Quantum Field Theory

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Module - 04
Quantum Electrodynamics
Lecture - 28
Compton Scattering II

So, we are evaluating the scattering cross section.

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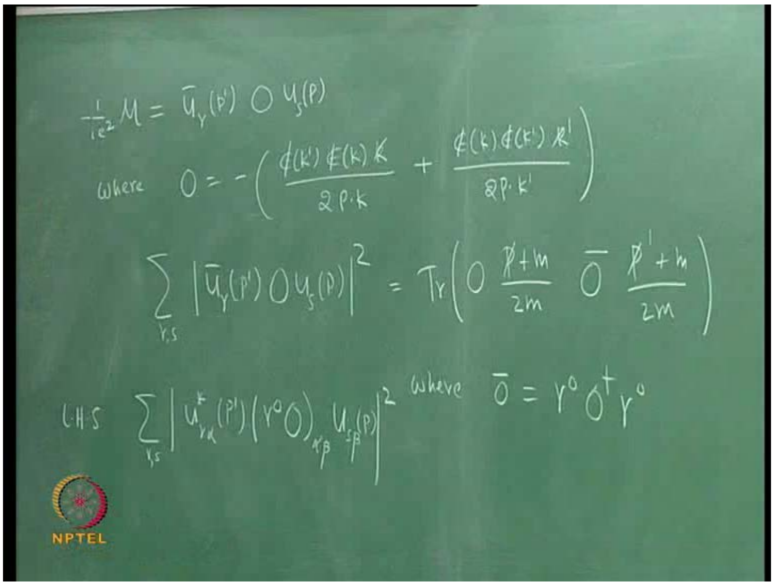

And then, what we saw in the last lecture is that, minus 1 over i e square times the Feynman amplitude is equal to u bar r p prime. Then, some operator O times u s p; where, the operator O is given by minus epsilon slash k prime epsilon slash k k slash divided by twice p dot k plus epsilon slash k epsilon slash k prime k prime slash divided by twice p dot k prime. What we need to evaluate is mod square of this quantity. And also, we are not assuming any polarization for incoming electron and outgoing electron. So, what we need to do is we need to evaluate this quantity half sum over r, s mod u bar r p prime u s p mod square. This is the quantity that we need to evaluate, because if you look at the formula for this scattering cross section, this is the only place where the electron polarization comes; the rest all are phase factors and integration measures and so on; there is nothing of that and that.

So, we will evaluate this quantity here. And, what I will show is this mod square is equal to trace of O p slash plus m divided by 2 m O bar p prime slash plus m divided by 2 m; where, O bar is defined to be gamma 0 O dagger gamma 0. This is what we will show in a moment. And then, I will evaluate this text. So, let us work out this expression and then show that, this is equal to trace of a bunch of gamma matrices. So, the LHS is sum over r, s mod of u bar r; I can write it in terms of its components. Remember – all these in addition to the spin indices – they also carry the Dirac indices. These are Dirac spinners – four-component spinners; these are Dirac matrices – product of a bunch of Dirac matrices – a linear of Dirac matrices. So, this quantity u bar r O u s is just a number in Dirac space. So, I can expand it in terms of the Dirac indices. Then, I will get u bar is nothing but u star of a – I will call it as r alpha p prime. u bar is u dagger gamma 0. So, I will consider this to be u dagger gamma 0 O – a single operator. And then, there is a vector here which is u s. So, alpha-th component of this – gamma 0 O alpha beta; and then, u s beta of p – this mod square. So, what is the mod square of this quantity? It is just the conjugate of this times itself.

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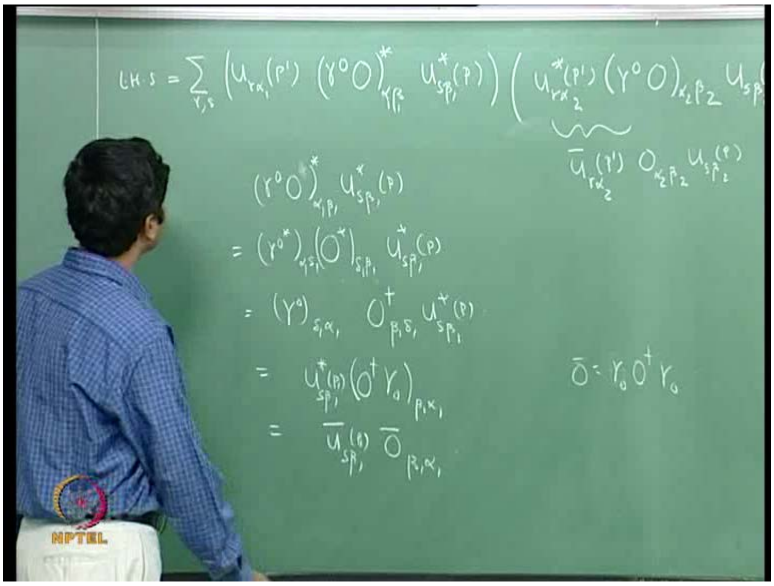

Therefore, LHS is equal to sum over r, s u r alpha 1 I will use for this one – p prime gamma 0 O star of alpha beta; and then, u s – alpha 1 beta 1 – u s beta 1 of p

Student: Star

u s beta 1 of p star; thank you. These times the same quantity, which appears there, except that we will use alpha 2 beta 2, etcetera for the domain indices. So, u r alpha 2 star of p prime gamma 0 O alpha 2 beta 2 u s beta 2 of p. These are components of Dirac gamma matrices; and, these are components of Dirac spinners. Since the components are numbers, I can freely move them wherever I want. So, I will rearrange these terms here.

