Quantum Field Theory
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Module - 3
Free Filed Quantization: Spinor and Vector Filed
Lecture - 22
Fermion Quantization VI

So, it is we will first discuss the helicity states of the Dirac spinor. Then we will continue on our discussion on discrete symmetries. So, to construct the helicity states, let us look at the Lorentz transformation properties of the Dirac spinor.

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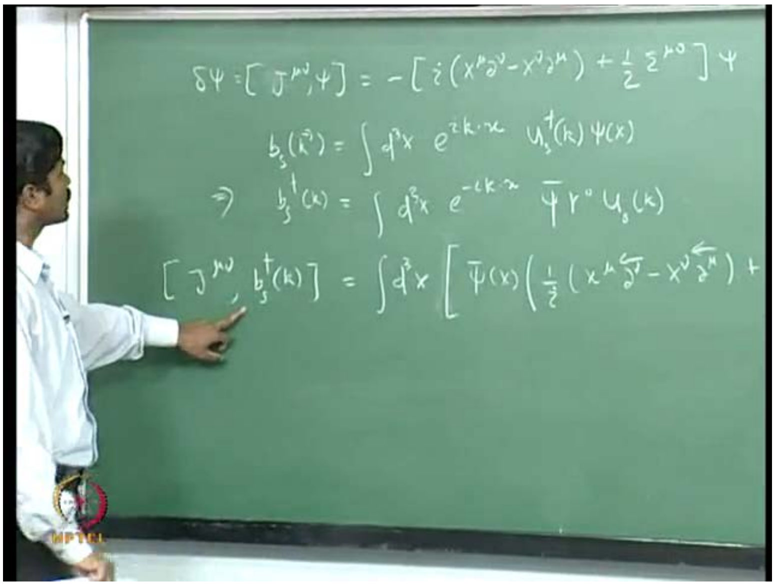

Delta psi which is nothing but the commutation relation between J mu and psi under Lorentz transformation is given by minus i x mu del mu minus x mu del mu plus half sigma mu mu acting inside and this is the orbital part of the angular momentum. This is the spin of the Dirac particle. From this relation here, we can derive the commutation relation between the angular momentum operators and the creation and annihilation operators that appear in the expression of psi.

So, to do that, let us pick up b s of k is a integration d cube x e to the power i k dot x u s dagger k psi of x and b s dagger of k from this expression is given by d cube x e to the power of minus i k dot x, then psi dagger which is psi bar gamma 0 u s of k. So, this basically implies that J mu nu b s dagger of k is given by d cube x. Then you have psi bar of x, this will act from the right. Therefore, this is so 1 over i x mu del mu minus x mu del mu plus half sigma m mu the gamma 0 u s of k e to the power of minus i k dot x.

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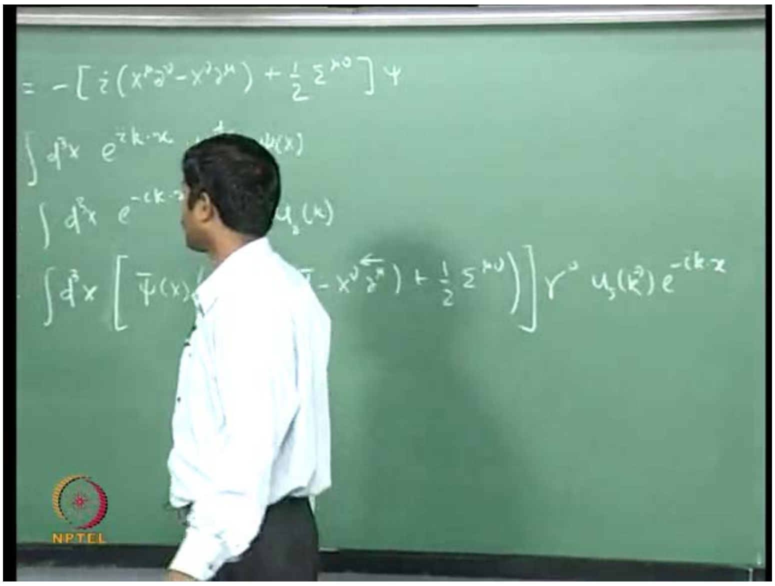

So, what I am interested in is in this commutation relation. Then you can see that if I say simply substitute it here, then if I simply substitute it here in this expression, I get this result here.

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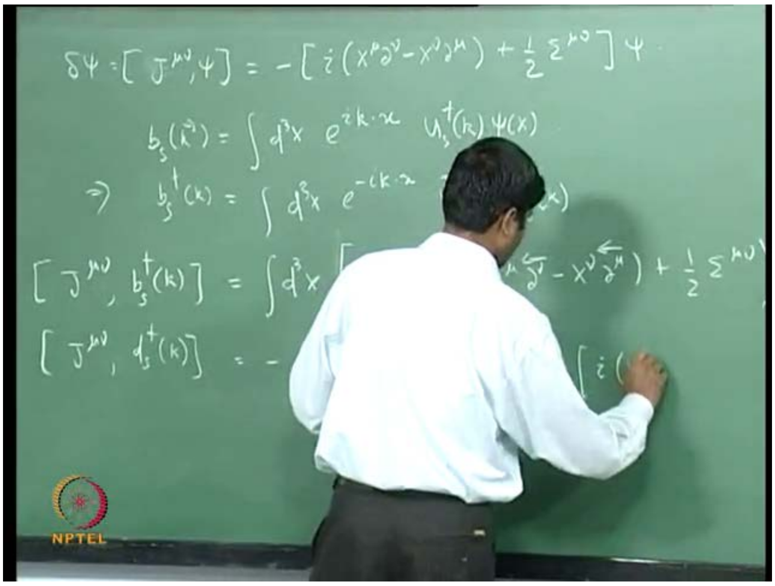

You can derive a similar expression for the commutation relation between J mu nu and the d s dagger of k.

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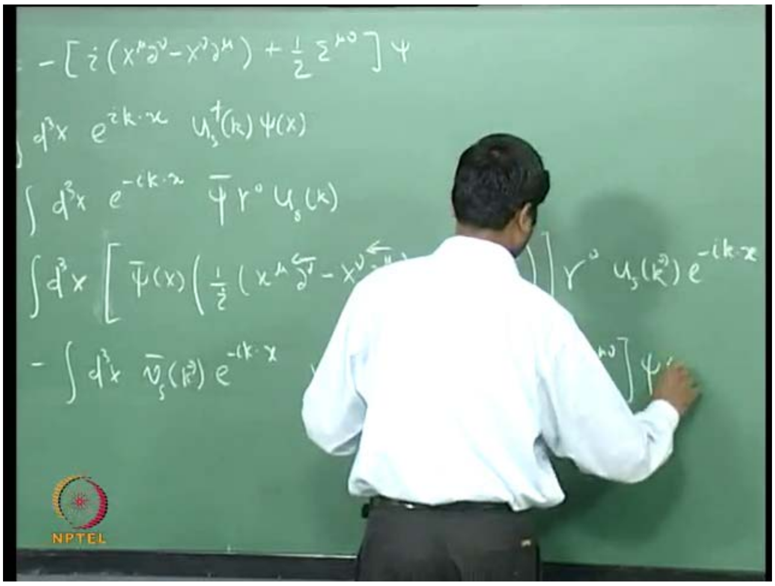

This commutation relation is given by minus d cube x v s bar of k e to the power of minus i k dot x gamma 0 I x mu del nu.

