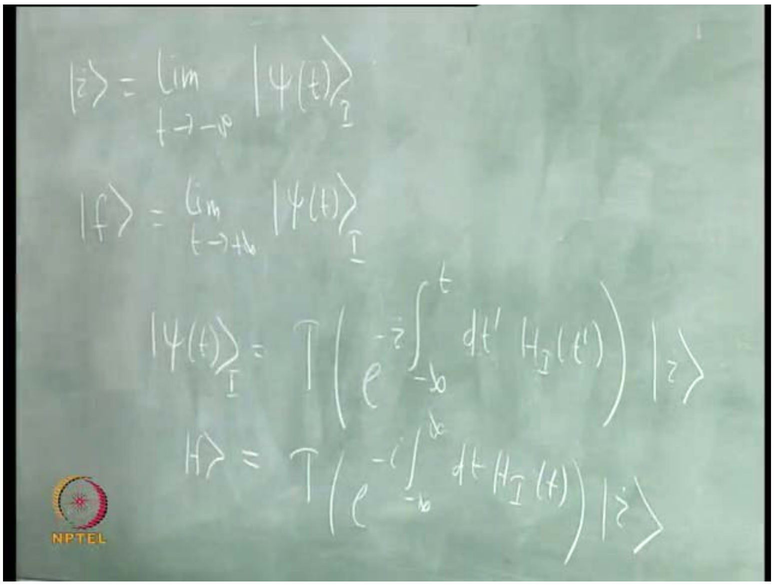

So, in this case you define the state i to be the limit t goes to minus infinity psi t I and you define the final state f to be limit t goes to plus infinity psi t I. Then from this expression, you can see that psi t I is nothing but time order product of e to the power minus i minus infinity to t d t prime HI of t prime acting on this state i. From this expression, what I can see is f is the state in the limit t goes to plus infinity.

So, this is time order product of e to the power minus i minus infinity to plus infinity d t HI of t acting on i. So, therefore, this operator contains the information about the final state. If we know this operator here, we can know what is the probability of this state i at t equal to minus infinity going to some state f at t equal to plus infinity? This is what is known as the S matrix.

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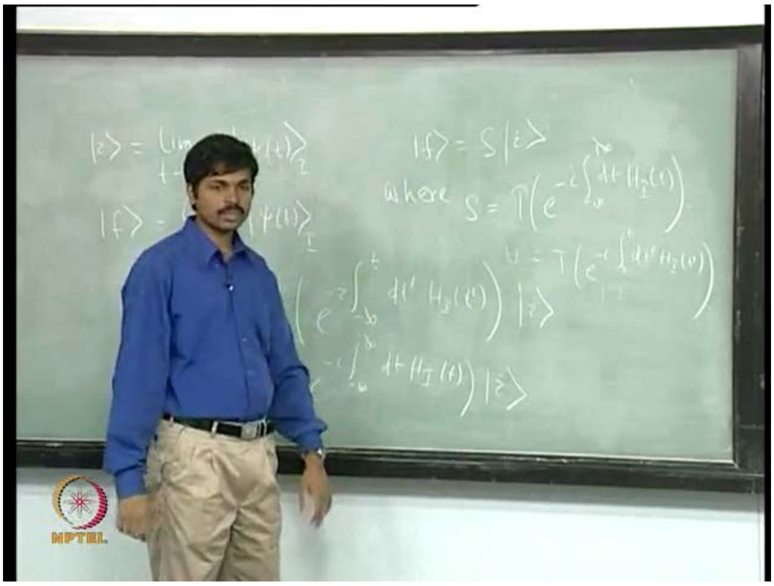

So, f is some S times i where S is time order product of e to the power minus i minus infinity 2 plus infinity d t HI of t. So, what would you like to find is we would like to we have already expressed this S matrix as well as the U matrix, which is time order product of e to the power minus i from some time 0 to t d t prime HI of t prime S some time order product. However, if we want to evaluate the matrix elements of these operators, the time order product is not a convenient expression.

In the next lecture, we will see that we can express this time order product of various operators in terms of normal sum of normal order products and some c numbers. That way, it is convenient to evaluate the matrix elements of the various terms in this expression. So, this is what we will be doing in the next lecture.