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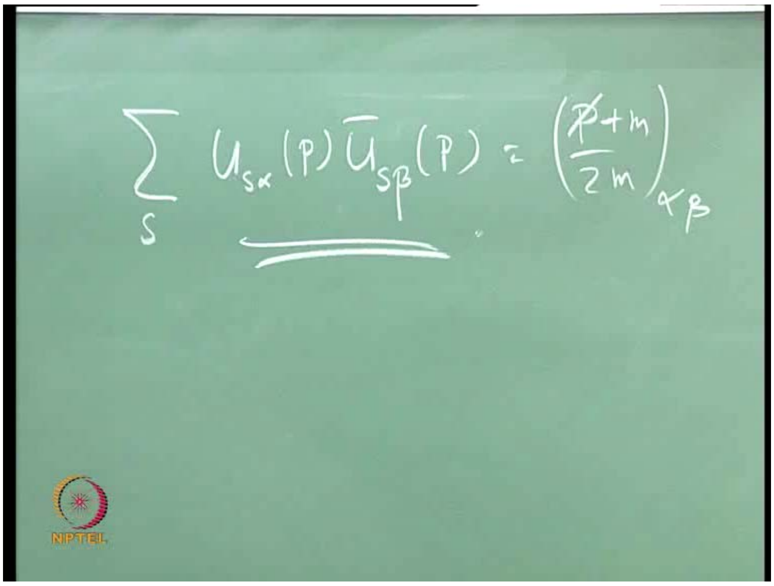

So, let us rearrange it appropriately, so that we can use this relation here – sum over s u s alpha of p u s beta of p bar is equal to p slash plus m over 2 m alpha beta. So, I will arrange the components in such a way that, we can use this identity repeatedly and then we will finally substitute the projection operator for that. You can see here for example, I can write u s beta 2 p; u s beta 1 p star; but, I need to write it in terms of u bar. So, the first term here I will substitute it as u bar gamma alpha 2 p prime O alpha 2 beta 2 u s beta 2 of p for this. Whereas, here I want to do a little bit more algebra to write it in terms of u bar. So, let us see if we can do that. So, let us split it up; gamma 0 O star of alpha 1 beta 1 u star s beta 1 of p. I will write it as gamma 0 O star – gamma 0 star of alpha 1 delta 1 O star of delta 1 beta 1 and u s beta 1 star of p.

Then, we will use the following property of the gamma matrix. Gamma 0 is Hermitian; therefore, gamma 0 dagger is gamma 0. So, what is gamma 0 star? So, this simply implies gamma 0 star is gamma 0 transpose. Therefore, gamma 0 star alpha 1 delta 1 is nothing but gamma 0 of delta 1 alpha 1; all right? And, here I will use the fact that, O star is O dagger transpose. For any operator O, its transpose of its Hermitian conjugate is just the conjugate of itself. Therefore, O star delta 1 beta 1 is nothing but O dagger of beta 1 delta 1; and then, u s beta 1 star of p; u s beta 1 star of p. So, this is simply u s beta 1 star O dagger gamma 0 of beta 1 alpha 1. Now, what I will do is; I have already told you that, I will introduce O bar, which is gamma 0 O dagger gamma 0. And, I will use the fact that, gamma 0 square is identity. Then, I will simply write it as u s beta 1 bar O bar of beta 1 alpha 1. Is this clear? u s beta 1 star p 1; I have introduced an identity operator here; and, I wrote this identity as gamma 0 square; one gamma 0 gives u O bar and the other gamma 0 gives you u s bar. So, for this, I will simply write it as this.

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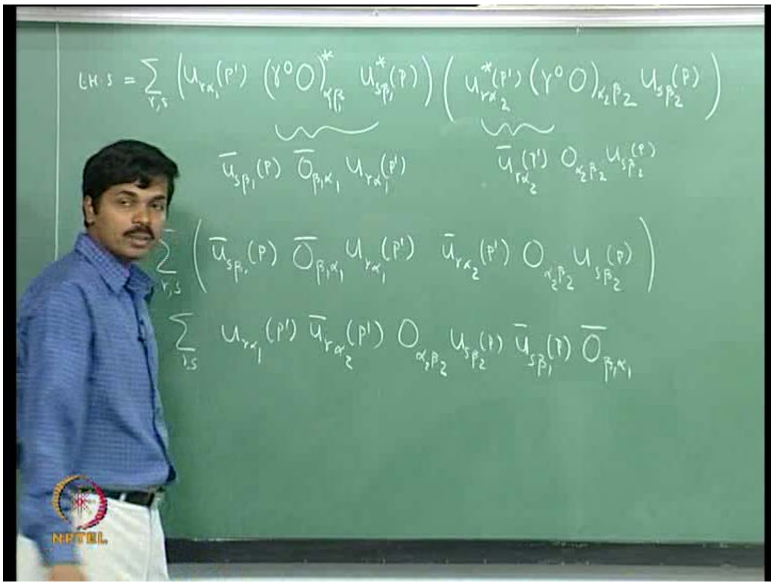

