# Quantum Field Theory

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## Module - 1
## Free Field Quantization – Scalar Fields
## Lecture - 4
## Quantization of Real Scalar Field – II

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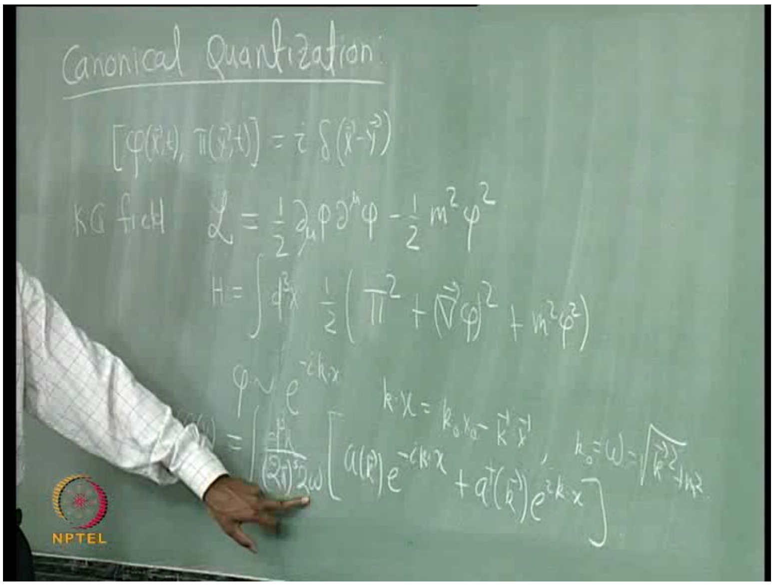


So, let me briefly review what we discussed in the last lecture. We are discuss the process of canonical cube quantization, and we started with the fundamental commutation relations, which is phi of (x, t), pi of (y, t) the commuter of phi and pi is i delta x minus y, where this is the Dirac delta function. And we are discussing the example of Kelvin Gordon field for its Lagrangian density is given by; L equal to half del mu phi del mu phi minus half m square phi square. We have seen that the Hamiltonian is a integration d cube x half phi dot sorry, pi square plus grade phi square plus m square pi square. And we need to find this spectrum for this Hamiltonian, in order to do that we were considering the plane wave solution to the equation of motion.

So, the plane wave solutions are this form is e to the power minus i k dot x, where k dot x equal to k 0 x 0 minus k dot x and k 0, which i also denominated omega h square root of k square plus m square. Then any general solution to the classical equations of motions can be written as a super position of these plane wave solutions. So, the most general phi is retain its integration d cube k over 2 pi cube 2 omega times a of k due to the minus psi k dot x plus a dagger k to the power i k dot x. Here, this 2 omega is for convince and when we cube quantize this phi pis are operators and hence the coefficients here a k and a dagger k are operators. We wanted to find the corresponding commutation relation for these operators, which are a k and a dagger k to do that we inverts this relation and to express the operators a k in terms o f phi and pi.

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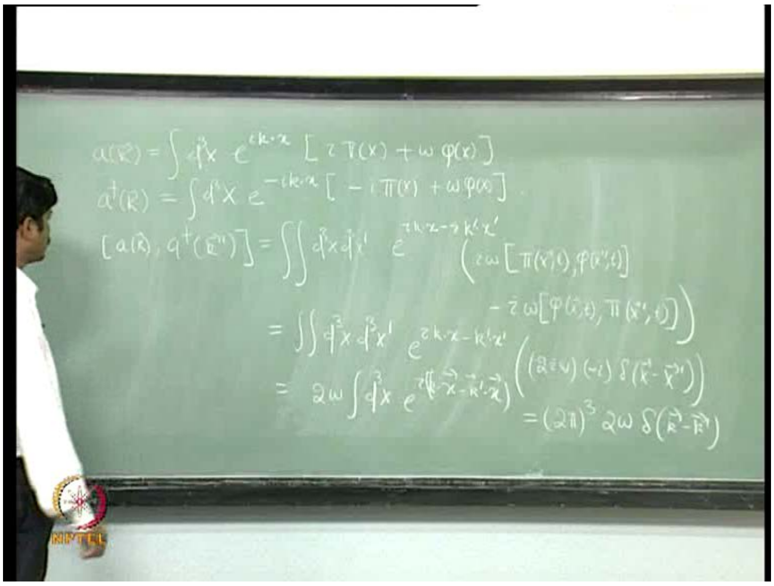

