Quantum Field Theory
Prof. Dr. Prasanta Kumar Tripathy
Department of Physics
Indian Institute of Technology, Madras

Module - 4
Quantum Electrodynamics
Lecture - 23
The S-Matrix Expansion in QED I

So, we have discussed quantization of a free electromagnetic field as well as a quantization of the Dirac field. What we will see from now is one of the interactions between the Dirac field and the electromagnetic field. So, the interaction is introduced, the minimal interaction is introduced in the following manner.

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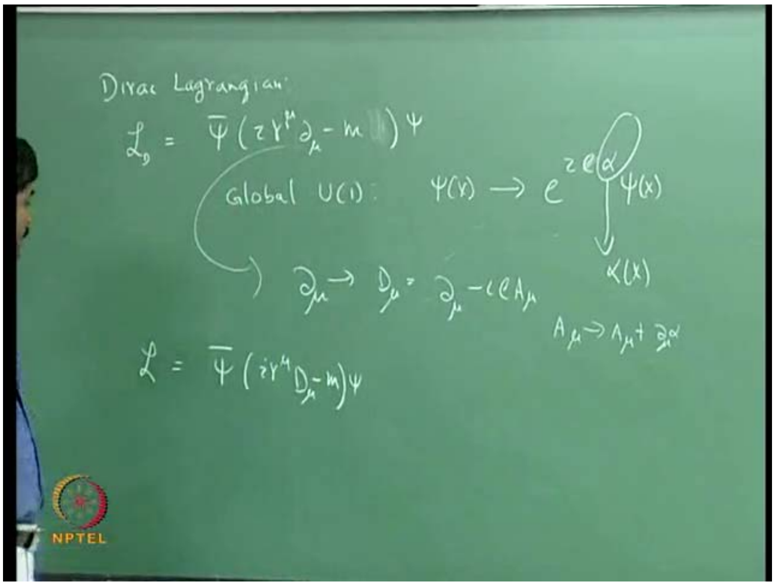

You consider the Dirac Lagrangian, which is given by LD equal to psi bar i gamma mu del mu minus m psi. Now, notice that this is the global u 1 invariance. So, if you make psi of x to transfer as e to the power i e alpha psi of x over e and alpha are co stamps, then this Lagrangian density is invariant under this transformation. Now, the question is if you want to make this global u 1 to a local u 1, then what do you do? You make this alpha to be actually A plus ion of x space time coordinates.

Of course, if alpha is x dependent, then this Lagrangian is no longer invariant. Under this u 1 transformation, what you do is you would modify this Lagrangian in such a way that it is invariant under u 1 transformation. How do you modify this? Is that same? This del mu the partial derivative to covariant derivative, so del mu goes to D mu, which is del mu minus i e A mu where A mu is the case field, which transforms under gaze transformation like A mu goes to A mu plus del mu alpha. Then this is Lagrangian density which I will call as L, which is given by psi bar i gamma mu D mu minus m psi, this is going to be invariant under the local gaze transformation.

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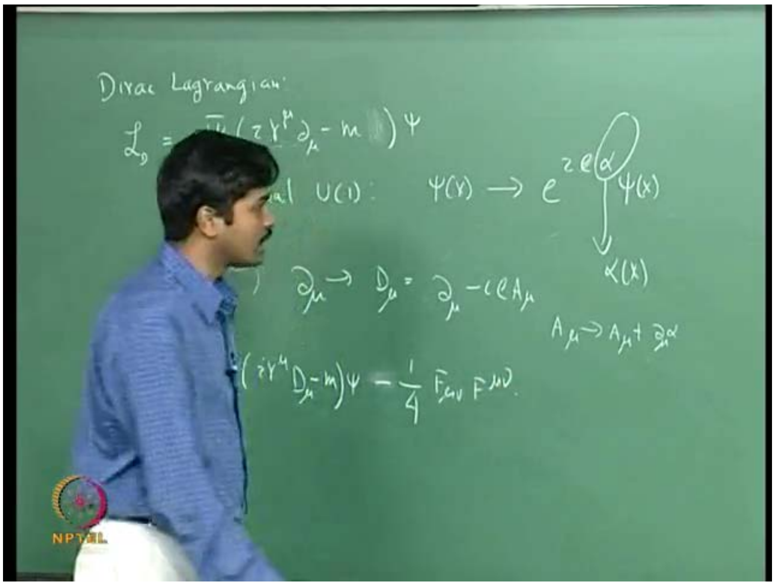

So, if you would like to incorporate the dynamics of the gaze field as well then we will add to this the Maxwell Lagrangian, which is given by minus one fourth F mu nu F mu nu.

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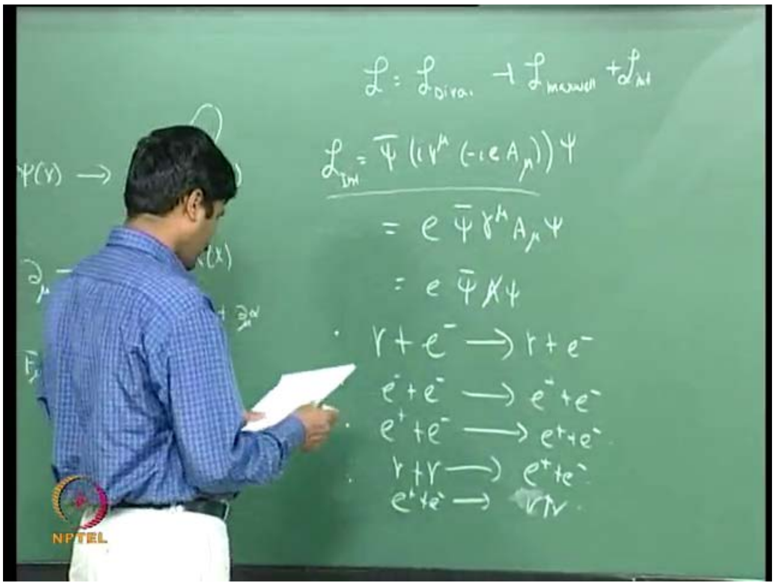

So, as a result, we get the Lagrangian density, which it contains all the free Dirac Lagrangian, the free Maxwell's Lagrangian and the interaction between them where L Dirac is given by this L, Maxwell is this quantity here, one fourth F mu nu r mu nu. What does L interact? The interacting part of the Lagrangian, H determined from this term here. This is given by psi bar i gamma mu times minus i e A mu psi. So, L interacting, it is this form. Let me use the convention. So, this is e times psi bar gamma mu A mu psi or another at e psi bar A slash psi. This is the interacting part of the Lagrangian density. We will see what do we get when we take this theory and quantize it.