So, let us do that. So, this is… So, the whole thing will be u s bar beta 1 of p O bar beta 1 alpha 1; and then, u r alpha 1 p prime; alright? So, let us rewrite the whole term and then rearrange them appropriately. So, this LHS is sum over r, s u s beta 1 bar of p O bar beta 1 alpha 1 u r alpha 1 of p prime. And then, here I will write it as u bar r alpha 2 of p prime O alpha 2 beta 2 u s beta 2 of p. Now, since these are numbers, I will write these numbers – these two numbers at the end. So, then it is simply sum over r, s u r alpha 1 p prime u bar r alpha 2 p prime O alpha 2 beta 2 u s beta 2 p u bar s beta 1 of p O bar of beta 1 alpha 1. But, now, you see that it is of this form here. Therefore, we will use this relation there; and then, we will substitute this for p prime slash plus m over 2 m. And similarly, here I will use that identity.

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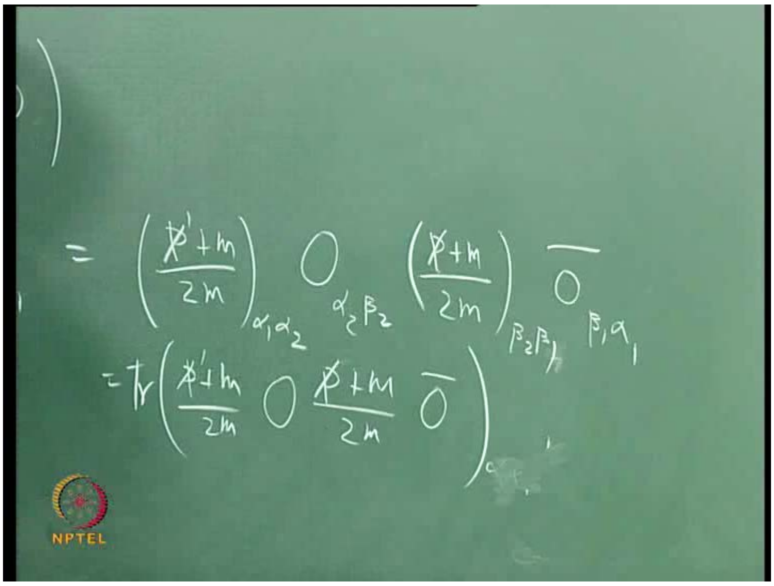

Then, this term here will become p prime slash plus m divided by 2 m; the alpha 1 alpha alpha 2 component of that – alpha 1 alpha 2. And then, I have O alpha 2 beta 2. And then, this is again p slash plus m divided by 2 m; the beta 2 beta 1 component of that. And finally, I have O bar beta 1 alpha 1. Remember these repeated indices are summed over. Therefore, what you get here is nothing but p prime slash plus m divided by 2 m O p slash plus m over 2 m O bar; the alpha 1 alpha 1 component of that; and, it is summed over alpha 1. Therefore, this is basically the trace of this operator. So, what is showed is that, whatever we had claimed, this is nothing but if you use the cyclic property of the trace, it is just this one. So, what we did is that, we prove that, the mod square of this quantity is equal to this. Now, what we will do is that, we will substitute this value for the operator O; and then, we will expand it. And, we will see what we get. So, let us put this for the operator O.

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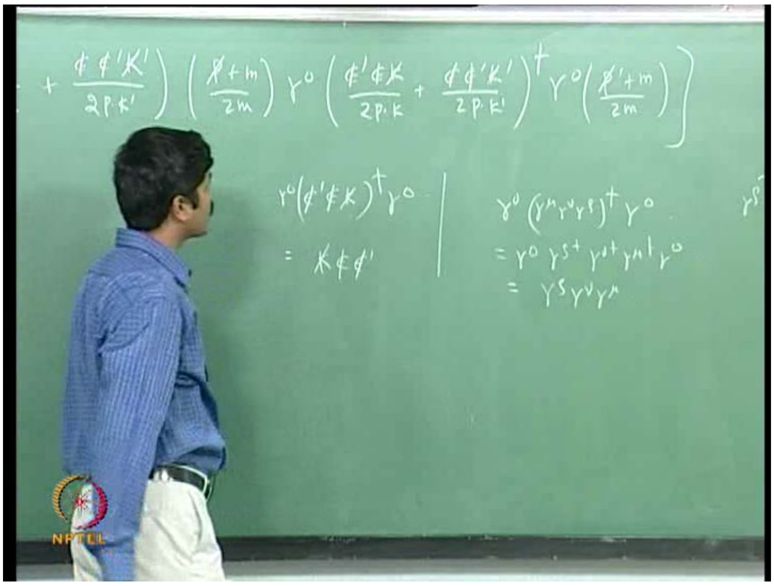

Then, I will denote this quantity to be X. So, X is trace of epsilon… I will repeatedly use these expressions. So, I will denote epsilon prime to be epsilon of k prime, so that we do not have to write every time epsilon of k prime. And, I will simply write epsilon for epsilon of k. This is the notation I will use from now on just for convenience; and, we will go on. So, first thing is this is just epsilon prime slash epsilon slash k slash over 2 p dot k. Then, p slash plus m over 2 m; and then, gamma 0…

Student: ((Refer Slide Time: 19:13))