So, this is what we have done in the last lecture. And the result is a k h integration d cube x e to the power i k dot x i pi of x plus omega phi of x, a dagger k it is a d cube of x e to the power minus i k dot x minus i pi of x plus omega phi of x. What we will now do is; we will find the, what is the commutation relation a (k), a dagger integral k prime, alright. So, this commutation relation will simply be integration, there 2 integrations in the first I will use x as the integration variable, and in the second expression I will use x prime as the integration variable. So, d cube x d cube x prime, and from this I will get e to the power i k dot x. Here, minus i k prime dot x prime because the argument of a dagger k prime in the integration variable for a dagger I am using as x prime.

And, then the commutate commutator of these 2 quadrates, while we evaluating the commutator you must remember that pi of x commutes with pi of x prime and similarly; phi of x commutes with phi of x prime. So, keeping this in mind the commutator of this 2 turns will become here, i omega and commutator of pi of x pi of x prime. So, i omega pi of x t phi x prime t, because e f considering equal time commutation relations. And similarly; here i will get the commutator of phi and x, so minus i omega phi of x t and pi of x prime. Now, we will use the fundamental commutation relation, which is this commutator here is i delta x minus x prime.

Similarly, here the commutator is minus i delta x minus x prime. So, when i use that what i get is integration d cube x d cube x prime e to the power i k dot x minus x prime dot x prime, and 2 i omega times minus i delta x minus x prime. So, here I will get a 2 omega and using delta x minus x prime I can integrate out one of these variables. So, ultimately i get 2 omega integration d cube x d to the power i k dot x minus k dot prime x. Now, when I integrate this out, ultimately this becomes 3 dimensional integration and this will give you 2 pi delta k minus k prime. So, after integrating you will get 2 pi cube 2 omega delta k minus k prime.

So, what we have seen is; if we evaluate this commutator here we get 2 pi cube this implies a (k) a dagger k prime commutator is equal 2 pi cube 2 omega delta k minus k prime. You can do similar calculation to show that the commutator involving 2 s or the commutator involving to a daggers integral. So, we now know what the commutation relation between this a s and a dagger, which are nothing but the coefficients in the expansion for the field phi and pi. And these coefficients are operators, because we are quantizing the system and these operators obey these commutation relations. Now, what we will do is we will start with the expression for the Hamiltonian of this system and we will express the Hamiltonian in terms of these operators a k and a dagger k. So, let us do that now.

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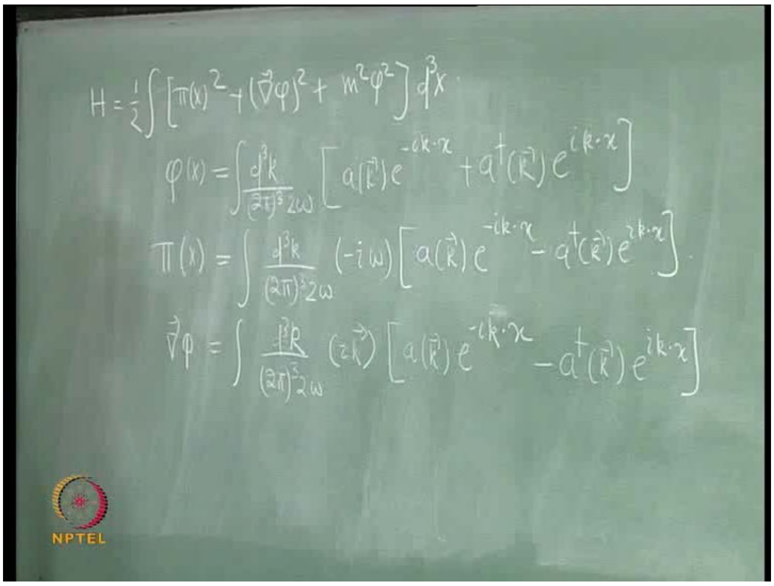


