Quantum Field Theory
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Module - 1
Free Field Quantization-Scalar Fields
Lecture - 2
Introduction to Classical Field Theory

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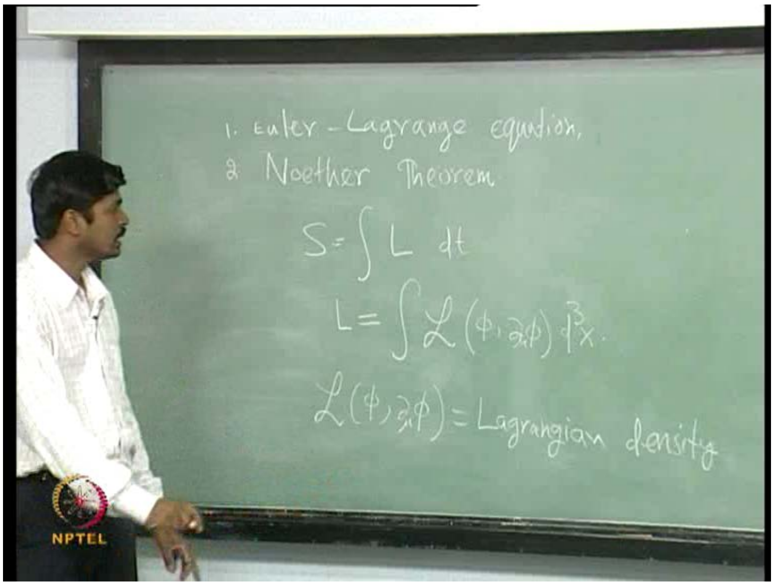

Today we will discuss Euler-Lagrange equations. And then we will discuss symmetric and conservation laws, and now the theorem so on the Euler-Lagrange and Noether theorem. So, just like in mechanics you start with the action which is integration of the Lagrangian with respect to time. But the Lagrangian L here is a taken to be integration of something which is called this is the Lagrangian density over the entire space; here this curly L is known as the Lagrangian density, L of phi Del mu phi is known as the Lagrangian density. And it is a functional of the filed phi and it is derivative.

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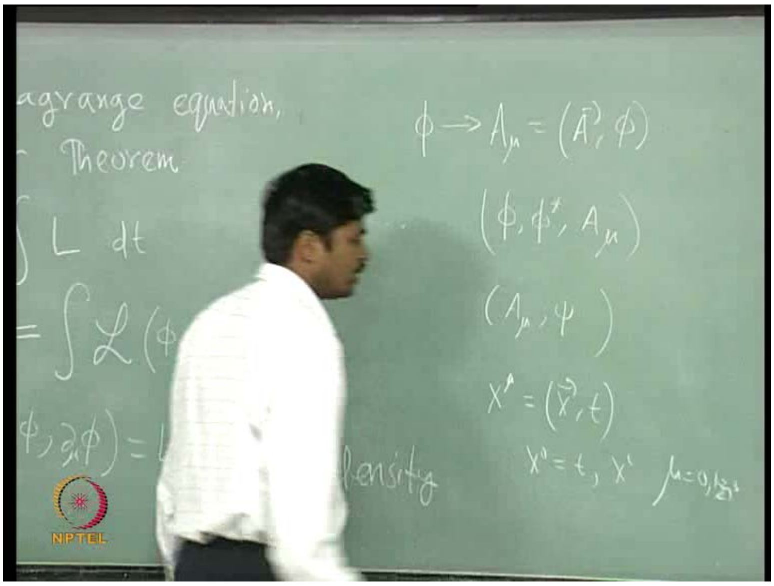

This filled phi here in this case depends what system we are describing; for example, if we consider system of electron magnetic fields then the field phi is a guess field A mu which is nothing but the vector potential A and scalar potential there is also be denoted as phi. If you consider let say electromagnetic field inter acting with phi or some other complex field; then in this case the field is question is a complex scalar field phi it is complex conjugates phi star. And the electromagnetic field A mu; if you consider electron interacting or electron inter acting with electron magnetic field then the field in this question is A mu and psi which represents the field for an electron.

So, depending on the system we have the field phi and it is derivative L phi; in general the Lagrangian density can contains finite number of derivatives our most of the system that we study in physics involve only 1 derivative of the field here. So, I will consider the Lagrangian density to be functional of the field phi. And all this first derivative then phi here I should explain what this notation means; the special coordinate I take them to be x mu to be x. And t where x denotes the special variables and t is the time; x 0 is represented x t and x i at the i components of this x so this index mu here density 0, 1, 2, 3, 4 variables.

