Quantum Field Theory
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Module - 05
Radiative Corrections
Lecture - 38
Self energy II

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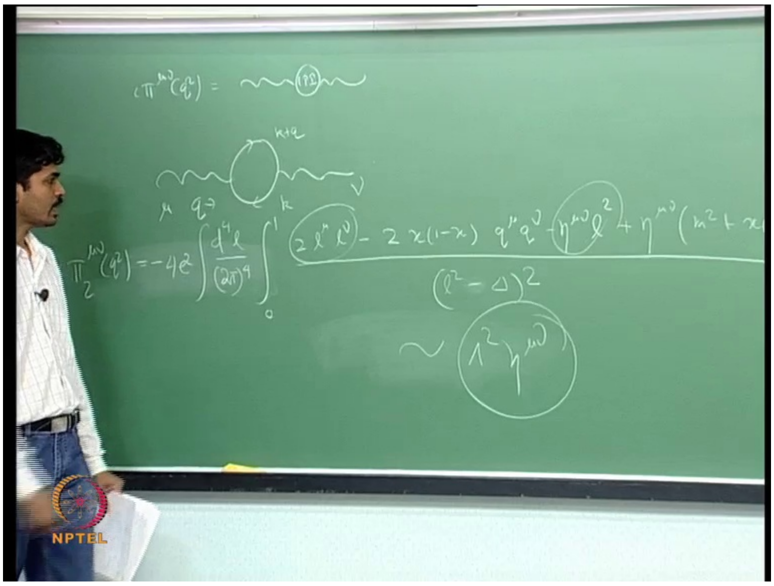

So, we have been discussing the photon self energy. In the last lecture what we did is we started computing the second order contribution to the one particle irreducible function for i pi mu nu q square, which is basically given by 1 p i. And the second order contribution is determined by the one loop diagram of this type where you have a photon of momentum q creates a virtual electron-positron pair k and k plus q of moment k and k plus q. And finally, they get annihilated. So, what we did is that using Feynman rules, we wrote the amplitude for this diagram, and then we have used the Feynman parameterization. We simplified the numerator, and at the end of the day we got the following expression for pi 2 of mu nu.

So, what we got is i pi 2 mu nu of q is equal to 2 l mu l nu minus 2 x 1 minus x q mu q nu minus eta mu nu l square plus eta mu nu m square plus x into 1 minus x q square. This with l square plus delta whole square and this has to be integrated; the variable x has to be integrated from 0 to 1, and the momentum l will have to be integrated through all the values it takes. So, there will have to be a d 4 l divided by 2 pi to the power fourth, and we have the overall factor minus 4 e square. So, this is what we got for i pi 2 mu nu of q square, and we can do a Euclidian continuation after doing a Euclidian continuation we can evaluate.

So, basically you will have to do a weak rotation, and what we get in this process is this is l square minus delta square; this will become plus, and then here you will get a minus sign, here you will get a minus sign, and that is all. And then you will get a factor of overall i here. You can evaluate this integration explicitly. What I would like to emphasis here is that when you consider the large momentum behavior of this integration in the limit when l tends to infinity, when the momentum is very large, these two terms are dominant compared to these two terms here. And you can do a power counting here.

The dimensional analysis basically says that the term here as well as here will be quadratically divergent, whereas these two terms here will not be quadratically divergent and we have discussed in the last lecture, this integration itself l mu l nu will simply be replaced by eta mu nu l square divided by 4. So, inside the integrant we can just replace this. So, what we get at the end of the day is simply half eta mu nu l square and finally, when you evaluate this integration the contribution from these two terms will go like lambda square where lambda is the cutoff introduced. So, you have lambda square eta mu nu which is these two terms are which is actually quadratically divergent. What is worse is that when you use this regularization that is when you introduce a cutoff lambda and evaluate this integration, it does not reserve ward identity.

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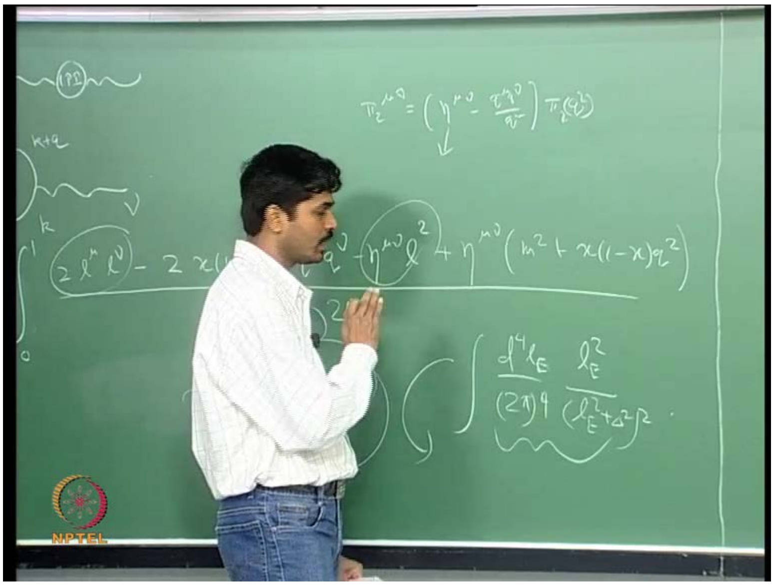

Remember when we discussed the ward identity, we saw the expression for pi 2 mu nu must be such that it is proportional to pi 2 mu nu must be equal to eta mu nu minus q mu q nu divided by q square times some pi two of half q square. That is what we were excepting to get, but in this regularization process we do not get the second term here; all that we get is eta mu nu times some quantity which is quadratically divergent. So, the ward identity is lost in this process. In order to preserve the ward identity what we will do is that we will introduce a new regularization process which is what is known as the dimensional regularization.