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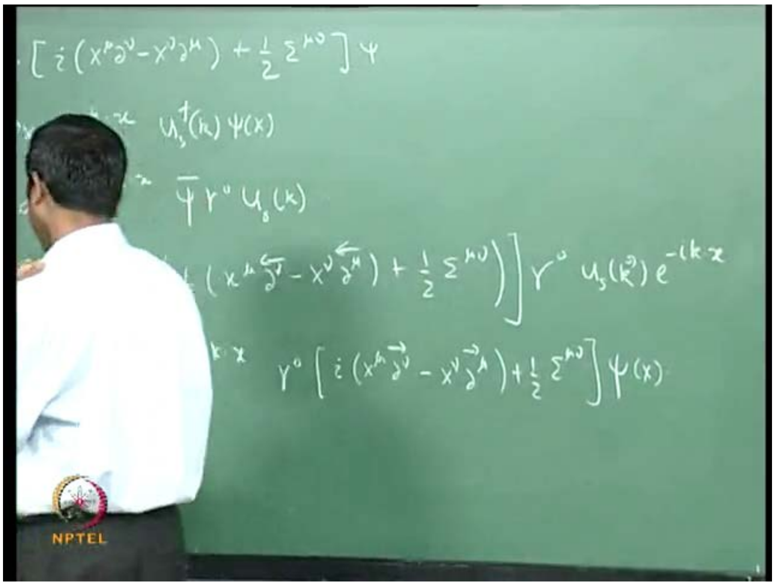

Now, this will act on psi from the left minus x mu del mu plus half sigma mu nu psi of x.

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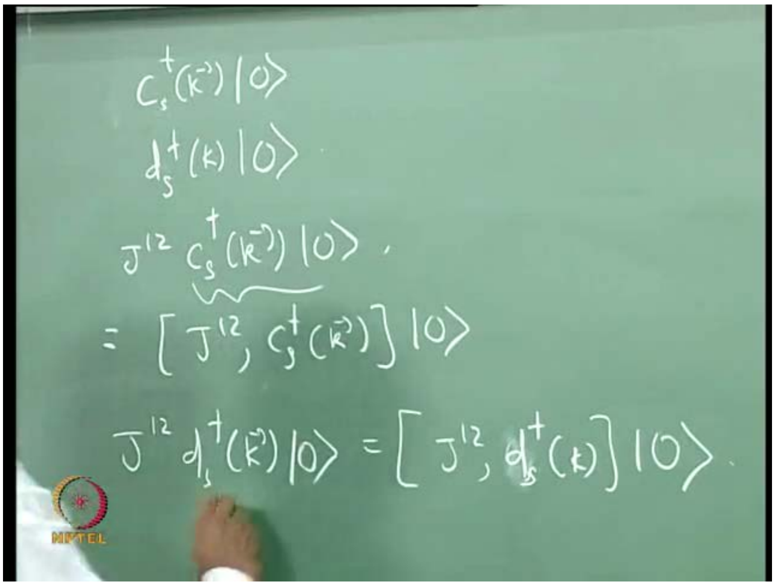

All you are interested in is to find is given this states that is b s dagger of k on vacuum or d s dagger of k acting on vacuum, what are the spins for all this state? So, that is what we are going to derive. Then to do that, we will just use this relationship here. You know that this is J c s dagger of k acting on the vacuum dagger. This vacuum has a spin 0. It is a state with spin is 0 state. This is 0 1 to acting on the vacuum state is 0. Therefore, this quantity here J 12 acting on c s dagger of k is nothing but the commutator of J 12 c s dagger of k acting on the vacuum.

Then, I can use this commutation relation here. This is c s, according to our notation, this is c s dagger. So, we can use this expression here and then you can act on this quantity on the ground state. Then we can derive what is the Eigen value. What is this 12 Eigen value of this state? Similarly, we can do a similar exercise for these states. So, I have J 12 d s dagger acting on k 0 is basically given by the commutator of J 12 c d s dagger k acting on 0. So, this relation here is going to give me the J 12 Eigen value of these states. So, to do that, let us consider what is known as the Pauli lubanski operator.

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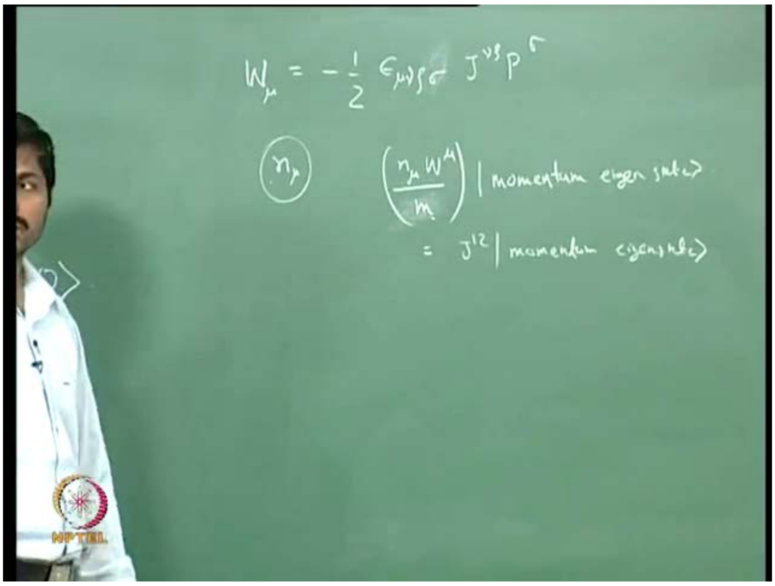

It is given by W mu; this is minus of half epsilon mu nu epsilon m mu rho sigma j mu rho p sigma because of the anti symmetrization here and because of the presence of the p sigma. What this operator will do is that it will project out the orbital angular momentum part. Then it will only care about this spin of any given state. Let us see this in more detail, especially we will see that if we have a unit vector, not a unit vector, but properly normalized space like vector this I will call as n mu.

So, what we will do is that we will that define a vector n mu and then we will show that n mu W mu divided by k divided by m acting on any Eigen state of momentum is equivalent to J 12. So, if you have a momentum Eigen state is same as J 12 acting on the state. So, our goal is to full one thing is this operation and some momentum Eigen state is same is J 12 acting on momentum Eigen state. That is number 1. Number 2 is we will have and will show that this in this expression, the orbital part the angular momentum of just drops out. So, the only thing you get here is the spin of the state so that that gives the spin of the state.