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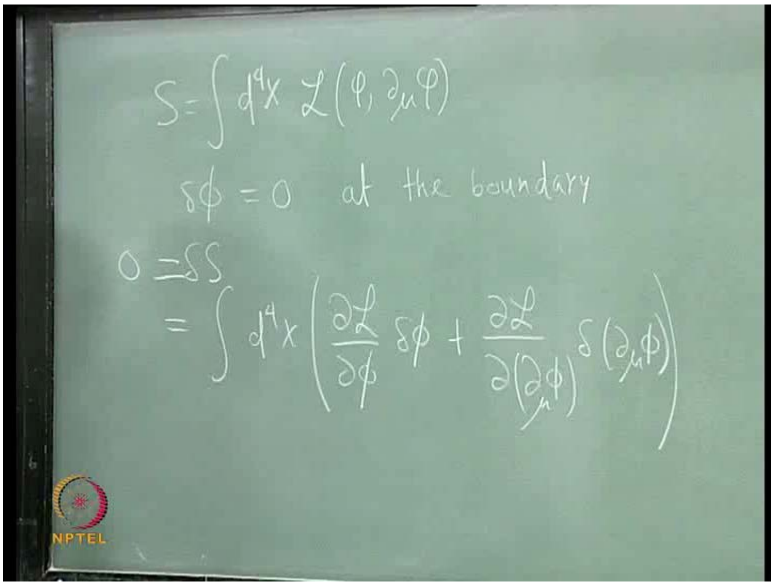

So, I will take this action from this we can see that the action is actually d 4 x times the Lagrangian density of phi and Del mu phi; we will vary this action is by keeping the fields variations of the filed delta phi equal to 0 at the boundary. And this will give me the field equation. So, let us do that. So, delta S by setting delta S to be 0 is equal to integration over d 4 x, del L over del phi plus del L over del of del mu phi delta of del mu phi; at given point delta phi is basically phi prime of x minus prime of x.

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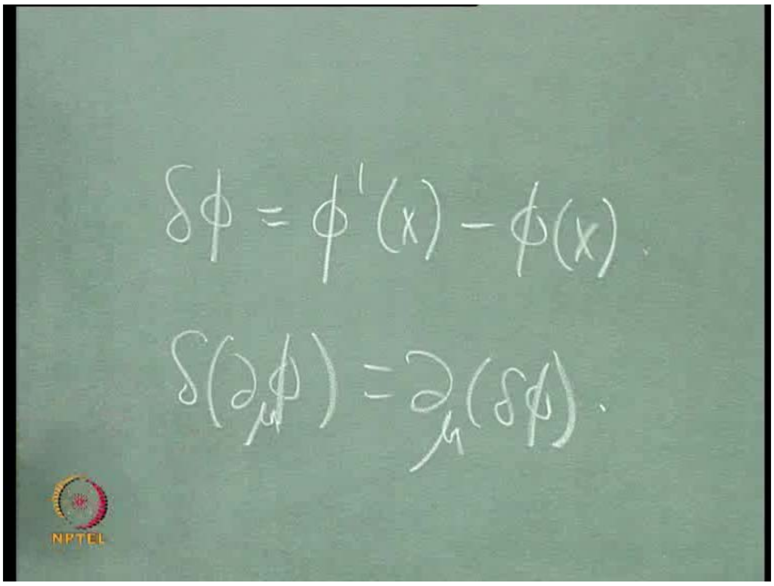

So, as you can see the variation does not depend on same point. So, therefore delta and Del mu can commute. So, I can write delta of delta mu phi is Del mu of delta phi; I will substitute this here and then I will do some rearrangement.

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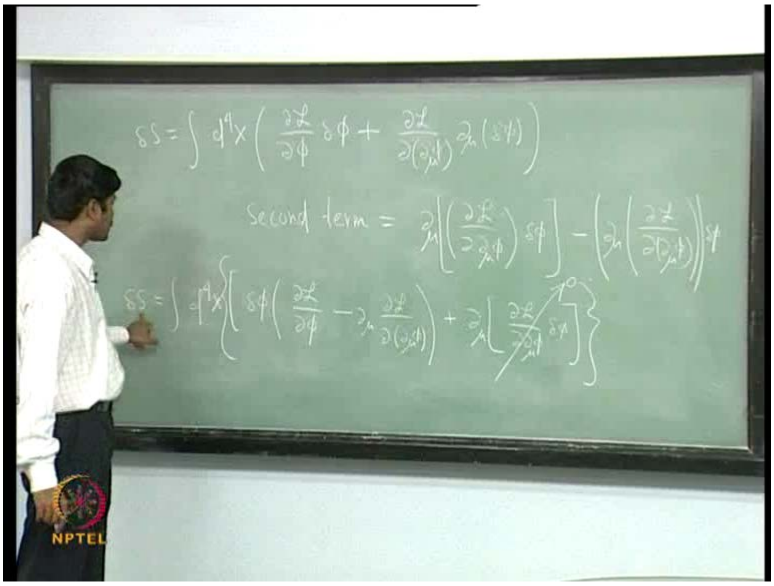