This is one of the most important properties of Lagrangian density that you will come across in physics. It describes a wide gaze of physical processes, some of the simplest ones or the Compton scattering, where photon interacts with electron gamma plus e minus. It can also describe electron electron scattering, electron electron going electron electron.

It can also describe electron positron scattering e plus plus e minus going to e plus plus e minus and pair creation, in the sense gamma plus gamma going to e plus plus e minus or pair annihilation, which is e plus plus e minus going to e plus gamma gamma and a number of other physical processes. We will discuss some of them. Then we will compute the cross section for that, but before that, we need to consider these physical processes. What we need to know is what the S matrix is. Then what are it is elements? Which terms in the S matrix contribute to what physical processes? All those things that we need to know. In other words, we can derive a set of very simple rules, which at the Feynman rules will tell us how to compute the amplitude for any physical process. What we will do is we will first consider the S matrix. Then from the S matrix, we will derive the Feynman rules.

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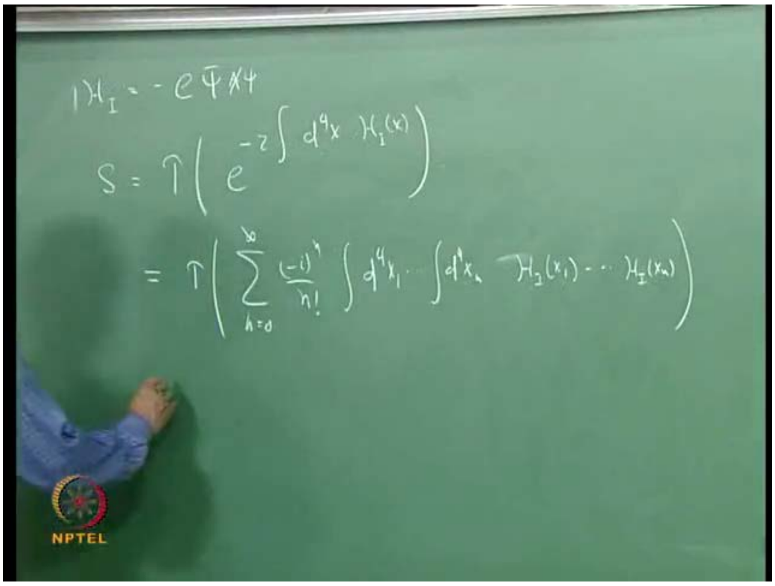

Of course, the interaction interacting Hamiltonian density which I will call as HI is minus of this, so minus e psi bar A slash i and the S matrix is given by time order product of e to the power minus i integration of d 4 x HI x. I can evaluate it order by order. So, this is nothing but time order product of sum over n equal to 0 to infinity minus i to the power n divided by n factorial d 4 x 1 up to d 4 x n and HI of the x 1, so up to HI of x n.

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So, I will denote them is sum over n equal to 0 to infinity S n where n equal to 0 corresponds to the first term in this expansion, n equal to 1 is the second term and so on. So, what is S0? S0, of course is identity. It just gives us the forward scattering part so you will forget about S0, and then we can consider the terms. So, let us look at the second term in the expansion, which I will call as the first order term. Look at this linear in e and the second term in the expansion, it is quadratic in e, which I will call as the second order term.

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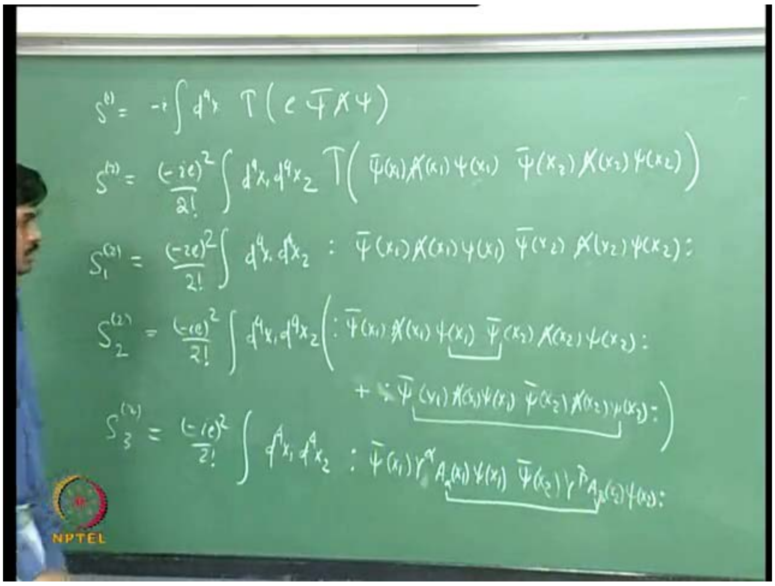

So, the first order term is S1. I will call according to my notation. S1 is given by minus i integration d 4 x time order product of e times psi bar A slash psi. We will discuss more about this term in a moment, but let us look at the second term in the expansion. So, S2 is given by minus i e square divided by 2 factorial d 4 x 1 d 4 x 2 and time order product of psi bar of x 1 A slash x 1 psi of x 1 and then psi bar x 2 A slash x 2 psi of x 2. Now, we can use wick's theorem to express this time order product in terms of normal order products.

So, what we will get when we express it into the normal order product? We will of course, get the first term is the normal order product of this and then all one contraction terms. So, what are all one contraction terms? This psi x 1 can contract with psi bar x 2 A x 1 can contract with A x 2 and psi bar x 1 can contract with psi of x 2. So, these are all one contraction term. Then you can have two contraction terms. What are the two contraction terms? They are these; this pair and this pair can contract or so on. So, there are finally, you will have three contraction terms. There will be one such term.

So, let us write down these quantities. I will denote them as S21 and so on with a sub script, which basically gives as various terms in the wick expansion d 4 x 1 d 4 x 2 and the normal order product of psi bar x 1 A slash of x 1 psi x 1 psi bar x 2 A slash x 2 psi x 2. Now, this does not enable any contractions. Now, we will consider the one contraction term. So, S22, I will call S22 is minus i e square over 2 factorial and integration d 4 x 1 d 4 x 2.