There is one more term plus epsilon slash epsilon prime slash k prime slash over 2 p dot k prime. Then, p slash plus m over 2 m and gamma 0. This quantity dagger epsilon prime slash epsilon slash k slash over twice p dot k plus epsilon slash epsilon prime slash k prime slash over 2 p dot k prime dagger gamma 0; and then, p prime slash plus m divided by 2 m. This is the trace that we need to compute. So, let us first see that, we can simplify this term further. This dagger look at a typical term gamma 0 epsilon prime slash epsilon slash k slash dagger gamma 0. So, what you get for this? This contains a bunch of gamma matrices. So, you have gamma mu gamma nu gamma rho dagger; and, gamma 0 is multiplied both sides. So, this will simply give gamma 0 gamma rho dagger gamma nu dagger gamma mu dagger gamma 0. But, gamma rho dagger is gamma 0 gamma rho gamma 0. So, gamma 0 square is identity. Therefore, you get gamma rho. And then, there are two gamma zeroes here: one from this dagger; one from this. So, this will simply give gamma nu gamma mu. So, this bar just reverses this order here. Therefore, this quantity will simply be k slash epsilon slash epsilon prime slash. Similarly, the second term here will have this; the order will simply get reversed and nothing else will change.

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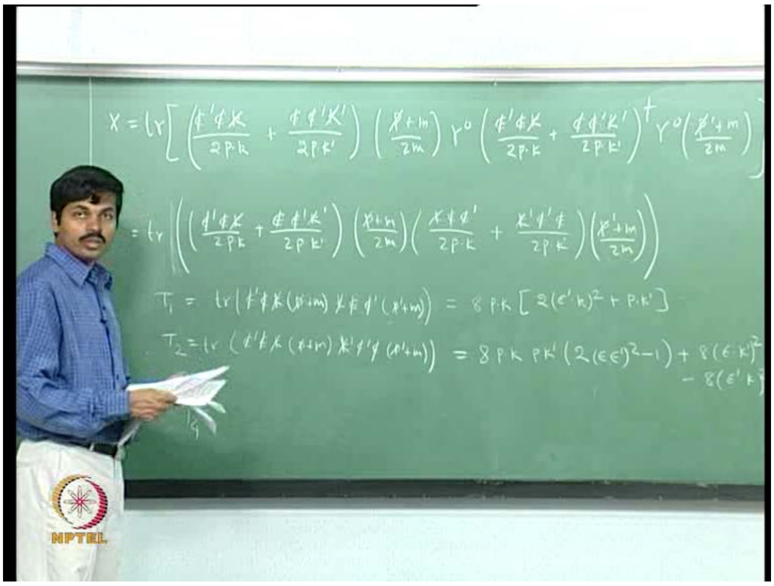

So, using this, we get trace of epsilon prime slash epsilon slash k slash over 2 p dot k plus epsilon slash epsilon prime slash k prime slash over 2 p dot k prime p slash plus m over 2 m. Then, this quantity as I said, the order will simply get reversed; so, k slash epsilon slash epsilon prime slash over 2 p dot k plus k prime slash epsilon prime slash epsilon slash over 2 p dot k prime; and then, p prime slash plus m over 2 m. So, this will give me four terms. I will denote them as T 1, T 2, T 3, T 4. For example, T 1 is nothing but trace of epsilon slash epsilon prime slash epsilon slash k slash; and then, p slash plus m k slash epsilon slash epsilon prime slash and p prime slash plus m. T 2 is just epsilon prime slash epsilon slash k slash p slash plus m; and then, the second term – k prime slash epsilon prime slash epsilon slash p prime slash plus m. And similarly, T 3 and T 4. T 3 is just the product of this term and this term with this and this; and, T 4 is the last two terms multiplied together. So, what we will do is that, we will evaluate T 1 and T 2. And then, T 3 and T 4 – we will see how we can evaluate them from T 1 and T 2 simply by some change of variables; and, by using some symmetry of these terms, we can just get T 3 and T 4 from T 1 and T 2.

So, what I will do is that, I will show to you that, we can evaluate this trace; and, at the end, we can show that, T 1 is equal to 8 p dot k times twice epsilon prime dot k square plus p dot k prime. And, T 2 we will evaluate and we will show that, the value of T 2 is somewhat complicated – 8 p dot k p dot k prime into twice epsilon dot epsilon prime square minus 1 plus 8 epsilon dot k prime square p dot k minus 8 epsilon prime dot k square p dot k prime. This is what we will show. And then, we will also argue that, T 2 has to be equal to T 3. And, T 4 can actually be derived from T 1. So, let us evaluate T 1 – the trace explicitly and then show that, this is in fact equal to the first term.