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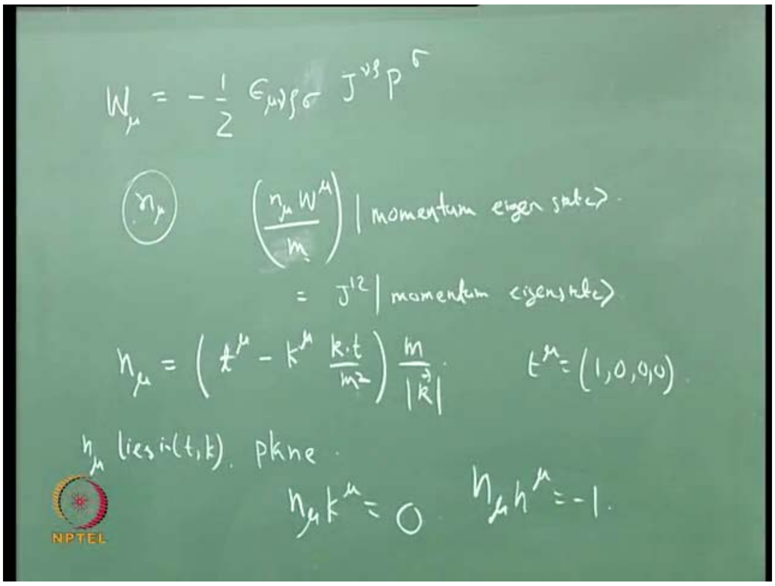

Let us define this vector n mu, which is a t mu minus k mu k dot t divided by m m divided by more k. So, the property of this vector is that it is of course this vector lies in the t k plane and where t is unit vector along the time axis t mu equal to 1, 0, 0, 0. So, this is orthogonal to k. So, this is vector in the t k plane and m mu lies in t k plane and it is an orthogonal to the vector k mu n mu k mu. You see that is actually 0. If you multiplied here by k mu you get t mu k mu here, so t dot k here k mu k mu, which is m square.

So, this m square cancels with m square and then you have dot minus k dot minus t, which is 0. So, this vector here is orthogonal to k so that you can normalize first that n mu n mu equal to minus 1. Therefore, it is a space like vector and it is this following property. Now, what is n mu k mu divide n mu w mu divided by m acting on a momentum Eigen state, let us try to see what do we get?

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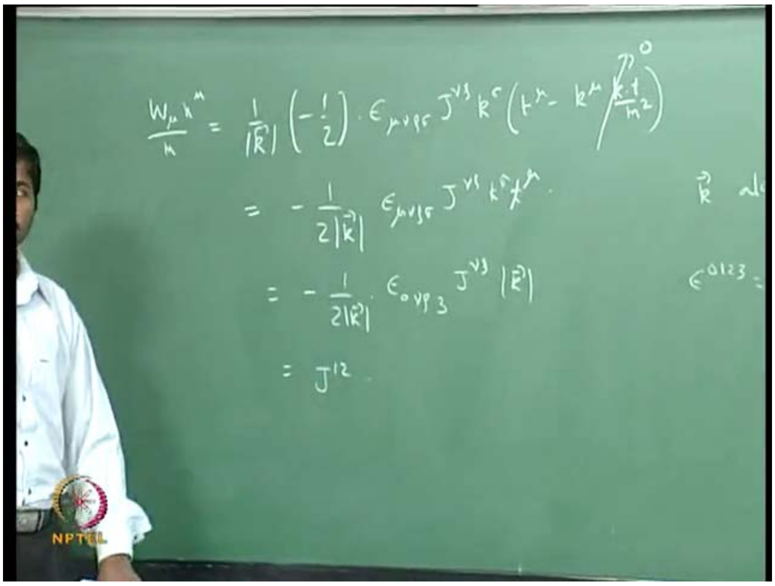

So, W mu n mu divided by m is nothing but 1 over mod k and then minus half from the definition of W mu. Then you have epsilon m mu rho sigma j mu rho. If we consider the operation of this is of this operator or momentum Eigen states, then I will simply get k sigma for p sigma where k sigma is momentum Eigen value this times t mu minus k mu k dot t divided by m divided by m square. But, it does not matter. As you can see this k mu, you can see here that we will simply drop out because of the anti symmetry between mu, nu and sigma.

So, this is done with some term. The second term here simply goes away. What is left here is minus 1 over 2 mod k and epsilon mu rho sigma J mu rho k sigma k mu. So, what is this quantity here? t mu is the unit vector along the t direction. So, this forces mu to 0. Then suppose we will choose k to be along the z axis, k along the z axis. So, the direction of propagation of particle is the along the z axis. Then sigma will take the value 3. So, what you have here is simply minus 1 over 2 mod k and epsilon 0 mu rho 3 J mu over k 3 is simply mod k because k is along 3 direction and t 0 is simply 1. Therefore, it is what it is and this will give the factor of 2 k.

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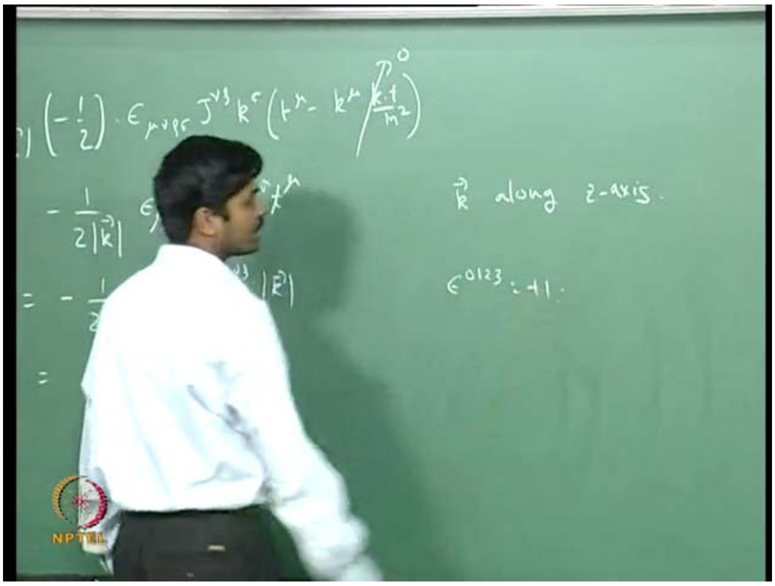

Then, I will use the notation were epsilon 0 1 2 3 is plus 1. So, with this convention, this is simply given by J 12. Therefore, this operator acting on momentum Eigen state same is equivalent to J 12 acting on momentum Eigen state. From the definition here, you can see that this orbital part will simply go away because of the presence of the of p mu.

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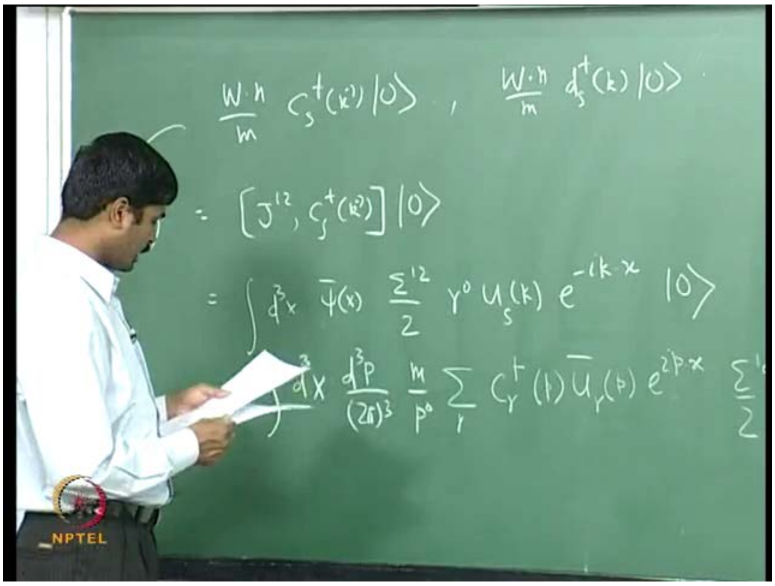

So, what we are interested now is we are just interested to see what we get when I consider this action on c s dagger k on a momentum Eigen state like this and also the action of operator w dot m divided by m on d s dagger k acting on the vacuum. So, consider the first. First this is nothing but it is just because as you have argued, this is the commutator of J 12 and c s dagger of k acting on the ground state. In the beginning of this lecture, we have already given this expression for J 12 and the commutator J 12 with c s dagger of k.