So, in the dimensional regularization basically what we do is we evaluate integrations of this type basically when you do a Euclidian continuation, you will get integrations of this type d 4 l E divided by 2 pi to the power fourth 1 over l E square plus delta square whole square, or you can have something l E square divided by l E square plus delta square whole square. So, instead of taking this integration in 4 dimension, you evaluate this integration in d dimension. And at the end of the day you take the limit d goes to 4; of course, when you take the limit d goes to 4, the integral will be divergent as it is obvious from here, but we can see in detail that this process of regularization do preserve the ward identities.

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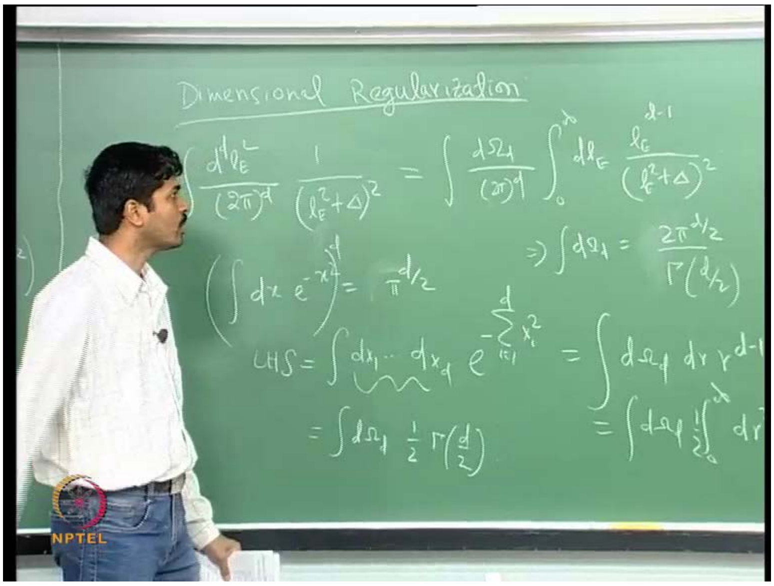

So, what we will do in today's lecture is we will introduce the dimensional regularization and, we will evaluate the one loop contribution using the dimensional regularization. Our goal here in the dimensional regularization would be to evaluate integrals of this kind, d d l E as I said just now that instead of evaluating the integration in 4 dimensions; we will evaluate it in general in d dimensions 1 over l E square plus delta whole square. This we will evaluate in d dimension. Since, the integrant depends only on the magnitude of l E; therefore, I can rewrite this as d omega d divided by 2 pi to the power d integration 0 to infinity d l E l E to the power d minus 1 divided by l E square plus delta whole square.

So, this basically when I integrate all the angular variables I we will get the area of a unit d minus 1 sphere which I can evaluate as follows. We know d x e to the power minus x square is square root pi. So, when I take this to the power d, what I will get here is pi to the power d by 2. The left hand side I can rewrite it as an integration d x 1 up to d x d e to the power minus sum over i equal to 1 to d x i square. This is my LHS, and this is just the volume element in d dimensional Euclidian's phase. And this is just the square of the radial distance in d dimensional Euclidian sphere. So, what I can I do is I can write it as d omega d times d r r to the power d minus 1; this is what is the volume element times this is e to the power minus r square.

Now I can easily evaluate the r integration here; that will give me what is the integration over all the angular variables. So, when I do the r integration what I get here is I can even rewrite it as d omega d integration times 0 to infinity d r square. Instead of taking the variable integration variable to be r, I can take the integration variable to be r square, then its r square to the power d by 2 minus 1. There is a factor of half; this will give me 2 r d r. So, that two will cancel this half, and this r here will add to this minus 1. So, at the end of the day you will get r square to the power d minus 1 e to the power minus r square, alright.

So, now, you can see that this is nothing but a gamma d by 2 where gamma n equal to integration 0 to infinity d x x to the power n minus 1 e to the power minus x. So, therefore, this integration here when I evaluate what I get is d omega d half gamma d by 2. This simply implies that the angular integration here is basically given by half. Sorry, this half will become 2, 2 pi d by 2 divided by gamma d by 2. So, so the first part here we have already evaluated, and this simply gives me 2 pi d by 2 divided by gamma to the power d by 2. What is left is the second term here. So, we will quickly evaluate the second term in this expression.