So, the Hamiltonian is H equal to half now, we have to write we have to substitute for the expression for pi of x and grade phi x in phi of x here. We have already worked out, we have already straighter that phi of is integration d cube k divided by 2 pi 2 omega times a k d to the power minus i k dot x plus a dagger k a to the power i a dot x. This is only relation you have keep in mind, from here we can derive what is pi of x pi is simply phi dot. So, when you take a time derivative; you will get the d cube k over 2 pi cube 2 omega here, all you will get is minus i omega, which has derivative with respect to time which is cube minus i omega here, plus i omega here and as a result you get this is a k d to the power minus i k dot x minus a dagger integral k e dot power i a dot x.

Similarly, what you will get for gradient of phi grade phi will give you a factor of i k d cube x over 2 pi cube 2 omega times i k, and this integral a k due to the power minus i k dot x minus a dagger k due to the power i k dot x. So, you will now substitute all these expression here and then do the integration x integration as well as one of the k integrations. And then express the Hamiltonian in terms of their operators a and a daggers. So this is d cube k.

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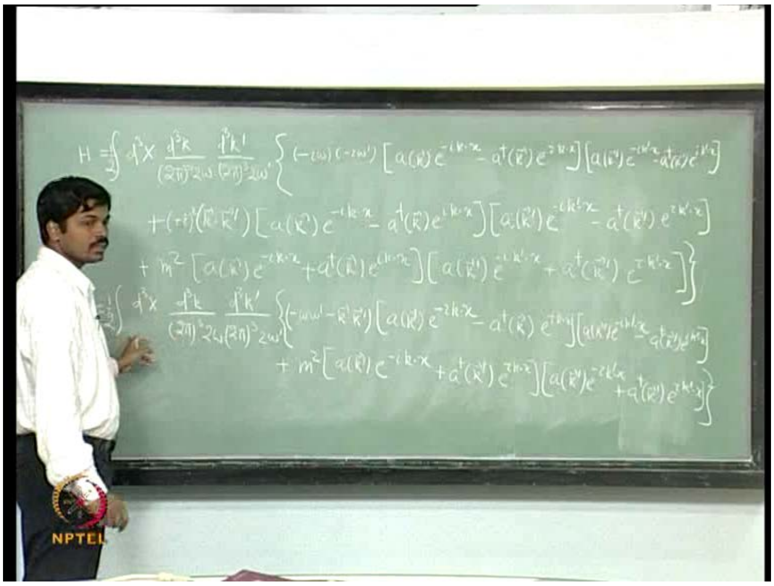

So, the Hamiltonian now is H which is integration d cube x, and you just see here you get pi of x square and so there are 2 pi s and for one pi i will use the integration variable as k and for the other pi i will use the integration variable is k prime, in order to keep them separate for one. Similarly, here you have grad phi square. So, for one of them I will use the integration variables as k and for the other great pi i will use the integration variable s k prime.

Similarly, for pi square also I will use k and k prime x variables. So, I have 3 integrations; one over d cube x and then I will get an integration over d cube k integration k over 2 pi cube 2 omega and alpha I have d cube k prime over 2 pi cube 2 omega prime. Then I have pi of x square so that will give me minus i omega for one of the pi s in minus i omega prime for the other pi time’s a k e to the power minus i k dot x minus a dagger k e to the power i k dot x that is for one of the pi’s.

And, the second pi I will get the same expression except that instead of k i will have a k prime. So, a k prime e to the power minus i k prime dot x minus a dagger k prime e to the power i k prime dot x. So, this is the first term, the second term involves grade pi square. So, that is exactly similar to the first line accept that here, I will get minus plus i square k dot k prime. And then again the same terms a k e to the power minus i k dot x minus a dagger k due to the power i k dot x times a k prime e to the power minus i k prime dot x minus a dagger k prime e to the power i k prime dot x. Then finally I have the pi square which will give me m square pi square simply; give me m square a of e to the power minus i k dot x plus a dagger k e to the power i k dot x times a k prime e to the power i minus i k time dot x plus a dagger k prime, e to the power i k prime dot x brackets.