So, that will give me delta of s equal to integration d 4 x; Del L over L phi delta phi plus Del L over Del of del mu phi Del mu of delta phi. Now, what I will do is I will take this second term and then I will write it as a total derivative minus some quantity; as you can see the second term can be written as del mu of del L over del, del mu phi times delta phi minus del mu of del L over del, del mu phi times delta phi. I will substitute this here and when I substitute what I get is delta S is equal to integration d 4 x; I will collect the term which come with the factor of delta phi together. And then I will write it is delta phi times Del L over Del phi. And this term here minus Del mu of Del L over Del mu phi; then I have this term plus del mu of L del L over del, del mu phi times delta phi.

Now, you can see the second term is total derivative. So, we can use gauss diversion theorem and then we will get and integration at the boundary; however, we have we require that the variation of the field at the boundary vanishes. So, therefore the contribution of this total derivative term is 0. And hence the variation of action actually depends on some term which is multiplied by delta phi delta phi is arbitrary is infinite this variations. And hence delta S will be 0; if and only if the quantity here in the bracket is 0. Hence, by varying the action we got the Euler-Lagrangian equation which is this quantity inside the bracket equal to 0.

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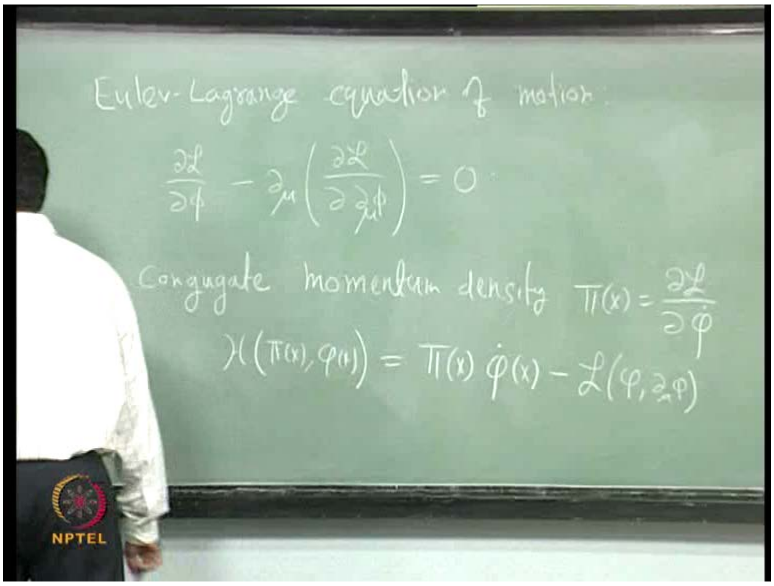

So, the Euler-Lagrange equation of the motion is given by Del L over Del phi minus Del mu Del L over Del mu phi is equal to 0. Now, we also need to introduce conjugate momentum and Hamiltonian in order to quantize; we will quantize the canonical way just the way you quantize in non relativistic quantum mechanics, you consider the coordinate variable. Then you consider the corresponding conjugate momentum; you start with the Kevin Gordon and relations and quantize the theory this will do exactly in the same way. And hence we need to introduce the conjugate momentum correspond to the field phi.

So, the conjugate in this case; what you will have is the momentum density. So, conjugate momentum density phi of (x) is equal to Del L over Del of phi dot. Then the Hamiltonian density H of (x) is define to be phi; note that phi dot here the dot represent derivative with respect to time minus L of (phi) the Hamiltonian density is define to this. Now, what we will do is; we will discuss Noether theorem in somewhat detail. So, Noether theorem states that for every continues symmetry there is a corresponding conserve quantity; we will explain what, when we call a quantity to be conserve as we go on.