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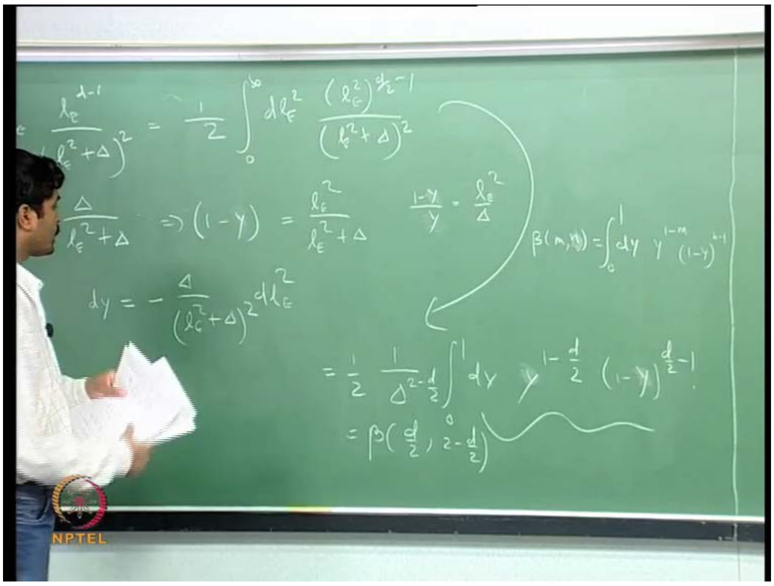

So, to do that what I need to do is I need to evaluate this integration here 0 to infinity d l l E to the power d minus 1 divided by l E square plus delta whole square. I will introduce this variable x or maybe I will introduce the variable y, which is delta divided by l E square plus delta. Then what I need here is I need this quantity here. So, this basically says that 1 minus x is nothing but l E square divided by l E square plus delta, and when I take the ratio 1 minus x divided by x; that is l E square divided by delta. So, I know this quantity here. Sorry, this is Y and. So, I know what this quantity here is.

This is simply given by and finally, d y is a minus delta divided by l E square plus delta whole square d l E square; that is what I get. So, when I substitute all this things what I will get here is 1 over 2 d l E square 0 to infinity. It is better to use the variable d l E square, because I have a d l E square here l square to the power d by 2 minus 1 divided by l E square plus delta whole square. And now l E square is nothing but 1 minus Y. So, you will get a factor of 1 minus Y to the power d by 2 minus 1 and d l E square over l E square plus delta square whole square is nothing but d y. So, when you pull all the factor of delta this is nothing but half one over delta to the power 2 minus d by 2 integration 0 to 1 d x.

Let us do this here. So, this is nothing but x to the power 1 minus d by 2 times 1 minus x to the power d by 2 minus 1. So, I have the variable Y here; this is Y. So, this is straightforward; you can just see by substitution that this is what you get. And what you have here is nothing but the beta function; beta m n by definition is a integration 0 to 1 d y y to the power 1 minus m   1 minus y to the power 1 minus n n minus 1; this is what is the definition of beta m n. So, when you use the definition of beta m n, what you see is finally, what you get here is a beta d by 2 2 minus d by 2.

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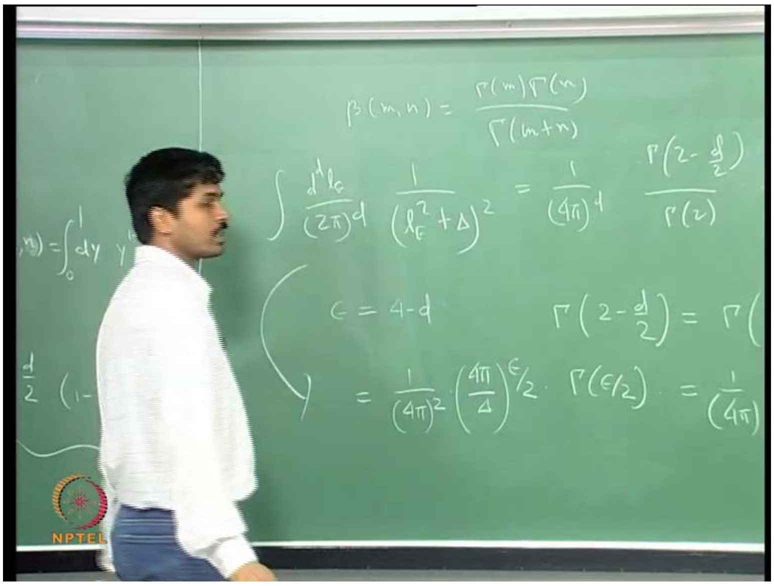

Finally, you can use the identity which is beta m n is equal to gamma m gamma n divided by gamma m plus n. And you can express this integration here in terms of the gamma functions. When you do that at the end of the day, what we will get is integration d d l E divided by 2 pi to the power d 1 over l E square plus delta whole square is nothing but 1 over 4 pi to the power d gamma 2 minus d by 2 divided by gamma 2, which is just to 1, 1 over delta to the power2 minus d by 2. So, this is what you will get, and then you can now see where the divergences. Because this gamma if you just consider a gamma of z, it has an isolated poles when z becomes 0 or minus 1 or minus 2, any of the negative integers. And z equal to 0 corresponds to 2 minus 2 d by 2 equal to 0 or d equal to 4.