You can see that this term is identical this term here. So, I can combine these 2 terms or the last term and I cannot combine. So, let us rewrite this expression in a simpler way, this is d cube x d cube k over 2 pi cube 2 omega d cube k prime over 2 pi cube omega prime times, this will give me minus omega prime minus k dot k prime. From these 2 terms and then I have this expression a k e to the minus i k dot x minus a dagger k e to the power i k dot x times a k prime e to the power minus i k prime dot x minus a dagger a prime or i a prime dot x. And then this is second term will be as usual; let us put the bracket here, plus m square a k e to the minus i k x plus a dagger k prime a dot k e to the power k i dot x, and a k prime e to the for minus i k prime dot x plus a dagger k prime proper i k prime dot x alright, now, the result over L factor of half thank you.

So, similarly; here I will write half so now I will multiply this term i will open out and I will simplify all the this term this term looks little bit lengthy, when we multiply it and then k carry out this integrations, we will get a simpler result. So, let us look at this term again. So, this forget about a case for a moment, this will give you an integration of minus i k plus k prime dot x. On the other hand; if you multiply this term with this term this it will give an integration over e to the power minus i k minus k prime dot x.

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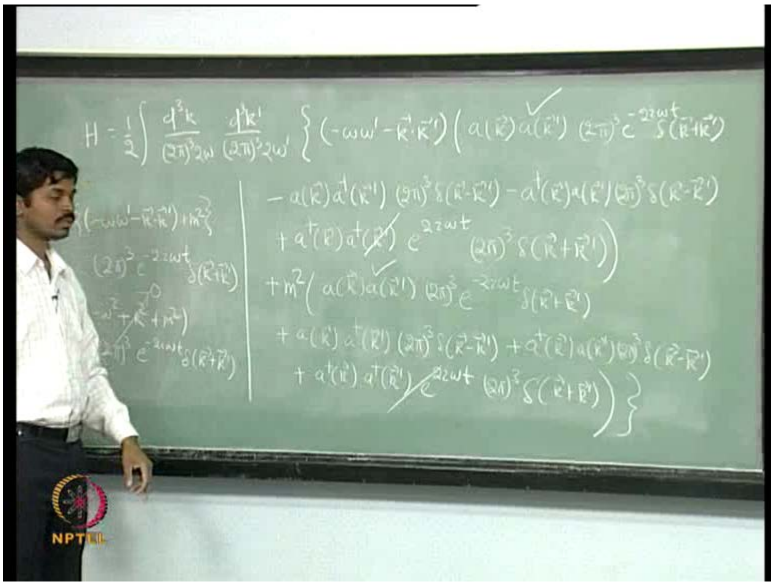


So, you have integrations of this type, when you expand you will get term. So, it is involves which have integration d cube x e to the power minus i k plus k prime dot x, and you will have integration d cube x e to the power minus i k minus k prime dot x. What will you get when you carryout this integration; in the first term you will get your first integration will give you remember here this dot products involves the 4 dimensional matrix eta mu new. So, you will get 2 terms here d cube x e to the power minus i omega plus omega prime t times e to the power i k plus k prime dot x.

Now, I can carry out this x integration here, when I carry out the x integration this term will give me e to the power minus i omega plus omega prime t delta k plus k prime. When I use this delta this result will simply become e to the power minus 2 i omega t delta k plus k prime. On the other hand; if there is a minus sign here then you will get a minus here and similarly, there is the minus here therefore; when you multiply this with the delta function the minus sign will cancel out omega and omega prime. So, this integration here d cube x e to the power minus i k minus k prime dot x, will simply give you delta k minus k prime, is this clear, there is the 2 pi cube. So, there is the 2 pi cube yes,

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This is you simply use simply use delta of, you know what is omega right, omega is simply square root of k square plus m square. Similarly, omegas prime is square root of k prime square plus m square. Now, you sue the expression f of x delta x minus a is f of a delta x minus a. So, that will simply give because there is the square involve here. So, k prime square here, you can simply replace it by k square and that will give me too. On the other hand here, because there is the minus sign omega and omega prime will cancel and you will get simply 2 pi cube delta k minus k prime.