So, there are three contractions, but out of which on enables the contraction of A x 1 with A x 2. The other terms enable the contraction of the firm means, so I will group the contraction of firm means into one term and then the contraction of the gaze fields, I will write it in a separate term. So, this is normal order product of psi bar x 1 A slash x 1 psi 1 psi bar x 2 A slash x 2 psi x 2, where these contracts with these. Then normal order product of psi bar of x 1 A slash S 1 psi x 1 psi bar x 2 A slash x 2 psi x 2, where this psi bar of x 1 contracts with this.

Finally, S23 enables the contraction of A x 1 with A x 2. So, this is minus i e square over 2 factorial integration normal order product of psi bar x 1 gamma alpha A l have x 1 psi x 1 psi bar of x 2 gamma beta A beta of x 2 psi of x 2, where A alpha x 1 contracts with

A beta x 2. This contains all one contraction terms. Now, we will look at all the two contraction terms. So, what are all two contraction terms?

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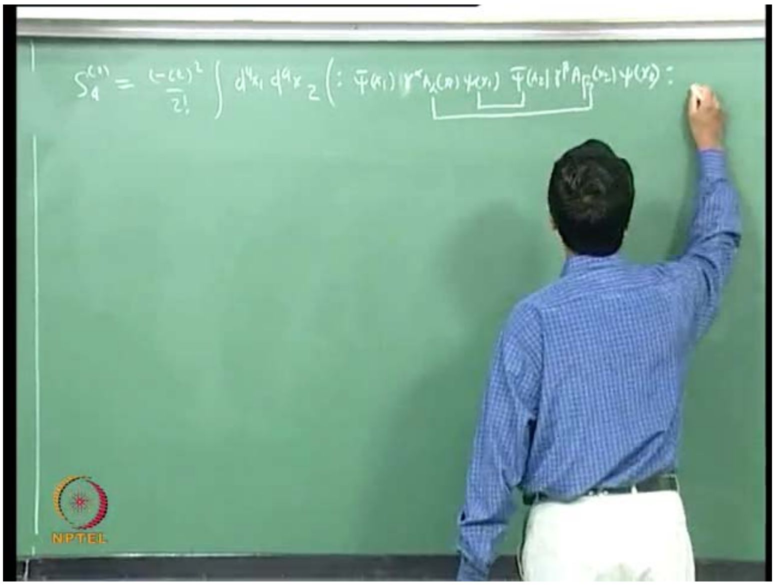

I will denote S24 as the one, which contains contraction of a pair of A of x 1 and A of x 2 and psi bar. So, it involves one pair from the gaze field and one pair from the normal field. Therefore, this is minus i e square over 2 factorial d 4 x 1 d 4 x 2 normal order product of psi bar x 1 gamma alpha A alpha x 1 psi x 1 psi bar x 2 gamma beta A beta psi of x 2, where psi x 1 contracts with psi bar x 1 and A alpha of x 1 contracts with A beta of x 2.

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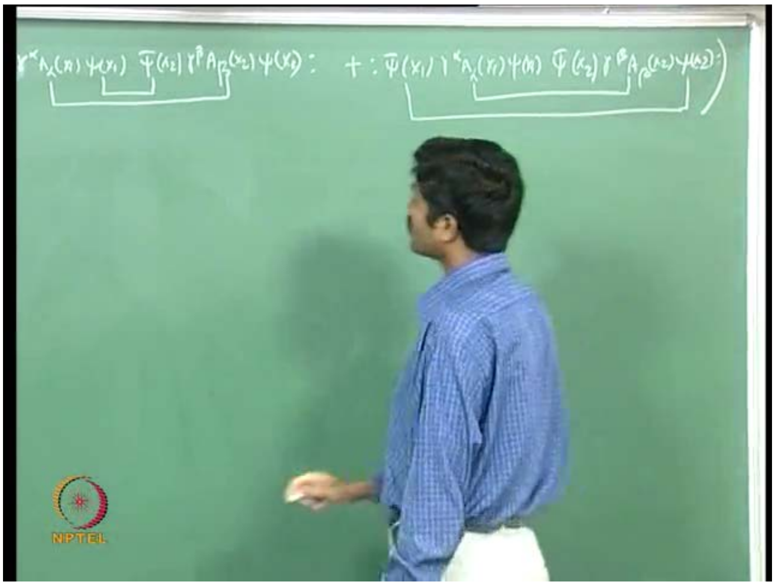

Then, I can have, similarly the normal order product of psi bar x 1 gamma alpha A alpha x 1 psi x 1 psi bar x 2 gamma beta A beta x 2 psi of x 2, where A alpha x 1 contracts with A beta of x 2 and psi bar x 1 contracts with psi of x 2.

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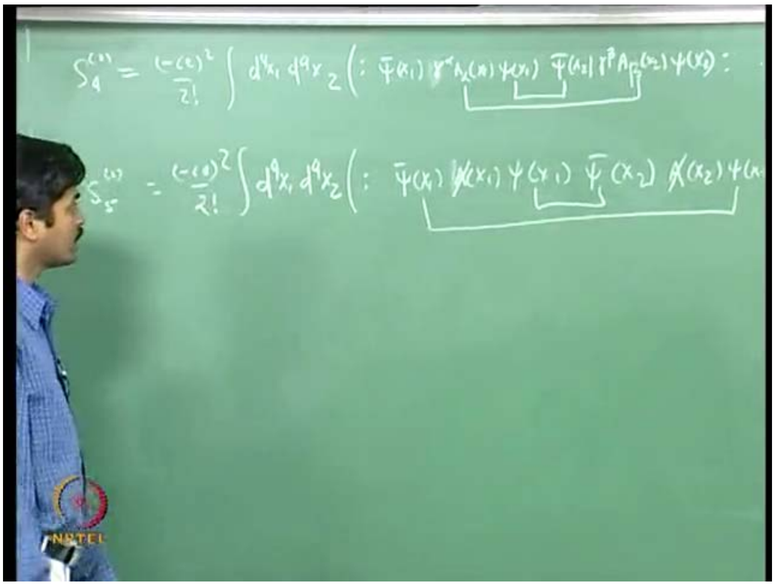

Now, I have one more term involving two contraction terms, which will contract both the pairs of fermions. So, S25 is minus i e square over 2 factorial normal order product. This contracts with this, and then this one contracts with this.