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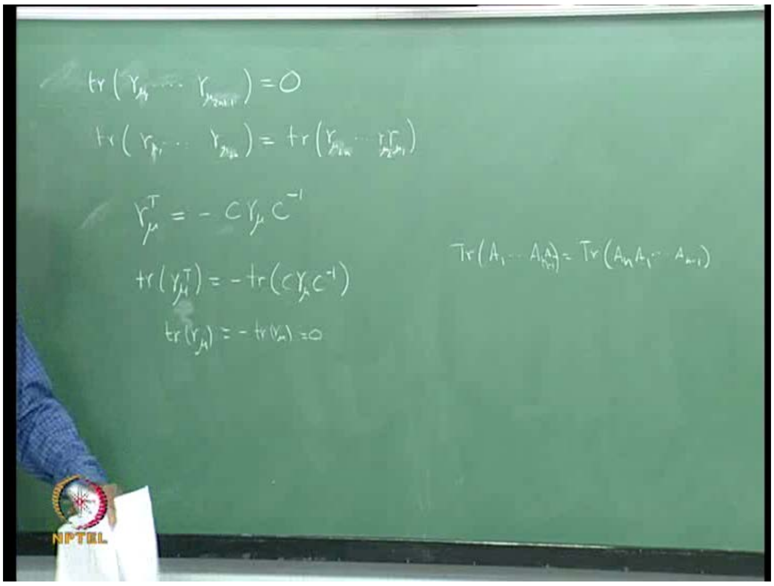

Before trying to evaluate any of these traces, I would like to prove you the following thing. First thing is trace of ((Refer Slide Time: 27:59)) number of gamma matrices vanishes. Trace gamma 1 up to gamma 2 n plus 1 is equal to 0. And, trace of gamma 1 up to gamma 2 n is equal to trace of gamma 2 n gamma 1. The order is reversed in the sense that… – gamma 2 and so on. These are the two identities I would like to prove it to you.

Student: Gamma is nu 1 to nu 2 n?

Absolutely. n number of gamma matrices and 2 n number of gamma matrices.

Student: It should be gamma nu 1…

You should write that way. Anything wrong with this?

Student: This means gamma ((Refer Slide Time: 29:01))

No, I said that, 2 n plus 1 number of gamma matrices; that is what I said; I stated this. So, let us make you happy. So, this is mu 1; this is mu 2 n plus 1; and, this is mu 1, mu 2 n; this is mu 2 n mu 2 mu 1. So, to do that, we will use the following thing. You know charge conjugation actually relates gamma matrices to their transpose. So, gamma transpose – gamma mu transpose in fact is equal to minus c gamma mu c inverse; where, c is the charge conjugation operator. So, we will use this relation to prove these identities.

First thing is immediately you can see that, this means trace of gamma mu; gamma matrices are traceless. So, you take the trace on the left hand side; it will give you gamma mu transpose, which is minus trace of c gamma mu c inverse. But, trace of gamma mu – trace actually is invariant under transpose. Therefore, this is equal to trace of gamma mu itself; trace of a matrix is equal to trace of the transpose of the matrix. And, here you can use the cyclic property of trace; you can bring it here. Trace of A 1 up to A n is nothing but A n minus 1 A n is trace of A n A 1 up to A n minus 1. This cyclic property of trace you can use. Then, you will get C C inverse, which is identity. So, that is minus trace of gamma mu. This implies it has to be equal to 0. So, gamma matrices are traceless.

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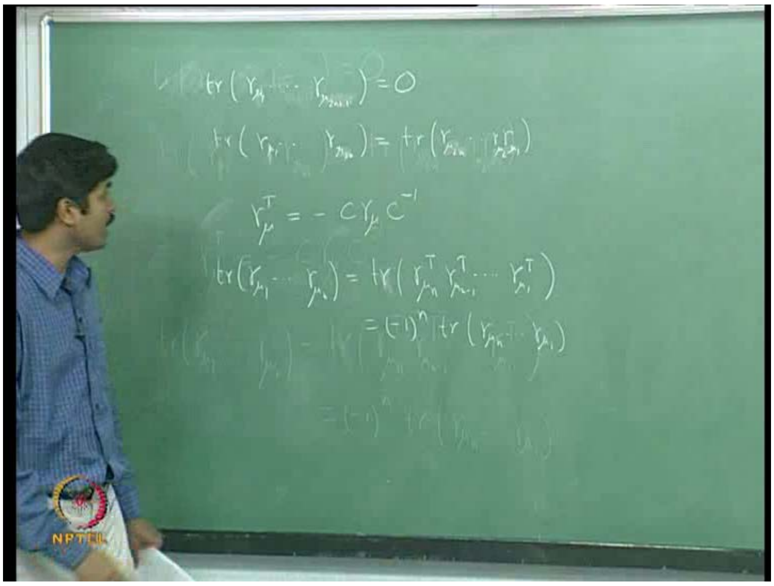

Then, what I can do is that, you can… Let us say take 3 gamma matrices – trace of gamma mu gamma mu gamma rho; all right? Or, you consider n number of gamma matrices – trace of gamma mu 1 up to gamma mu n; all right? So, this is equal to trace of transpose of itself. So, gamma mu n transpose gamma mu n minus 1 transpose gamma mu 1 transpose. Now, I will use again this relation and then I will use the cyclic property of trace. Then, this simply implies that, this is equal to minus 1 to the power n trace of gamma mu n gamma mu 1. So, the second of this relation is proved because if there are even number of gamma matrices, then this will simply give you 1. And hence, this identity holds.