If I use that, then what will we get is i d cube x psi bar of x sigma 12 divided by 2. I do not care about the orbital angular moment part because you have seen this W dot n does not carry because of the construction there. So, it does not carry the orbital part. So, I will just write here sigma 12 gamma 0 u s of k e to power of minus i k dot x acting on the ground states. Is this clear? So, now we can substitute for the expression for psi bar of x.

So, when I substitute this or we will get this d cube x n psi bar x will have an integration over d cube p divided by 2 psi cube m over p 0 and then sum over r c r dagger of p u r bar p e to the power i k dot x. So, this is for psi bar. Then now I have all these things sigma 12 divided by 2 gamma 0 u s k e to the power of minus i k dot x acting on the ground state.

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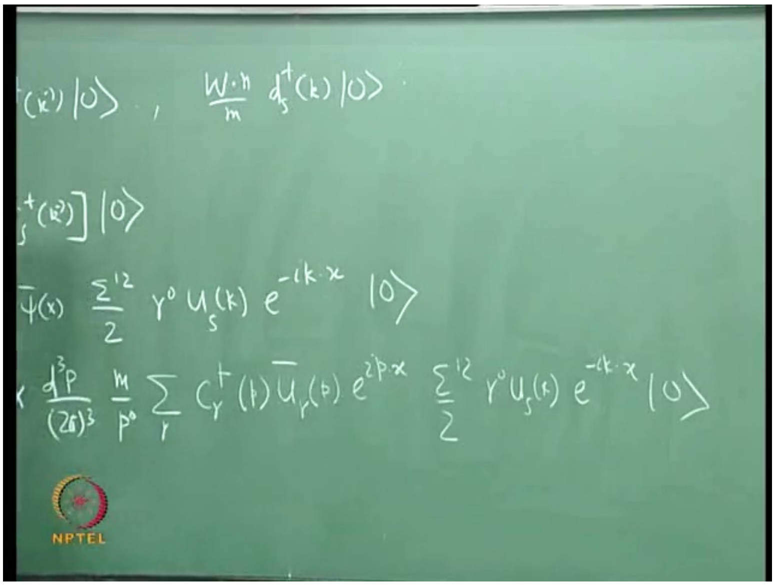

Now, what you can do is that you can see that you can carry out the p integration.

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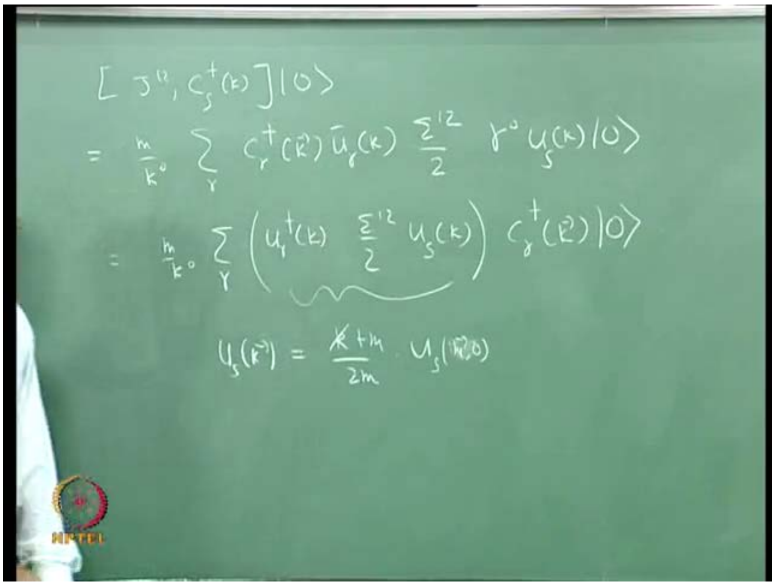

When you carry out the p integration r naught, you will be left with is m over k 0 sum over r c r dagger of k u r bar k and sigma 12 divided by 2 gamma j mu u s of k acting on the ground state. Now, this gamma 0 just compute with sigma 12 because sigma 12 involves product of gamma 1 gamma 2 sigma 12 is gamma 1 gamma 2 commutation after sum constant vector. Hence, sigma 12 gamma 0 is equal to 0.

So, this simply gives me m over k 0 sum over r this is u r dagger k sigma 12 over 2 u s of k and then c r dagger k acting on the ground states. So, what we will like to do is we like to evaluate this quantity inside the parentheses. We can now use an explicit spinor basis for this u s. We have already seen that u s k is nothing but k slash plus m divided by 2 m times u s 0 m u s m 0. That is what we have stated and then we know explicitly what the expression is for u s m 0.

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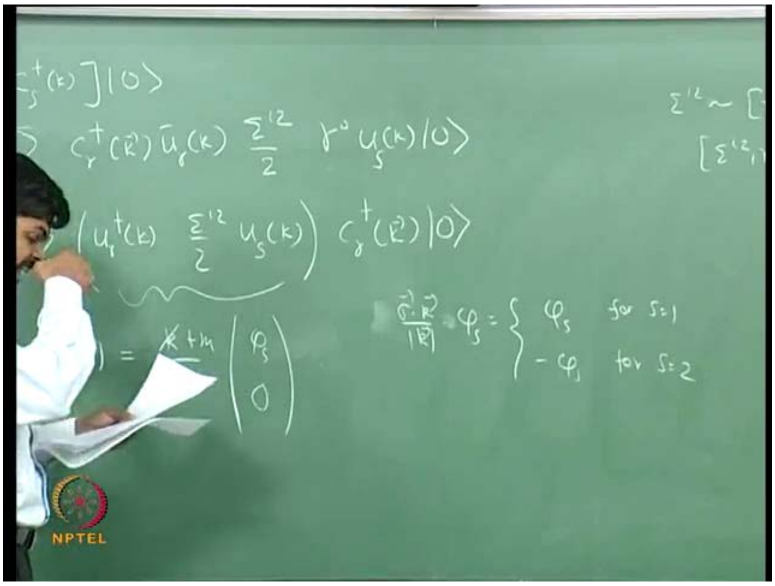

So, this u s m 0 as the form is last two components at 0 and then the first two components, I will call them as psi. So, this is what our u s k of 0 is. Then this phi s has this property that for s equal to 1, it is a sigma dot k. It is a spin off plus half. So, it is psi s equal to phi s for s equal to 1 and minus phi for s equal to 2. So, this is our choice of basis. What we will do is that we will substitute it in this equation. Then we will evaluate the quantity inside the bracket.