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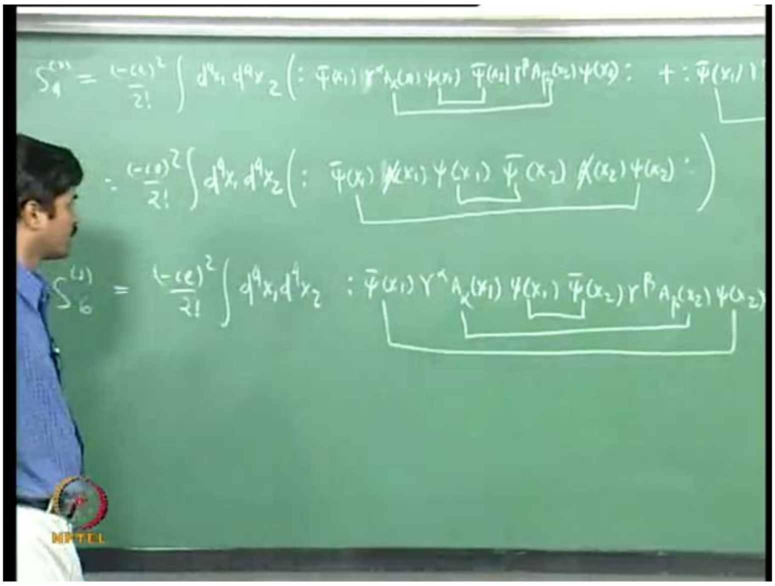

Finally, the last term which involves all three contraction terms where this contracts with this A alpha factor on contract with A beta x 2 and this. So, this is the list of all second order terms. We can go on like this, but we will stop with second order. Then we will see whether we can, I mean, we can find a simple to compute the amplitude for a physical process at any given order. So, let us look at each of these terms. Let us look at the first order term here, the first order term and then what kind of physical processes to which it can contribute.

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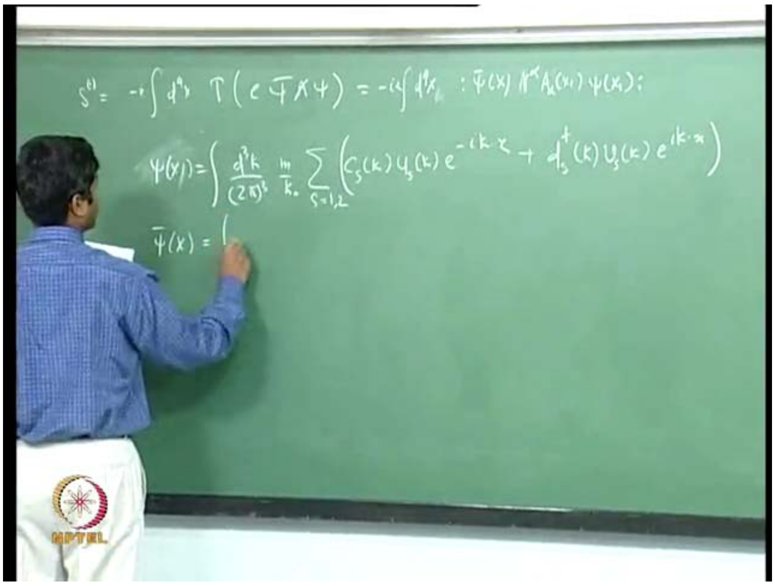

So, S1 is also equal to normal order product of itself minus i d 4 x 1 d 4 x normal order product of psi bar of x 1 gamma alpha A alpha of x 1 psi of x 1. So, this is the only term in the wick expansion. Now, what is the psi bar of x psi of x of 1? We know psi of x is the mode expansion integration d cube k over 2 pi cube m over k 0 sum over S equal to 1, 2 c S of k, u S of k e to the power of minus i k dot x plus d S dagger k d S k e to the power of i k dot x.

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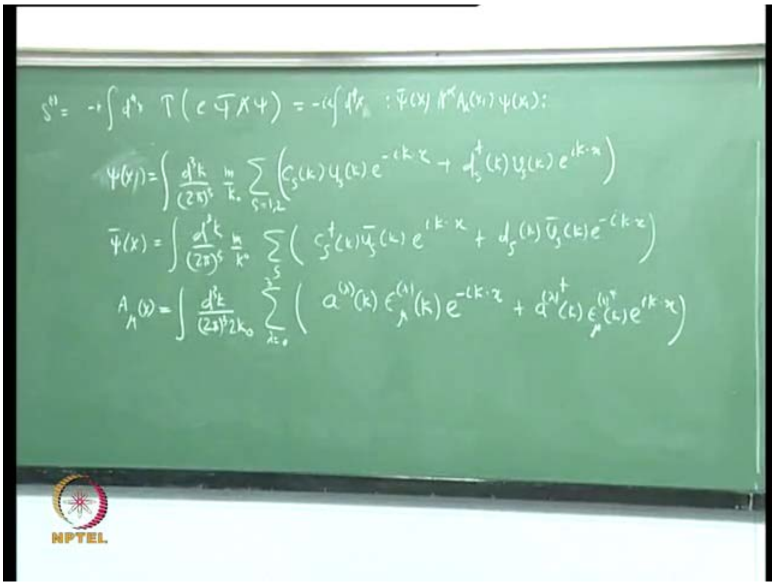

Psi bar of x is d cube k over 2 pi cube m over k 0 sum over S c S dagger u S bar k e to the power of i k dot x plus d S k d S bar of k e to the power of minus i k dot x. So, we will have A mu of x for is the mode expansion is given by d cube k over 2 pi cube 2 k 0 sum over lambda equal to 0 to 3 A lambda k epsilon lambda mu k e to the power of minus i k dot x plus A lambda dagger k epsilon mu lambda star k e to the power i k dot x. So, what are these? What do these terms represent?