So, when d equal to 4, 6, 8, and so on, you will have a singularity; the integration diverges, especially it diverges when d becomes 4. What we can do is that we can find the behavior of the integration here when d approaches 4. So, to do that what I will do is that I will introduce epsilon which is equal to 4 minus d and then I will see how this integration behaves when epsilon tends 0. So, when epsilon tends to 0, gamma of 2 minus d by 2 basically becomes gamma of epsilon by 2. And then you can use the expansion for gamma of epsilon to get this to be 2 over epsilon minus gamma plus and order epsilon term, where gamma is the Euler's constant. So, this is what is the expansion for gamma function, and then you can see that there is a pole here.

So, what I will do is that we will substitute this expansion here, and then you can rewrite this integration. This basically becomes gamma 2 is 1, and what I will do is that I will write here 1 over 4 pi square times 4 pi divided by delta to the power epsilon by 2 to combine this term and this term. And finally, this one here will give me gamma of epsilon by 2. And when I use this expression for gamma of epsilon by 2, what I will get is 1 over 4 pi square. This is nothing but 2 by epsilon minus gamma, and here this one is simply e to the power epsilon by 2 log 4 pi over delta. And for small epsilon I can just keep terms up to order epsilon, and this simply becomes 1 plus epsilon by 2 log 4 pi over delta.

So, when I substitute that here, this term here simply gives me 1 over 4 pi square. The first term here gives me 2 by epsilon, and then this is an order one term 2 by epsilon multiplied to this gives me log 4 pi over delta. And finally, when this term multiplies here, I will get a minus gamma; all other terms are of order epsilon, this plus order epsilon. So, this is what I get when I evaluate this integration in the dimension and I take the limit d goes to 4; in the limit d goes to 4 the divergence space here is separated out, and then there is a finite phase which is given by this.

It is very straightforward to evaluate when there is a l square in the numerator. Everything here will go as usual except that there is a l square here and the beta function here the arguments will sense; accordingly you will have factors of gamma matrices here. I will write on the integration the general integration and leave it as an exercise for you to evaluate this explicitly.

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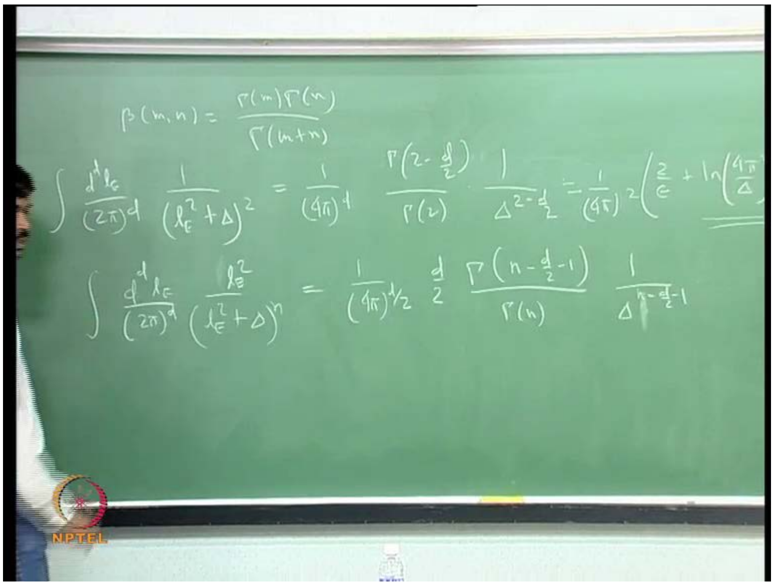

So, if you have d d l E divided by 2 pi to the power d 1 over l E square plus delta to the power n beta l E square in the numerator, what you will get is 1 over 4 pi to the power d by 2 d by 2 gamma n minus d by 2 minus 1 divided by gamma n 1 over delta to the power n minus d by 2 minus 1. So, now that we know how to evaluate this integration, we can consider pi 2 of q square and then do the integration explicitly. There is one thing that we need to be careful when we evaluate the numerator in pi 2 of q square.