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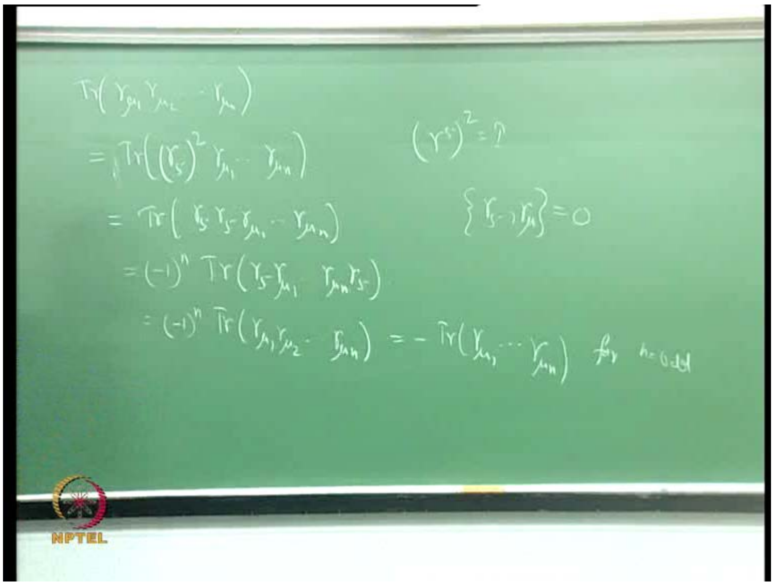

So, now, we have to prove the first relation, that is, the trace of odd number of gamma matrices is 0. And, this is very easy to show. So, let us consider this trace of gamma mu 1 gamma mu 2 up to gamma mu n; where, n is odd. This can be written as… Note that, gamma 5 square is actually equal to identity. So, we will insert an identity here. And, when we do that, this is simply trace of gamma 5 square gamma mu 1 up to gamma mu n; and, this is just equal to the trace of gamma 5 gamma 5 gamma mu 1 up to gamma mu n; and, retain one gamma matrix here; and, the other you take it to the extreme right. Remember – gamma phi anti-commutes with gamma mu's. So, when we take it to the extreme right, we will get minus 1 for each one of them. So, there is minus 1 to the power n. Therefore, this is equal to minus 1 to the power n trace of gamma 5 gamma mu 1 up to gamma mu n gamma 5.

And now, we will use the cyclic property of the trace. So, I can bring it back to here and use gamma 5 square equal to identity. And, when I do that, this is equal to minus 1 to the power n trace of gamma mu 1 gamma mu 2 up to gamma mu n. And, if n is odd, then what we get is this is minus of itself trace of gamma mu 1 up to gamma mu n for n – odd. Therefore, the trace of odd number of gamma matrices vanishes. Therefore, this is equal to 0 for odd n. So, now, we can look at T 1 and then we will evaluate it. And then, we will show that, it is the simpler expression, which I have given there.

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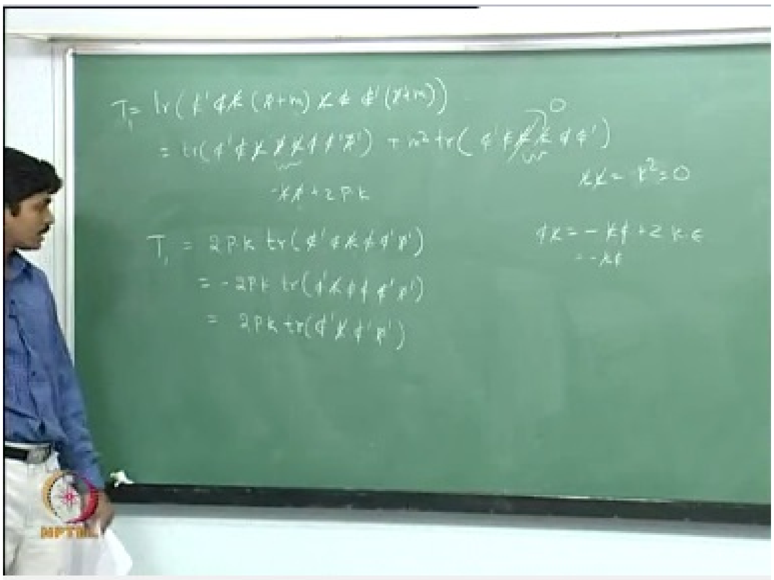

So, let us look at trace epsilon prime slash epsilon slash k slash and p slash plus m k slash epsilon slash epsilon prime slash; and then, p prime slash plus m. This is what is a T 1. And, as you can see, it will contain… If you just expand this term, it will give you four terms. So, the term which is linear in m – the two terms, which are linear in m contain 1, 2, 3, 4, 5, 6, 7 – 7 number of gamma matrices. So, that will simply be 0. Therefore, what I have here is trace epsilon prime slash epsilon slash k slash and p slash k slash epsilon slash epsilon prime slash p prime slash. That is first term. And, the second term is m square trace of epsilon prime slash epsilon slash k slash; and then, k slash epsilon slash epsilon prime slash. I claim that, the second term is 0 because of the presence of this term here. You have k slash k slash, which is equal to gamma mu gamma nu k square. I have done it in the previous lecture – k square. So, this is equal to 0. Therefore, the second term varnishes.