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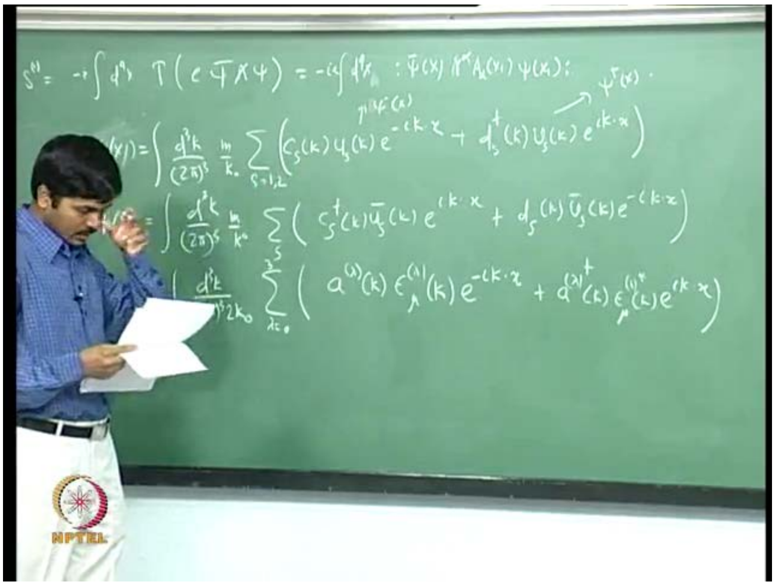

This I will write it is psi plus and psi minus where this I will call as i plus of x this term. I will call as psi minus of x. So, psi minus of x contains the annihilation operator part, one electron, sorry psi minus I will denote it is annihilation. This is the creation part. So, this I will denote is, I will denote as i minus.

This I will denote as i plus of x because this is the positive frequency part and this is the negative frequency part. This operator actually is a creation operator for an electron. This d dagger field is, this c S are the positively charged particle, this c S dagger create positively charge particle, whereas the d S dagger actually creates negatively charge particle. Therefore, this is i minus of x. I will denote this is the creation operator for an electron, whereas this i plus of x is it will describe the annihilation of a positron.

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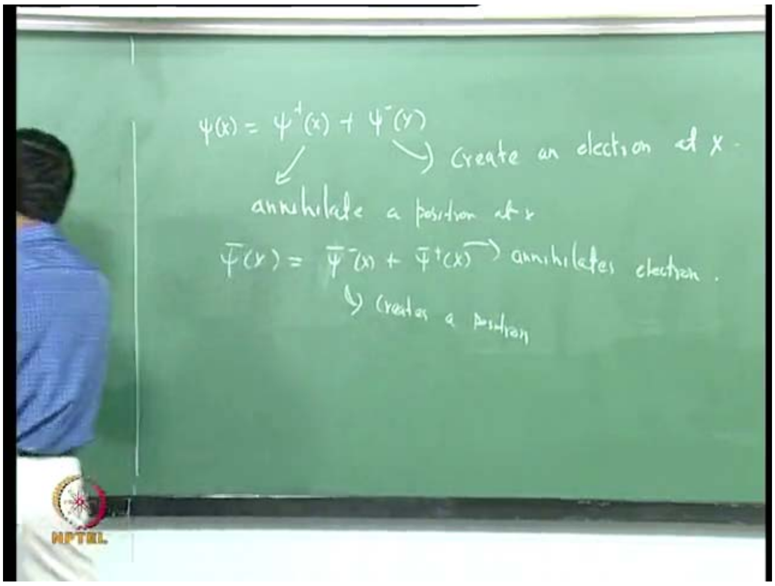

So, when I write psi of x. Psi of x, it is psi plus of x plus psi minus of x. This will create an electron at this to x time, whereas this will annihilate a positron at x. On the other hand, psi bar of x, I will write it is psi bar minus of x. At this term, I will write it is psi bar plus of x. So, psi bar minus of x creates an electron, whereas this annihilates a positron. So, psi psi bar minus of x plus psi bar plus of x, this one, the psi bar minus creates a positron. The psi bar plus annihilates an electron.

So, here this is the negative frequency component that actually creates an electron, whereas the positive frequency component annihilates a positron. Here the opposite thing happens in the sense that the positive frequency component here creates a positron. Then the negative frequency component creates positron, whereas the positive frequency component annihilates an electron.

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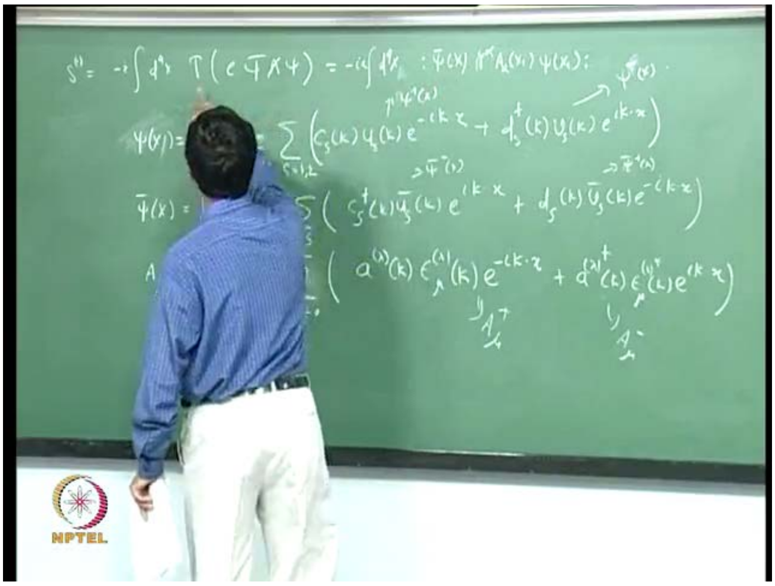

What about this? Here, this is the first term which I will call is A mu plus and this term I will call as A mu minus. So, this one annihilates a positron, whereas this one creates a photon. Now, what kind of terms are there in the first order term in the S matrix, let us look at it.