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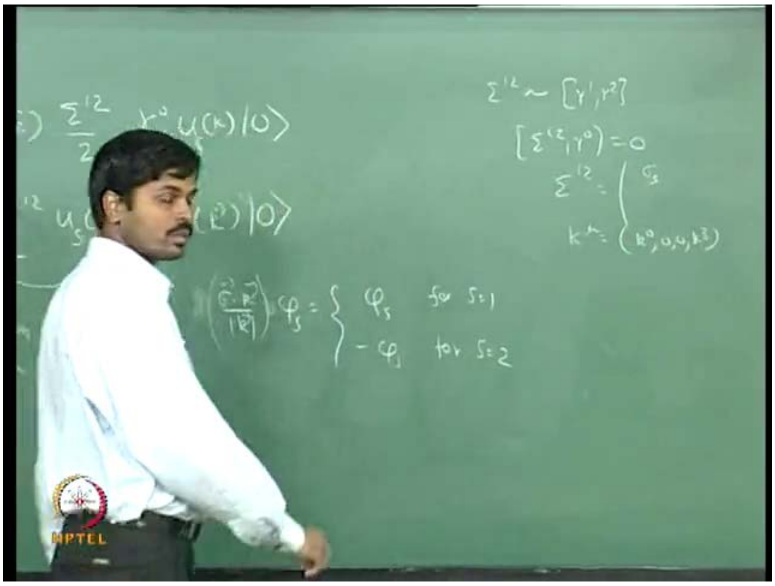

You can see that this when you substitute for u s this quantity here, we have a basis where k mu equal to k, 0, 0, k 3 the momentum is directed along the three axis. Therefore, this sigma 12 will pass through the operator and then it will act on this phi.

So, this action of sigma 12, if you remember the representation is for gamma 1 gamma 2 that we have used, and then this action is simply equivalent to this action on phi the first. So, argue is that you just consider this action and then you will see that here what you will get is just nothing but sigma dot k divided by mod k or what you will get is sigma 3 here because k is along the free direction. So, the result of this act, this acting on u s simply is equal to plus or minus of for s equal to 1 or 2.

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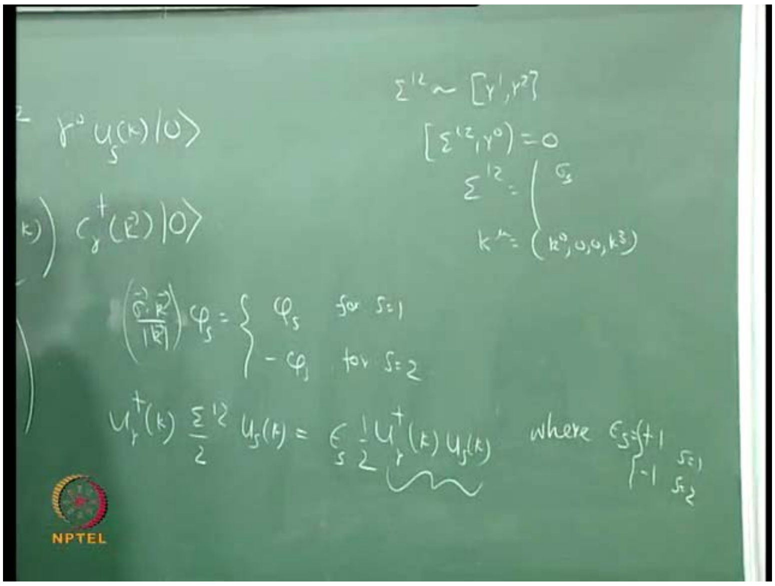

So, all that you will get here is this quantity u r dagger of k sigma 12 over 2 u s of k is equal to this. If I introduce notation epsilon s, this is nothing but epsilon s times u r dagger k u s k where epsilon s equal to plus 1, for s equal to 1 and minus 1 for s equal to 2. We have worked out the normalization factor here. Your dagger k u s k is nothing but k 0 over m times delta r s.

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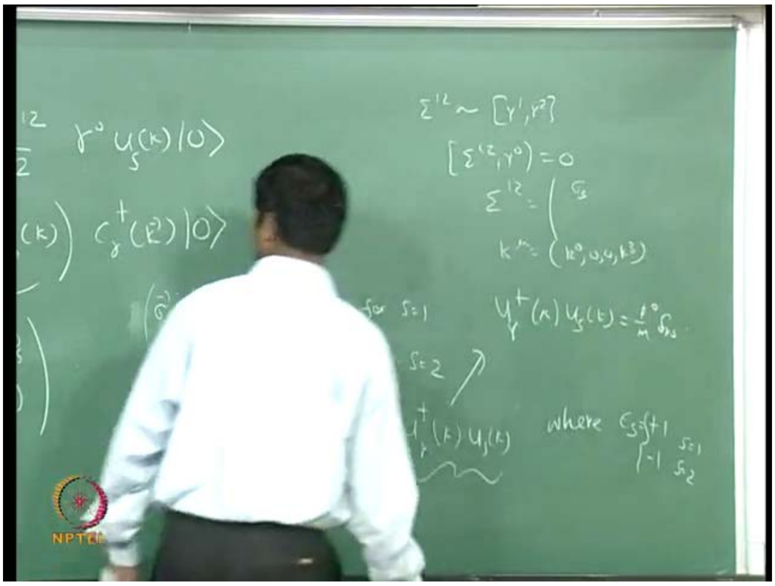

In one of the previous lecture, we have shown that u r dagger k u s k is k 0 over m delta r s. So, if I substitute this here, then what you get is this quantity. This is nothing but half epsilons r s.

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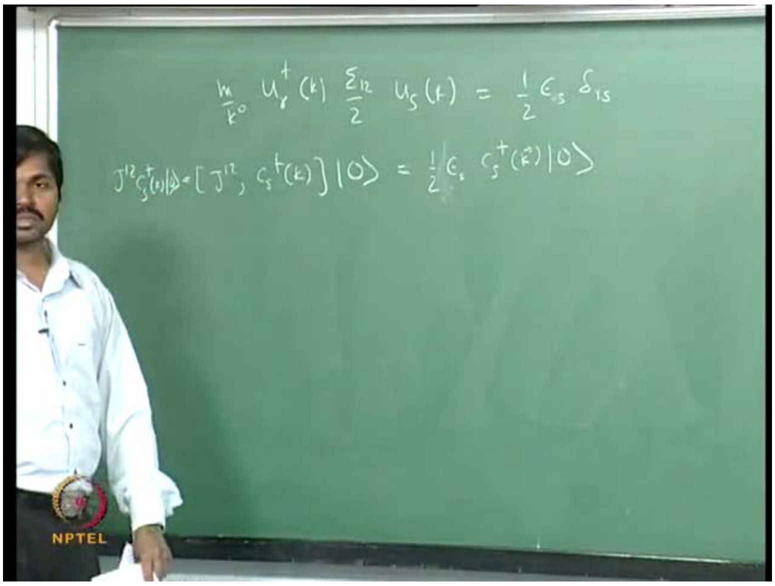

Let us write it here m over k 0 u r daggers k sigma 12 over 2 u s of k is nothing but half epsilon s delta r s. Here epsilon is already defined. Therefore, this quantity J 12 c s dagger of k acting on the ground state is given by half epsilon s c s dagger k acting on this. So, this is nothing but J 12 acting on c s dagger k ground state.