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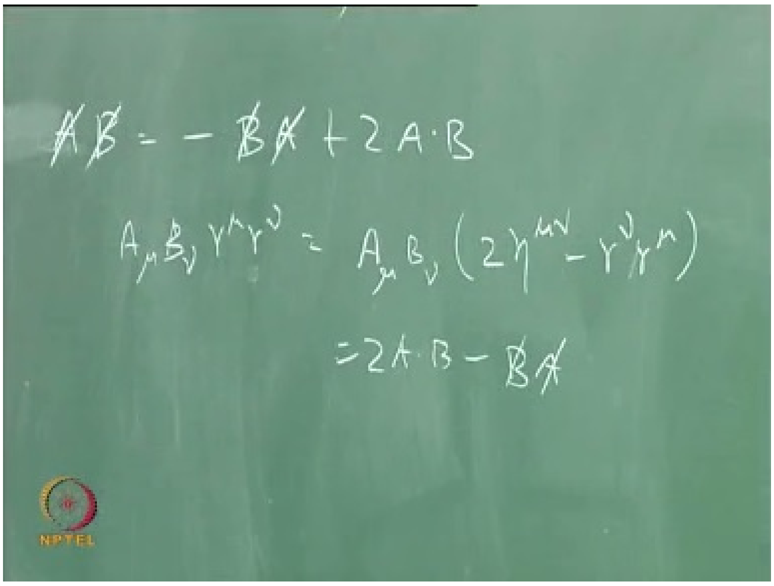

Now, what about the first term? First term – we can just look at this here; p slash k slash is there. So, I can write it as p slash k slash or any operator A slash B slash is equal to minus B slash A slash plus 2 A dot B, because this is A mu B nu gamma mu gamma nu. But, gamma mu gamma nu is A mu B nu times twice eta mu nu minus gamma nu gamma mu. So, the first term will give 2 A dot B. And, the second term will give minus B slash A slash. So, this identity we will use extensively. Let us use it here. This will give me minus k slash p slash plus twice p dot k. But, the first term, where there is a minus k slash p slash, this k slash is here. So, it will give k square, which will give 0. So, the first of these two terms is 0. Therefore, what you get is 2 p dot k trace of epsilon prime slash epsilon slash k slash epsilon slash epsilon prime slash p prime slash. So, this is what we get for T 1.

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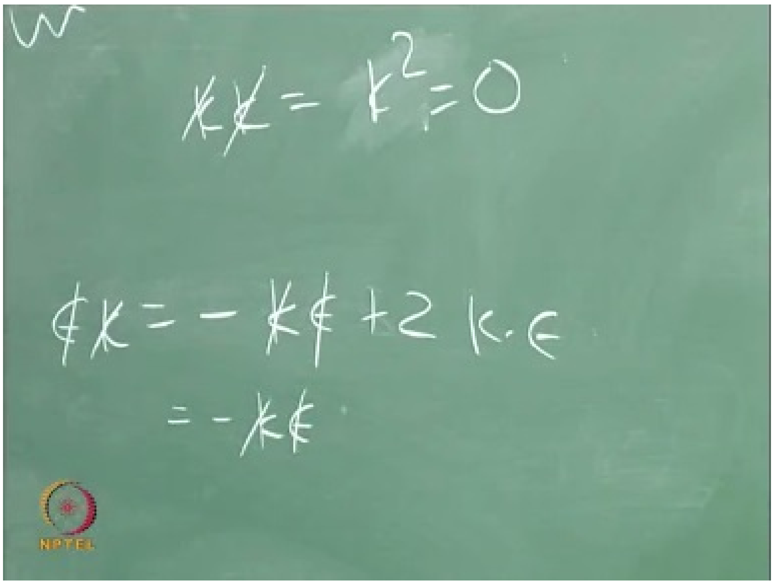

Now, we will use the fact that epsilon k is 0; epsilon slash k slash is minus k slash epsilon slash plus twice k dot epsilon. But, epsilon represents the transverse polarization for the photon. Therefore, k dot epsilon is 0. So, this is minus k slash epsilon slash. So, you can this relation. Then, this is minus twice p dot k trace of epsilon prime slash. This will give me k slash epsilon slash epsilon slash epsilon prime slash p prime slash. What is epsilon slash epsilon slash? It is minus 1. So, because of the metric that we have chosen, this is minus 1. So, this will give me simply twice p dot k trace of epsilon prime slash k slash epsilon prime slash p prime slash. So, all these complicated looking term we have just reduced it to the simpler expression.

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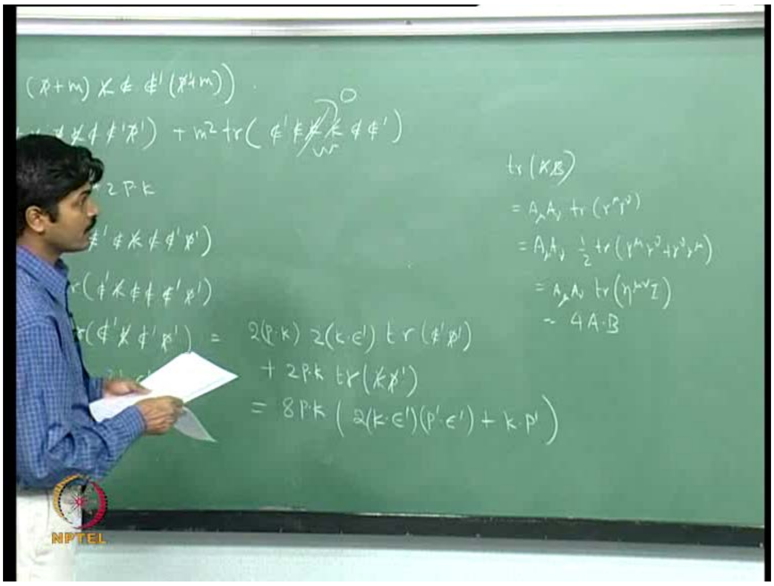

Again what I can do is that I can ((Refer Slide Time: 41:54)) this here. So, that will give me twice k dot epsilon prime and then minus k slash epsilon prime slash. But, then epsilon prime slash appears twice here. So, that will give me minus 1. So, this quantity here will be 2 p dot k times twice k dot epsilon prime trace of epsilon prime p prime epsilon prime slash p prime slash; and then, plus twice p dot k trace of k slash p prime slash. This plus sign here because there is a minus when this ((Refer Slide Time: 43:12)) And then, there are 2 epsilon primes, which will give minus 1. Therefore, this is a plus. So, you have this now.