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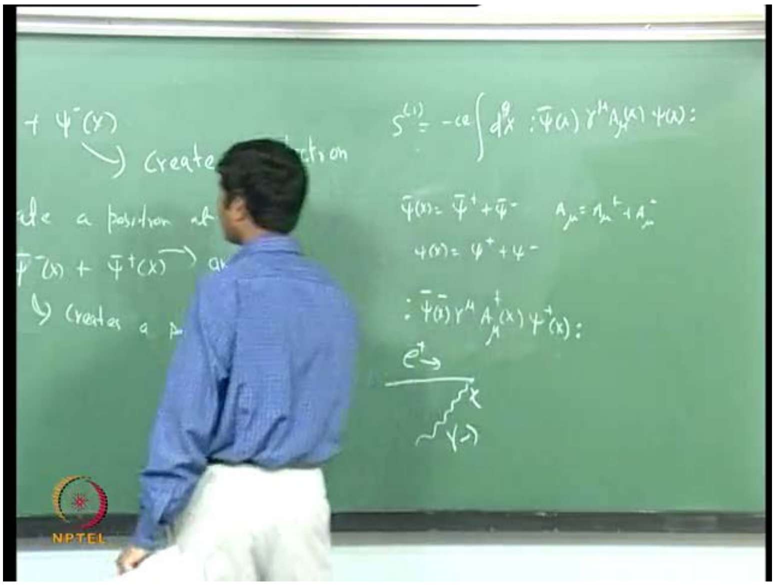

So, S1 minus i e d 4 x normal order product of psi bar of x gamma mu A mu of x psi of x, so what kind of physical processes are there to which this term will contribute? I mean contribute any way in the non zero contribution. This psi bar of x is total psi bar plus plus psi bar minus. Similarly, A mu is A mu plus plus A mu minus and psi of x. It is psi plus plus psi minus. So, a typical term of course, you can list all the terms, but typically it will be half done, so is that of this time normal order product of psi bar minus gamma mu A mu plus of x psi bar minus of x plus psi plus of x.

Consider a typical term, which looks like this. So, here this is psi plus of x. You just look at this. It annihilates a positron at x. So, you consider a positron, which is annihilated at this point x. So, this I will denote it is a positron e plus, I will use a different notation little later, but for the time being, it is and then A mu plus annihilates a photon at x of a photon is annihilated here. So, you have an e plus, there in an incoming positron, there is incoming photon. Then psi plus of x denotes, psi bar minus is basically denotes creation of an electron. The psi bar minus is creation of a positron.

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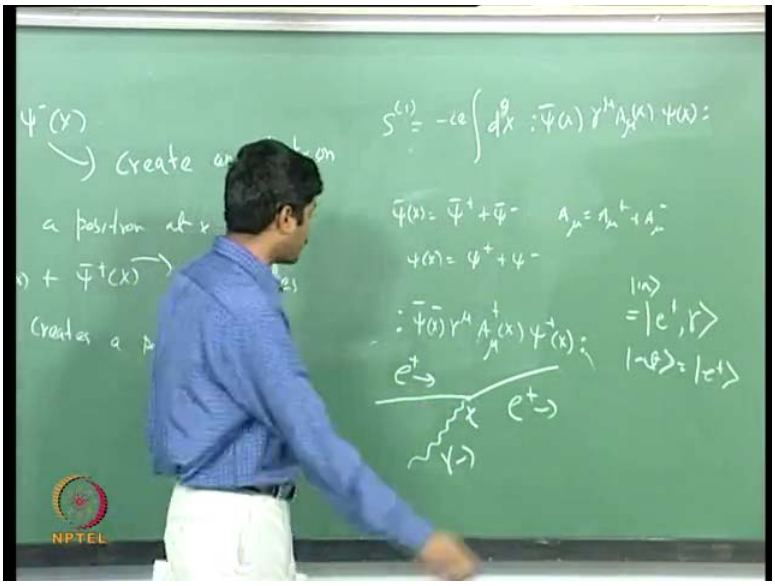

So, positron is created again at this face point x, so psi plus annihilation of a positron psi bar minus. So, this is what is the physical is the process to its term like this will give a nonzero contribution. We will see little later that in fact this, if we work out the amplitude, then the amplitude actually vanishes, but just looking at the term in the S matrix, you can say that this is what the process is. If you have in state, which is basically a positron and a photon and if you have an out state which is one positron, then this is the term in the S matrix.

So, it might give physical nonzero amplitude. So, similarly, there will be other terms here, which will describe there are physical processes, but we will see little later that all such terms actually at this order, all the terms give zero contribution to any physical amplitude that you can consider. So, this term does not do anything interesting. Now, we can look at the second order term. Let us consider a typical second order term, which is S22.

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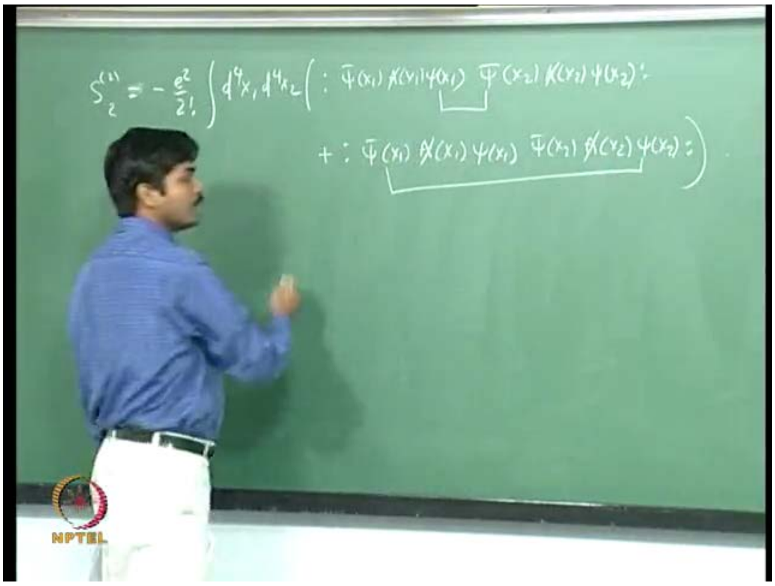

You will see. You can convince yourself that S21 also does not give a contribution to any physical process. S22 is the first nontrivial term, which actually gives contribution to many physical processes. We will see what are the physical processes to which this tem gives nontrivial contributions, and then on what way, it gives contribution to physical processes? So, S22 equivalent over notation is minus e square over 2 factorial d 4 x 1 d 4 x 2, and then normal order product of psi bar x 1 A slash x 1 psi of x 1 psi bar x 2 A slash x 2 psi of x 2 where this is contracted with this.

Then, psi bar x 1 A slash of x 1 psi of x 1 psi bar x 2 A slash x 2 psi of x 2, this has contracted with this. We can, can you simplify this term? Of course, we can simplify it a bit. You notice that these three terms contains two fermions operator and one boson operator. So, you can move it freely inside the normal ordering without any change of sign. What I will do is that I will take all these three operators to the beginning here.