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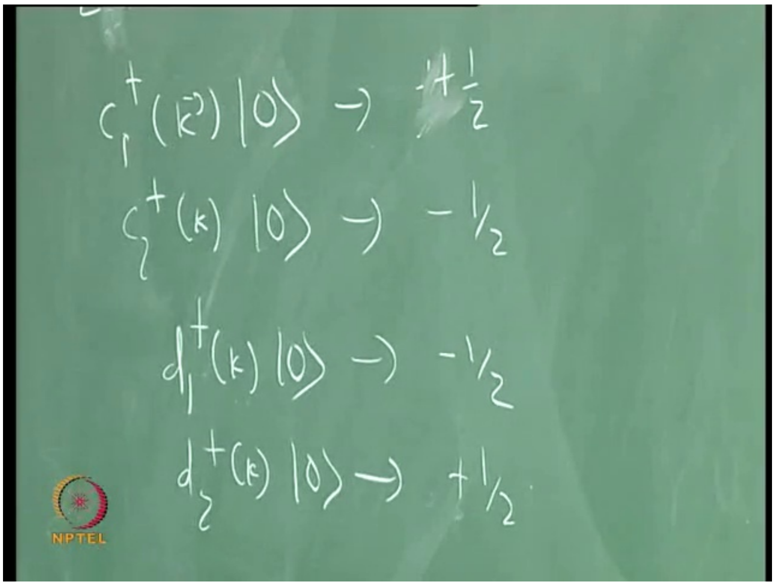

Therefore, c 1 dagger k is helicity plus half positive helicity state and c 2 dagger k acting on the ground state is helicity minus half. We can work in similar way for these two states, d 1 dagger k acting on the ground state. This will have helicity minus half and d 2 dagger k acting on the ground state will have helicity plus half.

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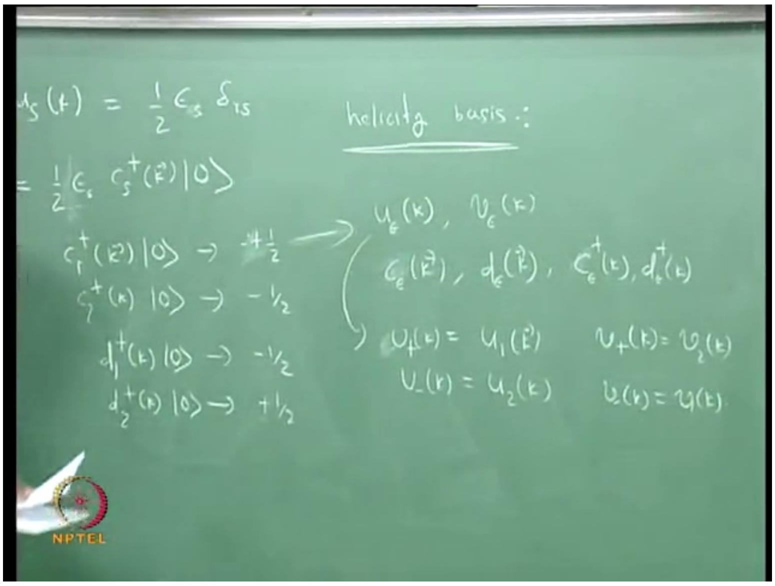

What we will do is that we will introduce, what we will call is helicity basis. This is just for convenience so that we will use it. One is discuss conjugation and time diversion. So, we will say that u epsilons k and v epsilon k instead of you think r s is induced. What you will do is that will use this epsilon and then epsilons will be plus 1 for positive helicity state and epsilon is minus 1 for negative helicity state, so this state.

For example, correspondingly we will use operator c epsilon of k and d epsilon k and their elements conjugate, which have c epsilon dagger k and d epsilon dagger k. Here, t denotes this state. So, in our notation u plus of k is nothing but u 1 k because this is the state with helicity plus half and u minus k is nothing but t u 2 of k where this is we plus of k is v to of k and v minus of k is v 1 k. This is just for a convenience that that we are introducing this notation.

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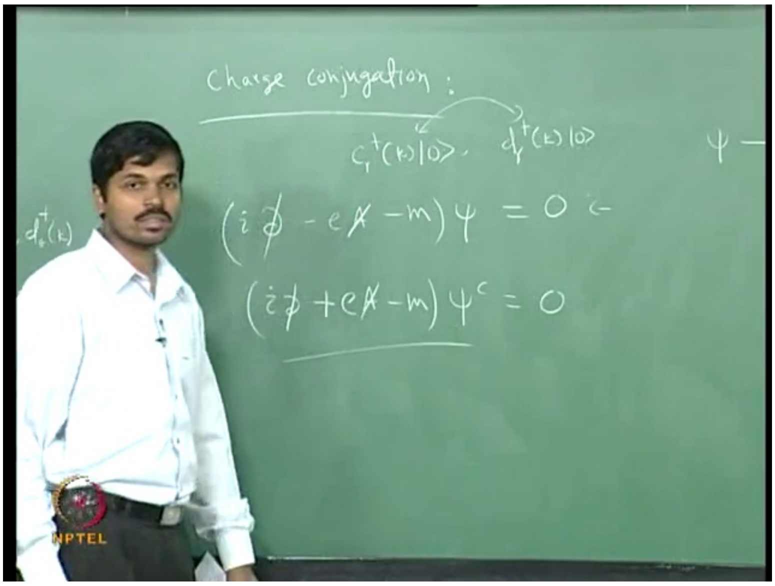

Now, let us consider charge conjugation. We have seen. We already discussed in the last lecture that in this vacuum, there are two types of particles, one with the positive charge and the other with the negative charge. So, what we have argued in the last lecture is that there must exist symmetric with relates this positively and negatively charged particles. Let us consider the example of electric charge. You consider electrons interacting with electromagnetic field.

In this case, the charges are electric charge and particles which c r dagger k acting on the ground states. They will give particles of charge plus 1, whereas d dagger k acting on the ground state gives particle of charge minus 1. So, what we will like to see is how these two particles are related. It is best to consider the equation of motion in presence of electromagnetic field. Then we can understand the charge conjugation better.

So, let us consider. They derive field interacting with electromagnetic field in presence of electromagnetic field derive equation is modify so that that is minus e A slash minus m acting on psi equal to 0. Let us say that charge conjugation take this field psi to sum field, which I have denoted psi c.

Then, what will be the equation of motion for psi c? So, charge conjugate field will be given by an equation, which is given by i del slash plus e A slash minus m acting on psi c equal to 0. So, we should start from this equation. Then we should do some operation here so that we will get an equation, which looks like this. That will tell you how psi and psi c are related with each other.

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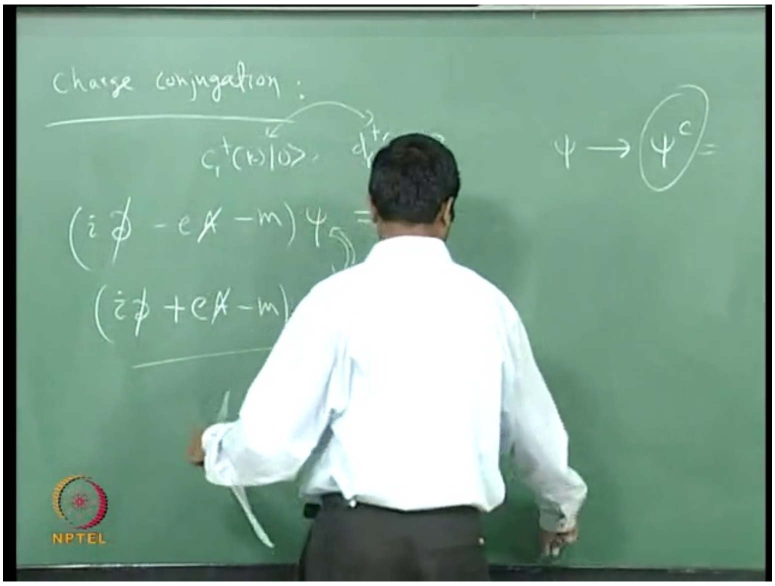

So, our goal is to find psi c, so to find the expression for the field psi c, it will be psi c so that these two equations are consistent with each other. So, let us start with this equation here.