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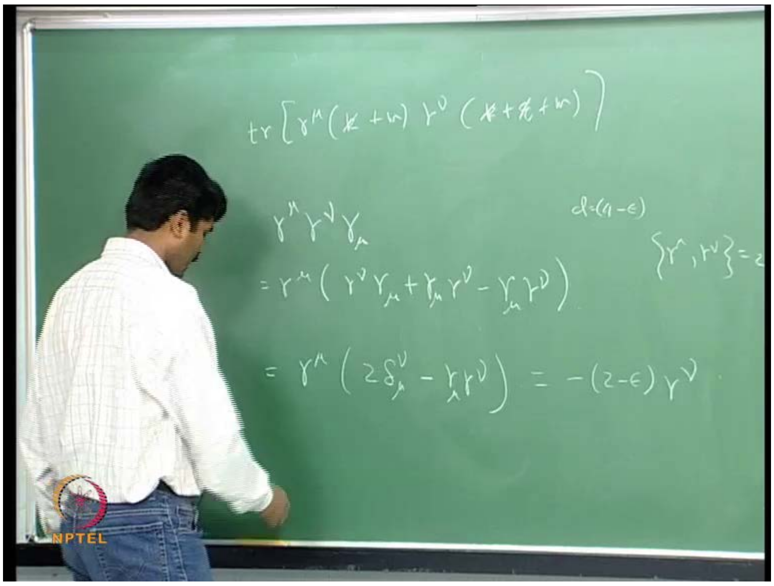

That is what we did is we started simplifying the numerator by evaluating trace of gamma mu k slash plus m gamma nu k slash plus q slash plus m. Assuming that this is actually happening in 4 dimension. When you do it in d dimension, you will get for example, let us say gamma mu gamma nu gamma mu. One typical term that is one typical expression that you come across when you evaluate trace of this kind. It is we have straightforwardly used this to be minus 2 gamma nu; however, in d dimension when d is 4 minus epsilon, you can see that this is nothing but gamma mu times gamma nu gamma mu plus gamma mu gamma nu minus gamma mu gamma nu.

And this we will use to be this is given by gamma mu. This is 2 delta mu nu minus gamma mu gamma nu we are using the same preferred algebra gamma mu gamma nu equal to 2 eta mu nu, but the Dirac matrices have different dimension. So, when you do gamma mu gamma mu, this will be trace of identity, and this trace of identity instead of 4 it is simply given by d which is nothing but 4 minus epsilon. So, when you substitute that at the end of the day, what you will get is minus 2 minus epsilon gamma nu. So, the epsilon will appear in identities like this.

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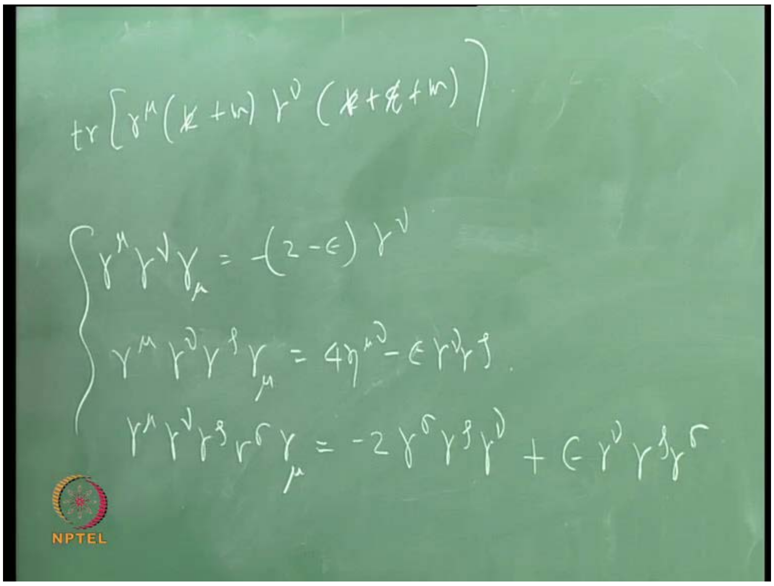

So, this one we have evaluated; we saw that this is given by minus 2 minus epsilon gamma mu gamma mu gamma nu gamma rho gamma mu is simply 4 eta mu nu minus epsilon gamma nu gamma rho. And when there are three gamma matrices gamma rho gamma sigma is equal to minus 2 gamma sigma gamma rho gamma nu plus epsilon gamma nu gamma rho gamma sigma. So, the epsilon dependence appears here explicitly when you evaluate the vertex corrections or whatever or the self energy diagram, but when you consider the physical amplitude then all these corrections simply drop out. So, we will not worry too much about the epsilon dependence here.

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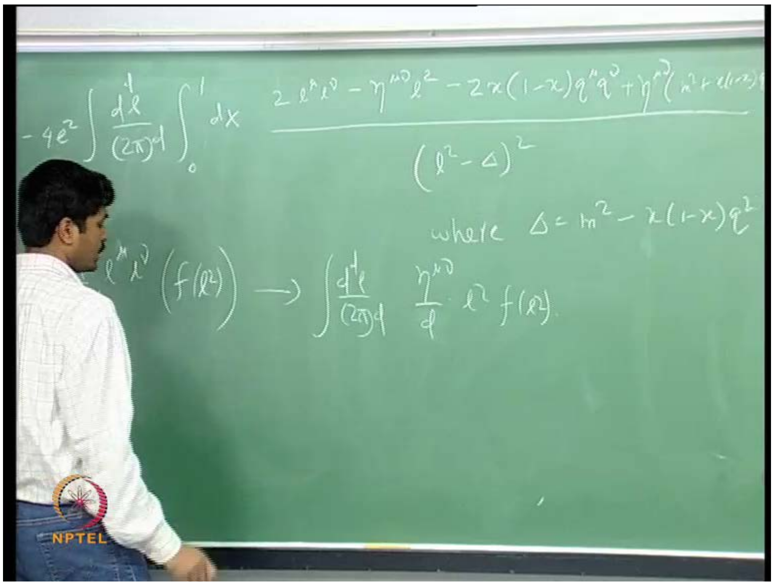