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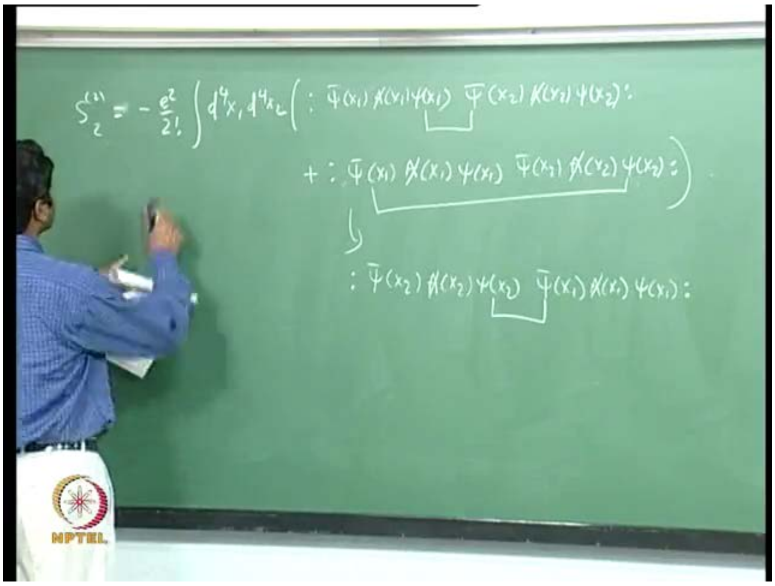

So, the second term here is normal order product of psi bar x 2 A slash of x 2 psi x 2 psi bar x 1 A slash of x 1 psi of x 1. If it had contained odd number of fermions, then you may or may not have change sign. Now, look at this. So, what, which term is contracted with what? Psi bar x 1 is contracted with psi of x 2. So, this is contracted with this here. Now, since x 2 is integrated over x 1 and x 2, you can change this variable x 2 to two x 1 and x 1 to x 2. Then you can see that this term is nothing but it is identical to the first term. So, you can write x 2 is minus e square and the second term here is removed. So, what are the physical processes to this? This can give non vanishing contribution again. So, we can look at psi bar intersect psi bar plus psi bar minus and so on.

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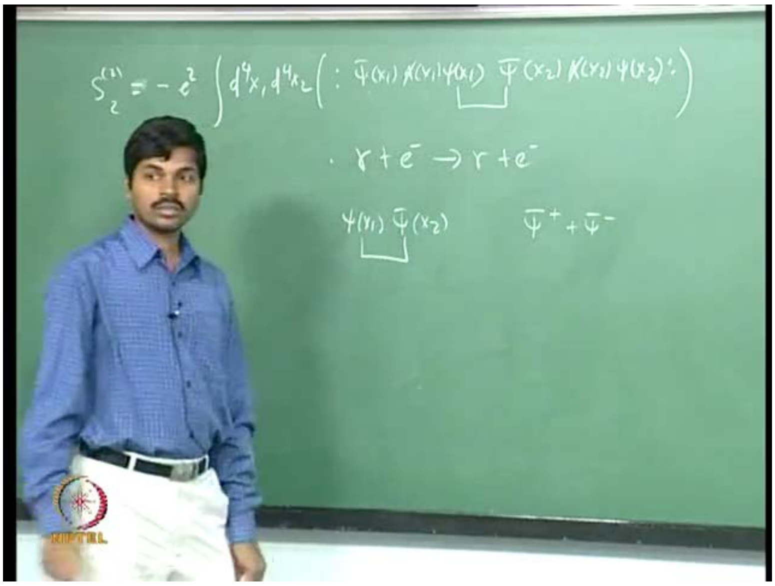

So, you can see that it of course, gamma plus e minus going to gamma plus e minus, it can give nonzero contribution to the Compton scattering because it can, so first of all this there is propagative here. This term psi of x 1 psi bar of x 2, this tells you the propagation of an electron from x 1 to x 2, whereas here this psi bar x 1 can contain psi bar plus plus psi bar minus. What is psi bar minus describing? Psi bar minus is describing creation of a positron psi bar minus an electron. So, you can have a process like this. Similarly, psi of x will contain psi. What I will use is the following.

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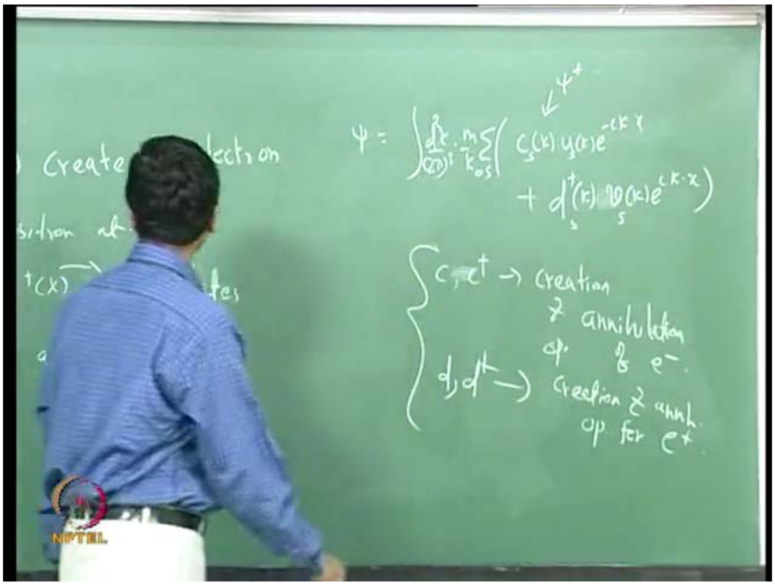

When I consider the mode expansion for psi, psi contains a bunch of terms. Then we have already seen c of k u k of x e to the power of minus i k dot x plus d dagger k v k e to the power i k dot x d cube k over 2 pi cube m over k 0 and sum over s here c s u s d s v s. So, I will denote this c and c dagger to be the creation and annihilation of an electron, so operators, whereas these d daggers, I will denote them as creation and annihilation operators for e plus positron. I will consistently follow this notation throughout the course of our lecture. So, then what happens is here the first term here is psi plus. So, psi plus describes the annihilation of an electron not positron.