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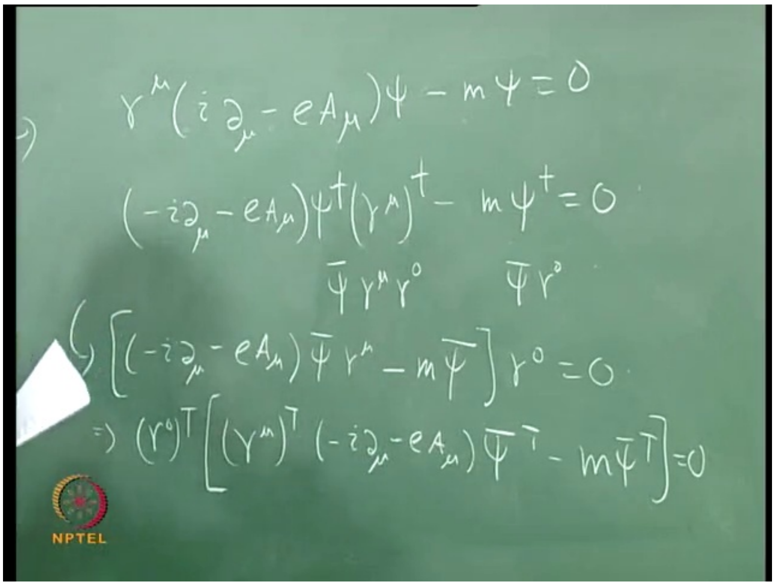

This equation I will rewrite it as gamma mu times i del mu minus e A mu psi minus m psi equal to 0 here. There is a relative sign between these two terms and the other hand equation; they are at the same sign. So, this I can elicit at list the sign. They will have the same sign if I do with fermion conjugate. So, let us consider this equation. Let us take the fermion conjugate of this equation here. So, what is the fermion conjugate of this equation? It is just minus i del mu minus e A mu acting on psi dagger gamma mu dagger minus m psi dagger is equal to 0.

What is gamma mu dagger? Gamma mu dagger is gamma 0 gamma mu gamma 0. So, I can write the whole thing interest of psi bar. It will look like psi bar gamma mu gamma 0 here. This one will look like psi bar gamma 0. Therefore, this equation here is minus i del mu minus e A mu psi bar gamma mu minus m psi bar gamma 0 equal to 0. So, I will take the transpose of this equation. So, this transpose of this equation will be gamma 0 transpose and then here gamma mu transpose minus i del mu minus e A mu psi bar transpose minus m psi bar transpose is equal to 0. So, I have not done anything fancy. I just took to this equation. I took fermion conjugate. Then I just transpose this equation. This is what I got.

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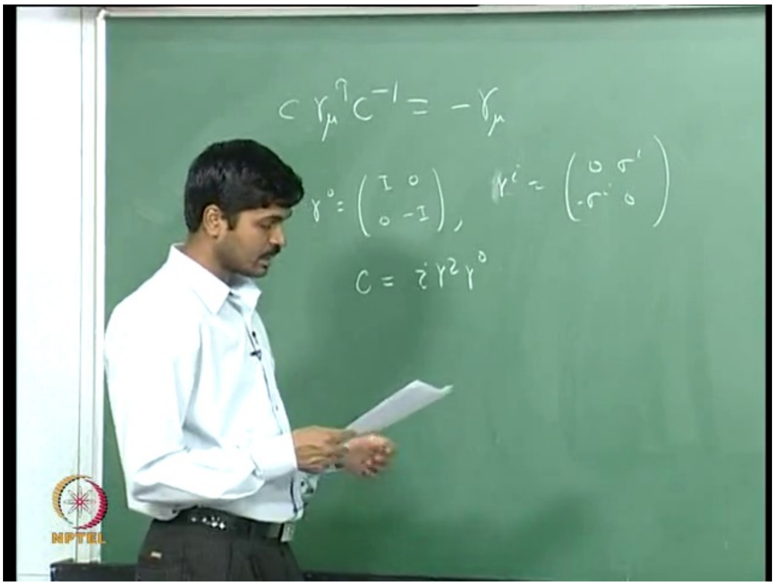

Now, you can see, if I take explicitly that in any representation, there always exists symmetric, which I will denote c gamma mu transpose c inverse is minus gamma mu and the usual representation that we are working within this representation, our c, in this representation where gamma 0 is identity 0 0 minus identity when gamma i is 0 sigma i minus sigma i 0, in this representation, c is i gamma 2 gamma 0. You can explicitly check it and verify that this relation is in fact true.

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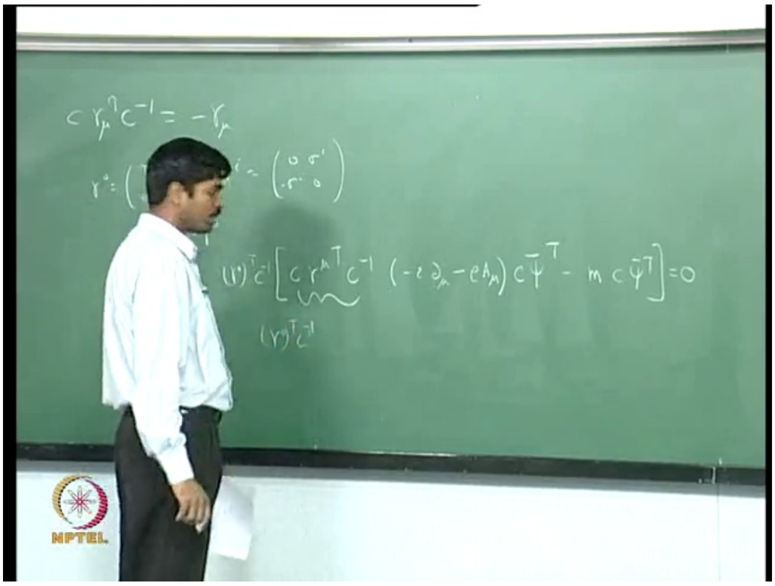

So, this simply means that I can take this equation here gamma 0 transpose and then c inverse c gamma mu transpose c inverse and then minus psi del mu minus e A mu c psi bar transpose minus m c psi bar transpose is equal to 0. Is this correct? So, what I did is that I have introduced an identity operator here, which is c inverse c and then c this goes through. Therefore, there c here, there is a c here, again here. I have introduced identity operator, which is c inverse c. So, as a result, I got c gamma mu transpose c inverse, which is nothing but minus gamma mu.