So, what we will do is that we will for pi 2 of mu nu, we will take this expression, i pi 2 mu nu; it is minus 4 e square. Instead of the evaluating the integration in 4 dimension, we evaluate it in d dimensions. So, this is d d l divided by 2 pi to the power d at 0 to 1 d x. We have already evaluated this expression here; I am merely rewriting it 2 l mu l nu minus eta mu nu l square minus 2 x 1 minus x q mu q nu plus beta mu nu times m square plus x into 1 minus x q square. This is the numerator, and in the denominator you have l square minus delta whole square, where delta is equal to m square minus x into 1 minus x q square; this is what we need to evaluate.

And I have already argued earlier that when you have l mu l nu with some function of l square here d d l over 2 pi to the power d. You can replace this integration by d d l divided by 2 pi to the power d eta mu nu over 4 l square f of l square. But here now because we are evaluating it in d dimension instead of 4 I will have a d here, and all other expressions will be the same. So, here instead of 2 l mu l nu what I will do is that I will write 2 divided by d l square eta mu nu.

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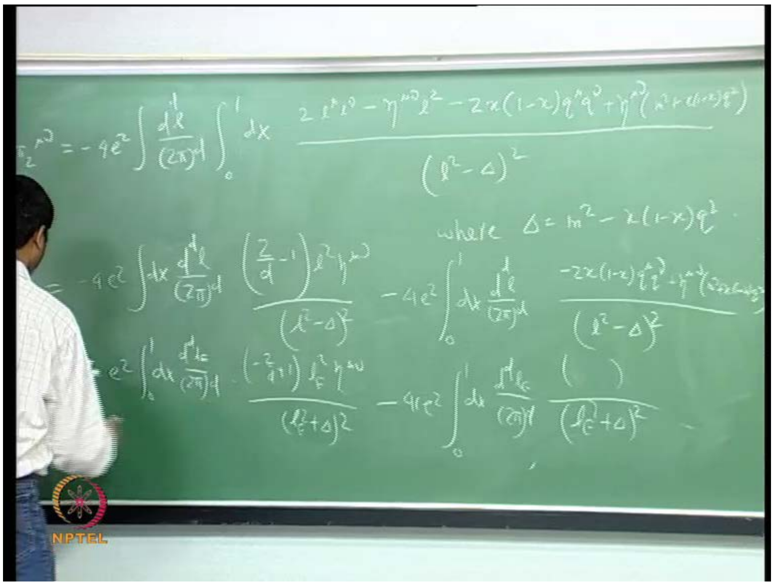

So, what I get here is if I write it into two separate piece, these two when I combine will give me minus 4 e square d x d d l divided by 2 pi to the power d times. This will give me 2 by d minus 1 l square eta mu nu divided by l square minus delta whole square. Remember this is the term which gave the quadratic divergent piece when we used the Pauli-Villars regularization earlier. And then the reaming term here is minus 4 e square 0 to 1 d x when d d l divided by 2 pi to the power d times this term here, which is nothing but minus 2 x 1 minus x q mu q nu plus eta mu nu times m square plus x into 1 minus x q square, which is this term here divided by l square minus delta whole square.

So, we will evaluate these two pieces separately. When I do Euclidian continuation, what I get here is I will get an overall factor of i minus 4 i e square 0 to 1 d x, and this l will become l E. So, d d l E divided by 2 pi to the power d, and here I will get a minus sign overall minus sign. So, I will get minus 2 over d minus plus 1 l E square beta mu nu, and here instead of minus this will become plus. So, l E square plus delta whole square this will be the first term. And in the second term I will have 4 i e square and here everything will be exactly as it is, expect that this will become l E d d l E divided by 2 pi to the power d.

And this is what it is; here I will have l E square plus delta whole square. So, you can see the first term here is given by this integration, and the first term here is determined by the second line here. And the second term here is determined by the first line here, which we have already worked out. So, we will use this result. When we use this result let us first consider the first term here which was quadratically divergent.