So, we will use these 2 results in the integration to value the Hamiltonian. And when we do that what we will get is the following H equal to half, the x integration is carried out. So, all you will have is d cube k over 2 pi cube 2 omega times d cube k prime over 2 pi k cube 2 omega prime minus omega prime minus k dot k prime. Let us now worry about the details of the operator ordering, because now a s and a dagger are operators we have to worry about their ordering also, where as e to the power i k x is number. So, you do not worry about whether it is, where it is place inside the integration. Now, this multiply by a k a dagger a k a times a k prime, and the integration will gave me e to the power 2 pi cube e to the power minus 2 i omega t delta k plus k prime.

So, this will have 4 terms the first of the four terms is this. And then I will have minus a k a dagger k prime and the integration over x will give me over to pi x cube delta of k minus k prime then I will get minus a dagger k a k prime again to 2 pi cube delta of k minus k prime, and finally; the last term minus a dagger k a dagger k prime e to the power 2 i omega t 2 pi cube delta of k plus k prime, this is first term. And this is plus fine then i will have plus m square again; this will have 4 terms and these 4 term are haves a k, a k prime 2 pi cube e to the power minus 2 i omega t delta k plus k prime that is the first term.

The second term is plus e a dagger k prime plus a dagger k k k prime 2 pi cube delta of k minus k prime and finally, I have this term plus a dagger k a dagger k prime e to the power to i omega t 2 pi cube delta k plus k prime. So, now we have to combine all this terms; you can see this term here can combine with this term similarly, this term this term can combine, this term this term will combine, and this will combine. What will you get when you combine? You see there is the minus omega so let us consider the first term here, in the first term you have minus omega omega prime minus k dot k prime. And there is the plus m square here, all of this are multiply by 2 pi cube e to the power minus 2 i omega t times delta k plus k prime, right. Now, you again you use the fact that delta k plus k prime is multiply to this whole term.

So, for omega you see sorry, omega prime you write omega, but for k prime you write minus k because there is the plus sign k here. So, when you do that this term will become minus omega square plus k square plus m square multiplied by the whole thing; 2 pi cube e to the minus 2 add omega t delta of k plus k prime. What is this? This is 0 because omega square is k square plus m square. So, therefore; this term can cancel. So, this term here cancels with this term, similarly; the last term which involves 2 add a dagger you can convince yourself that cancels with the last term here which also involves 2 a dagger.

So, this cancels with this terms, what remains is the terms. So, it is involves a s and a daggers. So, one a one dagger you will see that this term does not cancel with this term and similarly, this 2 terms do not cancel for each other. And the reason is that instead of delta k plus k prime you have a delta k minus k prime and therefore; there is the minus sing i am difference here which this terms do not cancel in also there is the relative minus sign here. The sign workout in such a whether they a up. So, let us now write down what are the terms which survive.

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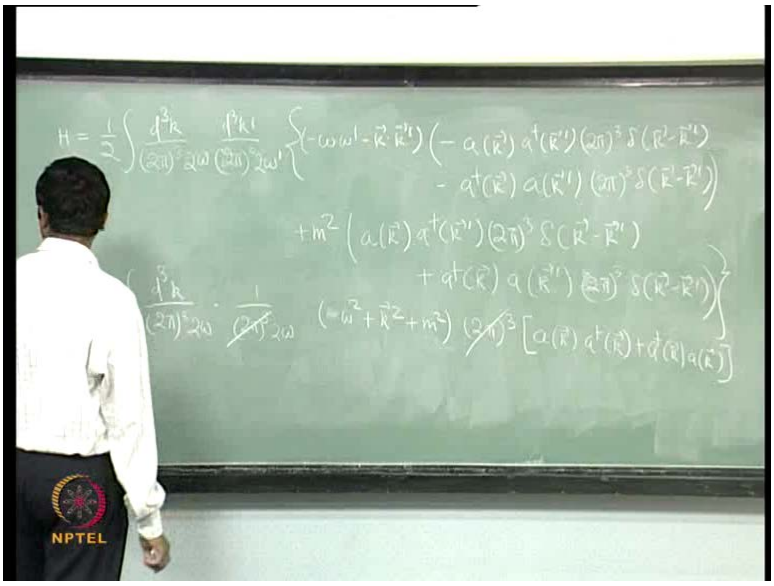