What is this? Let us compute trace of A slash B slash for any two operators. This is A mu A nu trace of gamma mu gamma nu. But, this I can write it as A mu A nu half times trace of gamma mu gamma nu plus gamma nu gamma mu. I will now use the relation that, this is equal to twice eta mu nu. So, this is A mu A nu trace of eta mu nu times the identity operator. So, this will give you 4 A dot B because trace of the identity operator is 4. So, this is four eta mu nu; this will give you A dot B. So, we will use this relation here. Then, you see that, this quantity is equal to 8 p dot k times twice k dot epsilon prime; then, p prime dot epsilon prime plus k dot p prime. This is still not the desired form.

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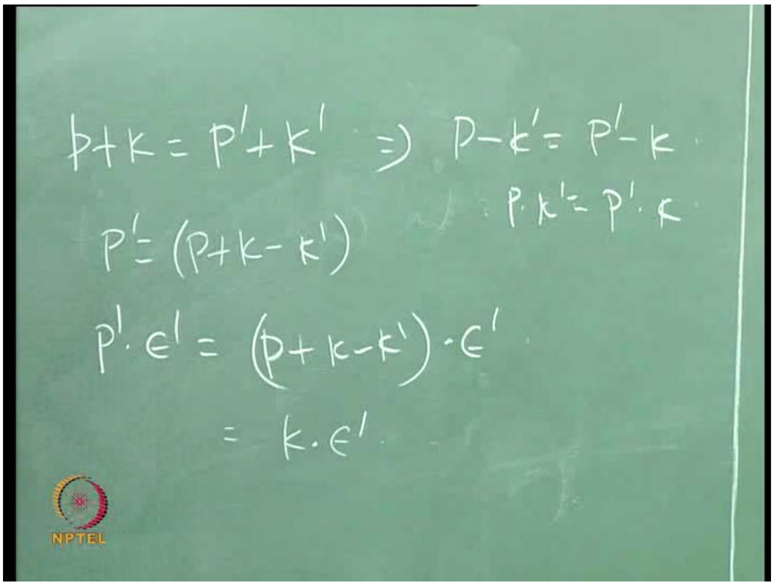

But, remember that, the energy momentum conservation implies p plus k is equal to p prime plus k prime. And hence, this quantity here – p prime is p plus k minus k prime. And hence, p prime dot epsilon prime will simply give you p plus k minus k prime dot epsilon prime. But, we are choosing the time axis along p and also… because epsilon prime is transverse; it is orthogonal to k prime. So, this is simply k dot epsilon prime. Therefore, p prime dot epsilon prime is simply k dot epsilon prime. And, for k dot p prime, you can again see that, this simply implies that, p minus k prime is equal to p prime minus k. Now, if you square this, then you will get p dot k prime is equal to p prime dot k.

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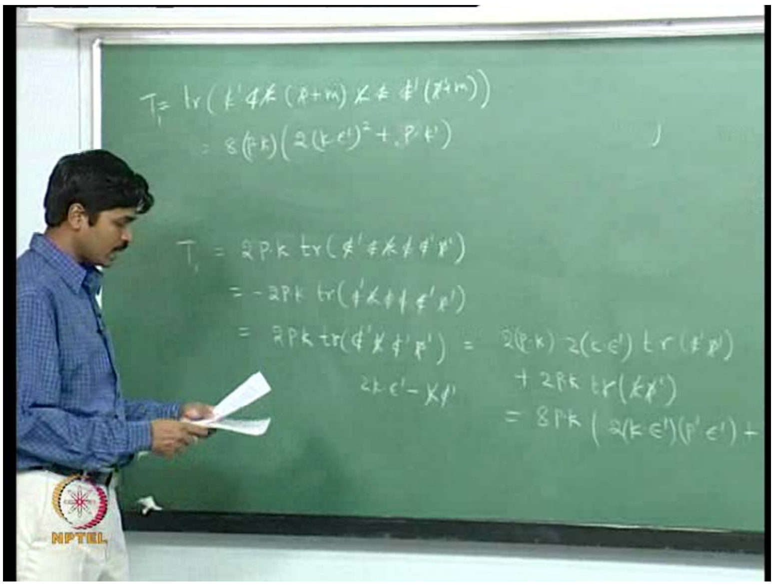

So, k dot p prime is k prime dot p. Therefore, this quantity here T 1 is nothing but is equal to 8 p dot k times twice k dot epsilon prime whole square plus p dot k prime. So, what we will do in the next lecture is we will evaluate T 2 and then we will prove that, T 2 is equal to this. Then, we will quickly compute T 3 and T 4 from T 1 and T 2. We will substitute the whole thing here; and then, this in the formula for this scattering cross section. And then finally, we will have a nice form for this scattering cross section.