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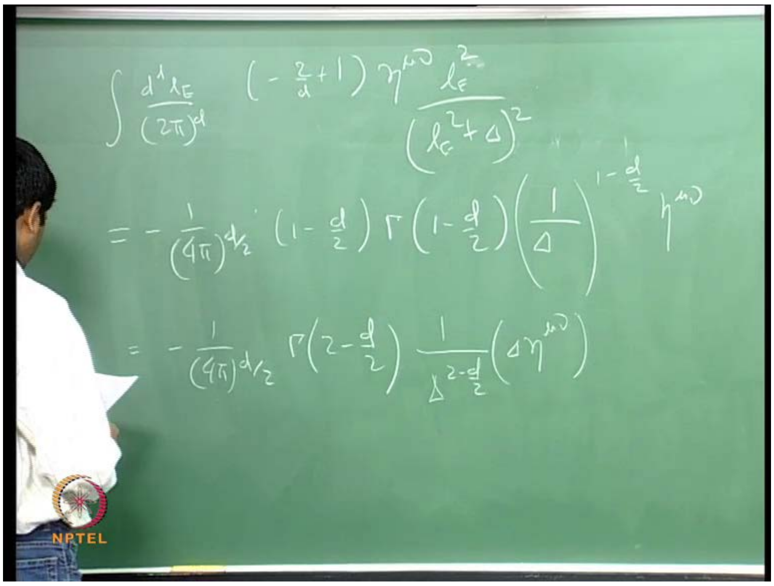

So, what I get is d d l E divided 2 pi to the power d minus 2 over d plus 1 eta mu nu l E square divided by l E square plus delta whole square. So, we have evaluated this integration, and I will simply write the answer. This is simply given by minus 1 over 4 pi to the power d by 2 1 minus d by 2 gamma 1 minus d by 2 1 over delta to the power 1 minus d by 2 eta mu nu. Now what we will use is that we will use this relation n gamma m equal to gamma n plus 1.

To write it as minus 1 over 4 pi to the power d by 2, this will simply be gamma 1 plus this which is gamma 2 minus d by 2. And here if I write delta to the power 2 minus d by 2 in the denominator, then what I will get is beta mu nu times delta. And so, I will use this then at the end of the day. So, if I want to consider i pi 2 mu nu, what I will get is let me write a factor of i here, and let us make it e minus E E; this is plus E and this is l E.

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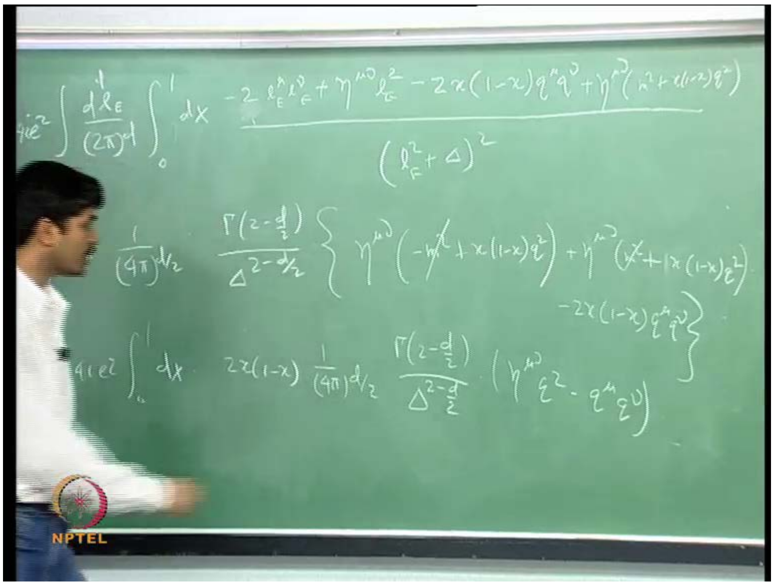

So, I will use this result here to rewrite this expression to be following. This is equal to minus 4 i e square 0 to 1 d x 1 over 4 pi to the power d by 2. And what I got there is gamma 2 minus d by 2 divided by delta 2 to the power 2 minus d by 2, and I will combine all the terms here in the numerator. To get this term here will give me eta mu nu times delta which is nothing but minus m square plus x into 1 minus x q square. That will come from these two terms here; here it is 2 by d. And this term here plus eta mu nu m square plus x into 1 minus x square. And finally, you have minus 2 x 1 minus x q mu q nu, but now you can see that this m square here cancels with this m square, and this becomes 2 x 1 minus x q square.

So, at the end of the day what you get here is there is a common factor of 2 x times 1 minus x. So, minus 4 i e square 0 to 1 d x 2 x into 1 minus x 1 over 4 pi to the power d by 2 gamma to the power 2 minus d by 2 divided by delta to the power 2 minus d by 2. And here you have eta mu nu times minus q square; sorry, here you have eta mu nu times q square, and there is a minus q mu q mu.

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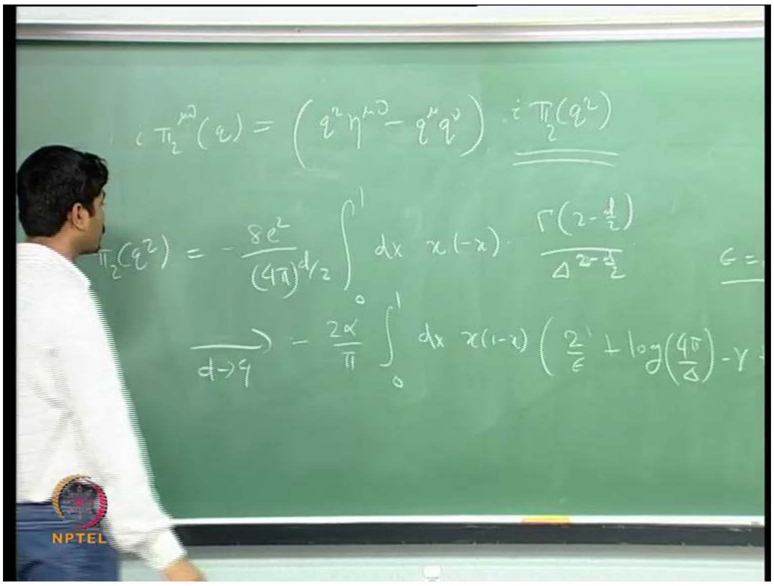