So, when we simplify; what we get is the Hamiltonian H half integration d k cube over 2 pi cube 2 omega times d cube k prime over 2 pi cube 2 omega prime then minus omega omega prime minus k dot k prime plus m square. Again, let us write it in more detail this minus a k a dagger k prime 2 pi cube delta of k minus k prime minus a dagger k a k prime 2 pi cube delta of k minus k prime. And then plus m square again the same set of terms; a k this time with the plus sign, a dagger k prime. So, I just rewritten all the terms just survive. Now, you can simplify this term by k out this k prime integration.

So, let us carryout the k prime integration then you can see that here, you will get half integration d cube k over 2 pi cube 2 omega and here, I will get 1 over 2 pi cube when I carry out the delta function I will get again 2 omega for this. Here I will get minus omega square minus k square and then there is the minus sign here that will make me omega square plus k square and there is the m square here. So, this will gives me plus m square times 2 pi cube and then a k a dagger k, when you evaluate the second term again you will get the same thing here, in the sense there is minus that minus multiplying with this minus will give plus signs, and this m square will add.

So, you will again get omega square plus k square plus m square times 2 pi cube but instead of a k a dagger k you will get a dagger k a k. So, you have all this term here times this is plus a dagger k a k. So, the complicated looking Hamiltonian. Now takes a very simple form; you will again you can use the relation that omega square is equal to k square plus m square. So, this will give you 2 omega square, this 2 pi cube will cancel this 2 pi cube this 2 omega here and there is the 2 omega square here that will cancel to give me simply omega.

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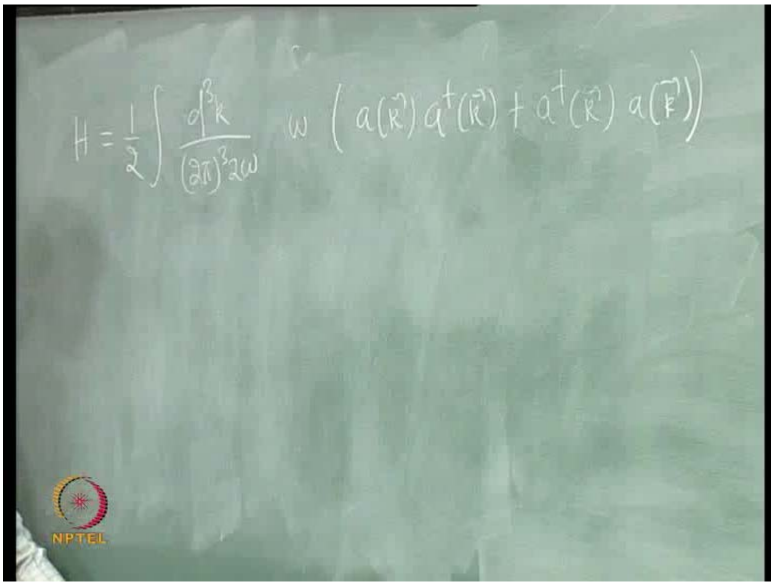


So, when i simplify what i will get is that the Hamiltonian of the system is half integration d cube k over 2 pi cube omega times a k a dagger k plus a dagger k a k. So, this is the Hamiltonian that we got at the end. Now, our task is to find this spectrum of this Hamiltonian. So, what we will do is that; we will start with this Hamiltonian in next lecture and then we will derive spectrum, and we will discuss some of the difficulties associated with this m l, how to integrate them and so on.

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This is for convenience, you can cancel this omega here, but I want to like here write it like this, because this term here the whole combination here is Laurence in variance. So, in order to have faced many Laurence in variance I just one to keep the integration measure always is d cube k times 2 pi omega. Any other cube questions then we will stop here today.