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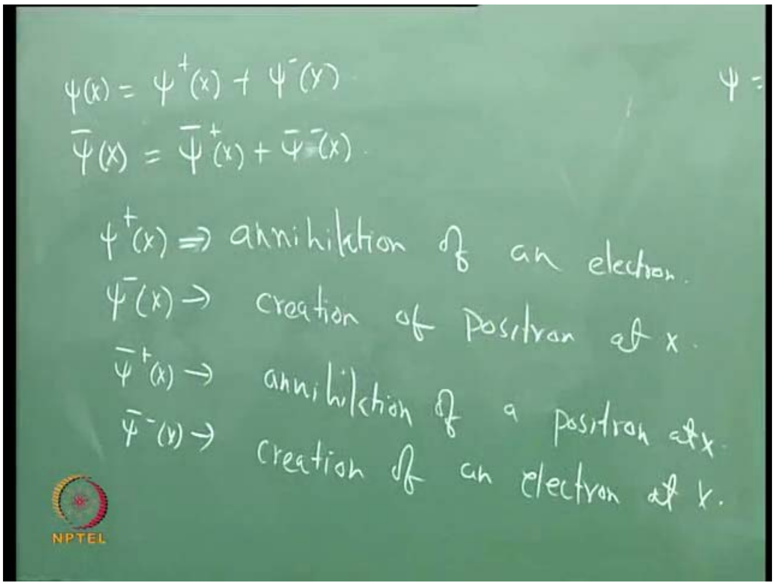

So, psi bar of x is psi bar plus of x plus psi bar minus of x, whereas psi plus of x which contains the c s is describing annihilation of an electron. Psi minus of x describes the creation of a positron at x, whereas psi bar plus of x describes annihilation of a positron at x and psi bar minus of x describes creation of an electron at x. So, this is the notation that we will be using throughout. So, there are of course, these physical processes.

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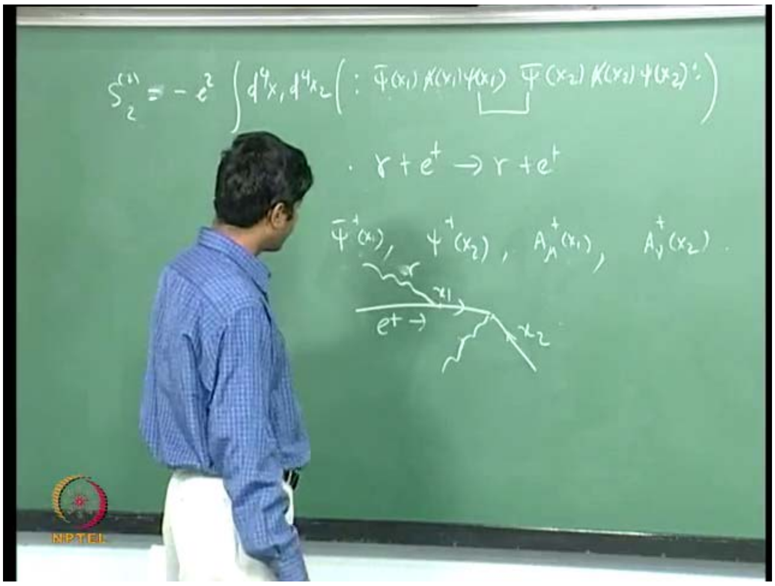

There you can also have psi bar plus of x 1 and you can have psi plus of x 2 A mu plus of x 1 A nu plus of x 2. It may contain this; I mean one such term will contain all these fields here. So, what physical process this will describe? What is psi bar plus of x 1 annihilation of an electron, annihilation of a positron at x 1? So, you have a positron. This is let us a x 1, an incoming positron is there, which is annihilated. What about psi bar plus of x 2? It creates a positron psi plus creates annihilation of an electron. So, there is an incoming electron with it. It is annihilation of an electron x 2. So, you have a x 2 and this describes the annihilation of a photon at x 1. This describes annihilation of a photon at x 2.

So, there is an incoming photon here e. This is gamma. So, there will be some such process like that which will and what kind of processes psi minus of x. If there is a psi minus of course, it will create an electron, but it will not conserve charge. Psi minus creates a positron. Psi minus creates a positron. So, then this will of course describe something like a positron, electron scattering. There are others also. So, what I suggest you is look at all the terms. You write psi bar in terms of psi bar plus psi bar minus and so on.

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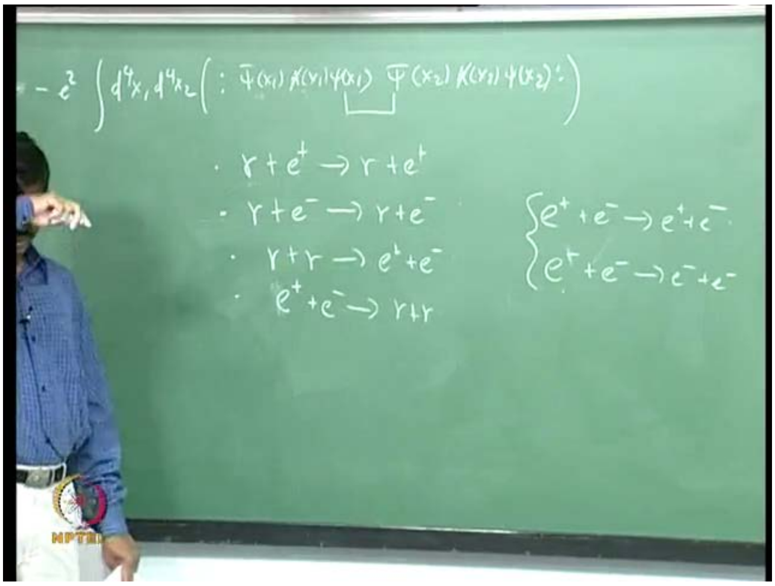

These are the physical processes which will be described in gamma plus e minus going to gamma plus e minus. It will also describe a pair creation gamma gamma going to e plus plus e minus and pair annihilation, which is e plus plus e minus going to gamma gamma. So, what we are struggling to see is whether e plus plus e minus going to e plus plus e minus, something like that or whether e minus plus e minus going to e minus plus e minus, you can see. You convince yourself that processes like this will not be described by this term in S matrix. So, these are the physical processes, which will be described by this.

Then, in the next lecture, we will see how we can describe these physical processes. We will see if there is a simpler way, if we have a set of rules, which will tell us how to instead of looking at these terms each and every time, if we can directly compute the matrix elements for any physical process.