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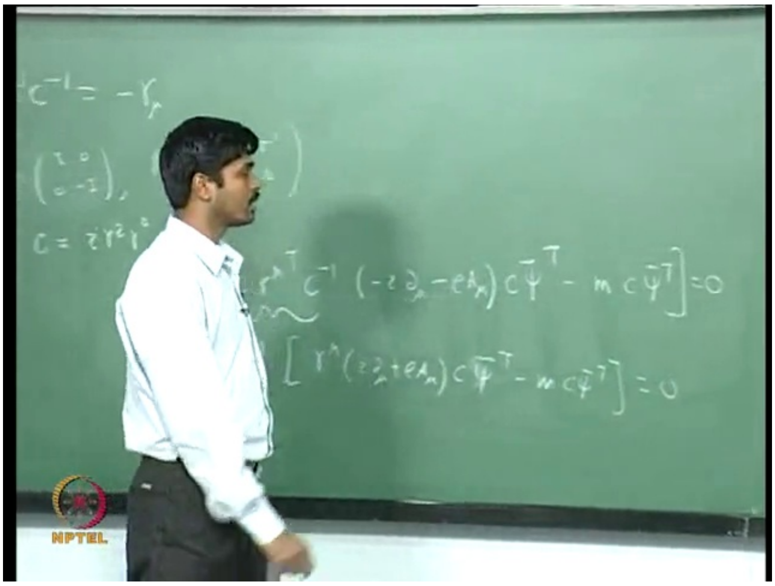

Therefore, this equation can be rewritten as gamma 0 transpose c inverse times gamma mu i del mu plus e A mu c psi transpose psi bar transpose minus m c psi bar transpose is equal 0.

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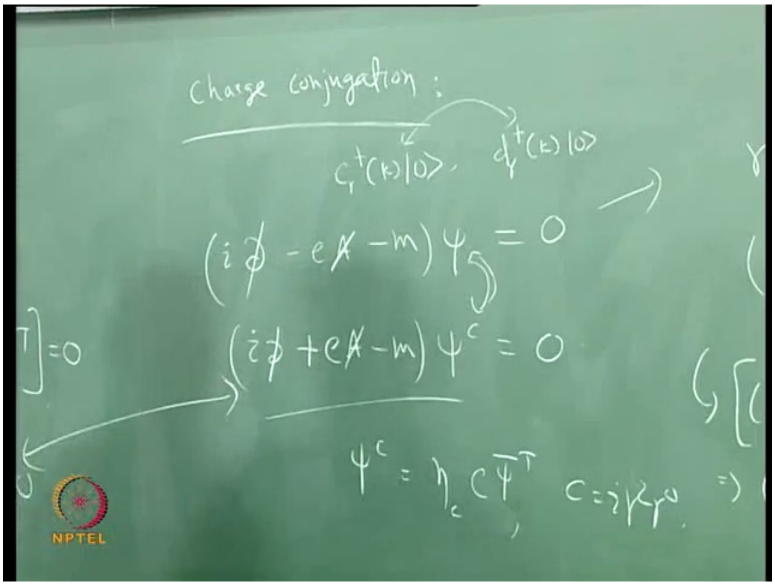

Now, you see that this equation is just identical to this equation here provided we can identify psi c with some, up to some phase factor as c. This is equal to c psi bar transpose where c is equal to i gamma 2 gamma 0 then impact to get this equation. So, this operation, therefore what we have concluded is that the charge conjugation operator operation takes the Dirac field psi to psi field, which is given in terms of psi by this relation here. Now, of course, our job is very straightforward. What we can do is that we can consider the mode expansion for the field psi in the elicit basis.

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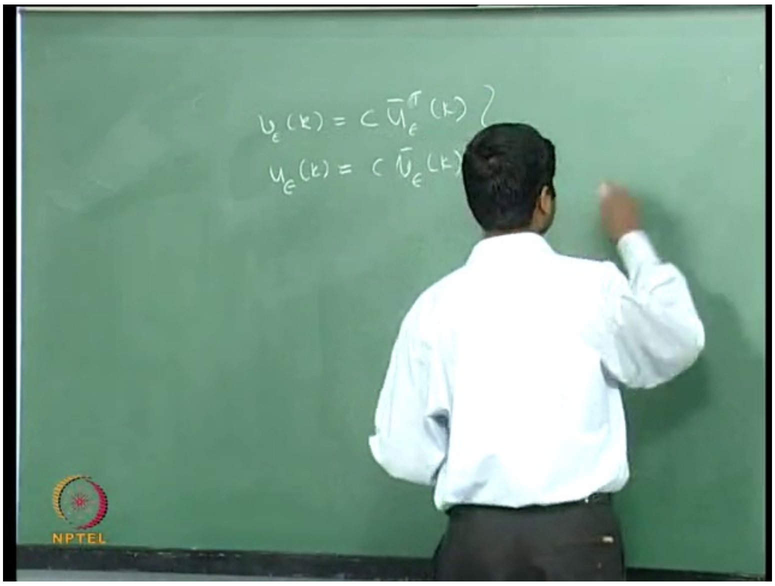

You can show that especially the v epsilon k is nothing but c u bar f transpose epsilon k and u epsilon k is c v bar epsilon k.

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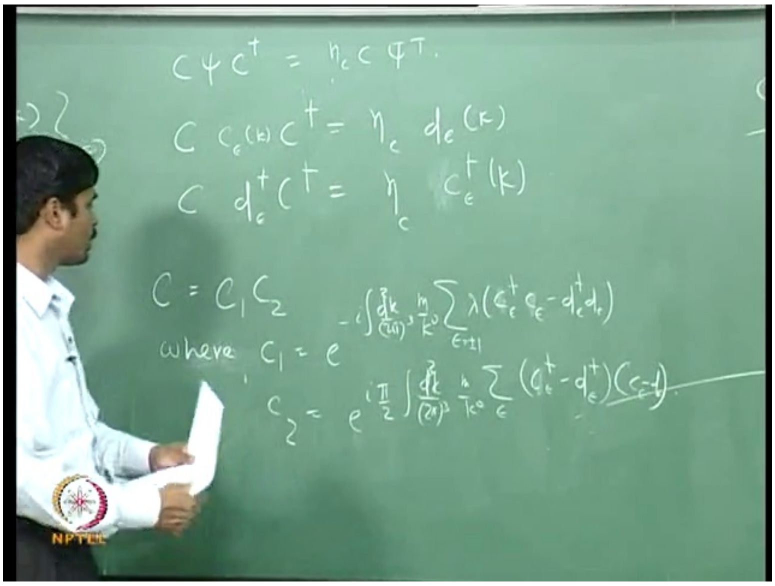

Then, this will imply that the operators c epsilon k and d epsilons are k related in the following manner, c c dagger is eta c d epsilon k, whereas k d dagger epsilon c dagger is equal to eta c d c dagger epsilon of k. So, as expected, what it does is that, what the charge conjugation does is, all you have do is that you have to consider this c psi c dagger and then you have to relates this with eta c c psi bar transpose.

Then, you compare the coefficients in this equation. You will see that the creation and annihilation operators, c, d, etcetera satisfies this relation. Therefore, when you do this charge conjugation, it takes a particle and to the anti particle. This is what it basically means because this c under charge conjugation gives a d and d under charge conjugation gives here c as expected. You can explicitly write an expression for c in terms of creation and annihilation operator. I will give this expression to you and then I will expect you to verify this.

You can in fact show that this is equal to c 1 c 2 where c 1 is e to the power minus i d cube k over 2 pi cube m over k 0 sum over epsilon which is plus or minus 1 lambda times d epsilon dagger d c epsilon dagger c epsilon minus d epsilon dagger d epsilon, whereas c 2 is equal to e to the power i pi 2 d cube k over to pi cube m over k 0 sum over epsilon d epsilon dagger minus d c epsilon dagger minus d epsilon dagger times c epsilon minus d epsilon. So, this is homework for you. You take this expression for c 1 and c 2, and then you see that this in fact satisfies this equation.