So, what you get is finally at the end of the day when we did the dimensional regularization, we got i pi 2 of pi 2 mu nu of q to be of this form q square eta mu nu minus q mu q nu times. Whatever the reaming vectors that I will denote it as i pi 2 of q square, and this is at a scalar quantity. So, this is of the form that that is required by the ward identity. So, therefore, what we saw by doing dimensional regularization is that this pi 2 mu nu in fact satisfies the ward identity. If you just take q mu pi 2 mu nu at the second order because of the presence of this term here, it simply becomes 0. Of course, this piece here is divergent, but it is logarithmically divergent. And then we can know what pi 2 of q square is; pi 2 of q square when I take all the vectors into account is simple given by minus 8 e square divided by 4 pi to the power d by 2 0 to 1 d x x into 1 minus x times gamma 2 minus d by 2 divided by delta to the power 2 minus d by 2.

We can take epsilon to be 4 minus d, and then we can consider terms up to order one. Then this is simply given by in the limit when d goes to 4 pi 2 of q square is simply given by minus 2 alpha divided by pi 0 to 1 d x x times 1 minus x. And this term here is simply given by 2 by epsilon plus log 4 pi divided by delta minus gamma plus order epsilon term. This is what we get for pi 2 of q square, and then the divergent piece is given by this term here 2 over epsilon.

So, from here we can just calculate the shift in the electrical charge, and the order alpha shift in the electrical charge especially is simply given by pi 2 at q square equal to 0. You can see that pi 2 at q square equal to 0 is. So, the q square dependence appears here in delta. It is simply given by some constant vector times 1 over epsilon which is actually a divergent piece. From this expression for phi 2 of q square you in fact can compute the q square dependence of electrical charge or in other words the q square dependence of the fine structure constant.

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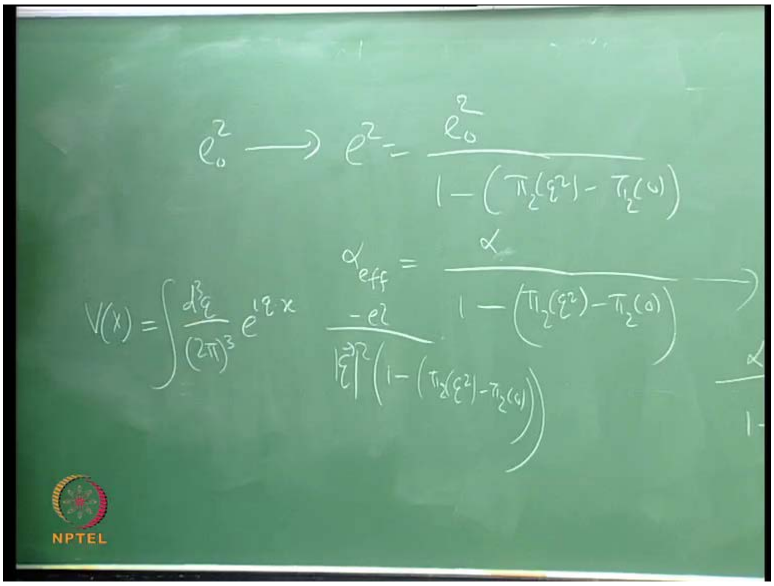

So, because of this correction here the effective electric charge up to this order is basically given by e 0 square is basically replaced by e square which is e 0 square divided by 1 minus pi 2 of q square minus pi 2 of 0. Or in other words the effective fine structure constant alpha effective is given by alpha divided by 1 minus pi 2 of q square minus pi 2 of 0. And you can use this explicit expression for pi 2 of q square to compute the q square dependence of the final structure constant. I will leave it as a homework to show that at large q square, when q square is much much greater than m square, the effective fine structure constant will simply be given by alpha divided by 1 minus alpha over 3 pi log minus q square over m minus 5 alpha over 9 pi.

So, you can see the q square dependence of the electric charge. Similarly, you can take the non-relativistic approximation and then you can compute the effective potential; the pi 2 of q square basically changes the electromagnetic interaction and the effective potentials is basically given by v of x. You can just take the Fourier transform here. So, this is is d q q over 2 pi q e to the power i q dot x minus e square divided by q square times 1 minus pi 2 of q square minus pi 2 of 0. So, this is what is basically is determining the effective potential up to one loop order, alright.