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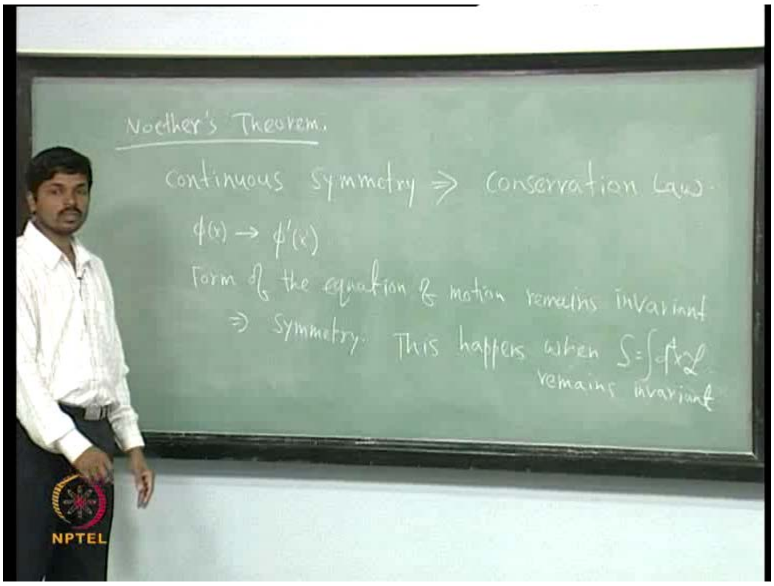

So, let us start with the Noether theorem; the statements is that continuous symmetry implies conservation law when do we say that the symmetry theory? You consider the field phi of (x) suppose you make a transformation the field goes some transform filed phi of (x prime). If the equation of motion has the same for if the form of the equation of motion does not change; then you say that this transformation of the field phi of (x) to phi prime of (x prime) is actually a symmetric of the theory.

So, form of the equation of motion remains in variant implies you have a symmetric; this will happen if the action when S remains in variant and this transformation. So, you make this transformation and then if the action does not change under this transformation then classically you have the symmetry with the theory. In this case what we will discuss is we will discuss continuous symmetries; the symmetry transformations the set of symmetry transformation which are actually parameterize by continuous parameter these are known as the continuous symmetries.

So, what we will see what happens when the system remains in variant under a continuous symmetry. Now, the theorem states that continuous symmetries implies set in quantity remains conserve and this is what we will see in the subsequence discussion. So, what kind of symmetry transformation that we come across in physics, mostly we will discuss relativistic quantum field theory.

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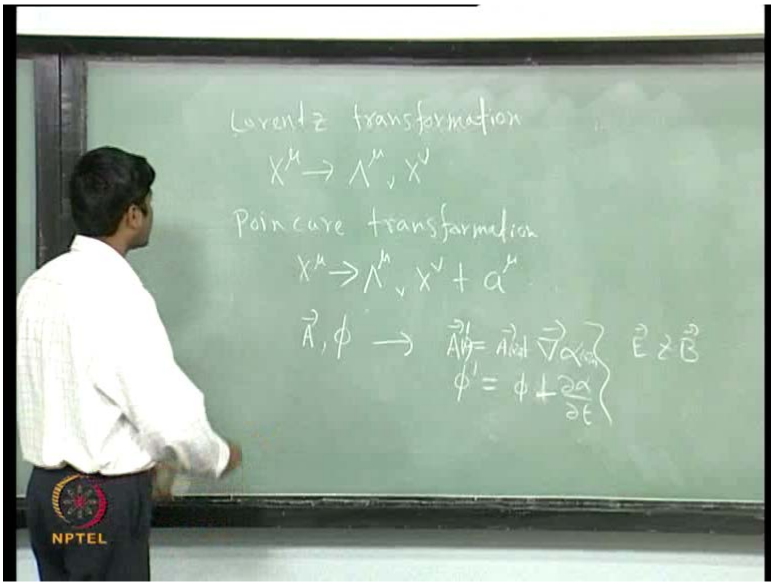

So, we keep seeing Lorentz transformation where the space time coordinates actually change like x mu going to lambda mu nu, x mu this is also known as the homogenous Lorentz transformation; we have in homogenous Lorentz transformation or Poincare transformation oasis this is Lorentz transformation flows a translation. So, x mu under Poincare transformation goes to lambda mu nu, x mu plus some translation A mu; also in electro dynamics you see there are this vector potential A and scalar potential phi. And if you make the transformation such that A goes to A prime; which is A plus gradients of some scalar which I will call as alpha. And if phi goes to phi prime which is phi plus Del alpha over Del t; I am not sure about the minus sign I think this is minus sign here. Then the electric and magnetic field E and B in the mention variant unlike this transformation does not depend on these wave parameters. So, this happens at the same point, at the same special point at same time.

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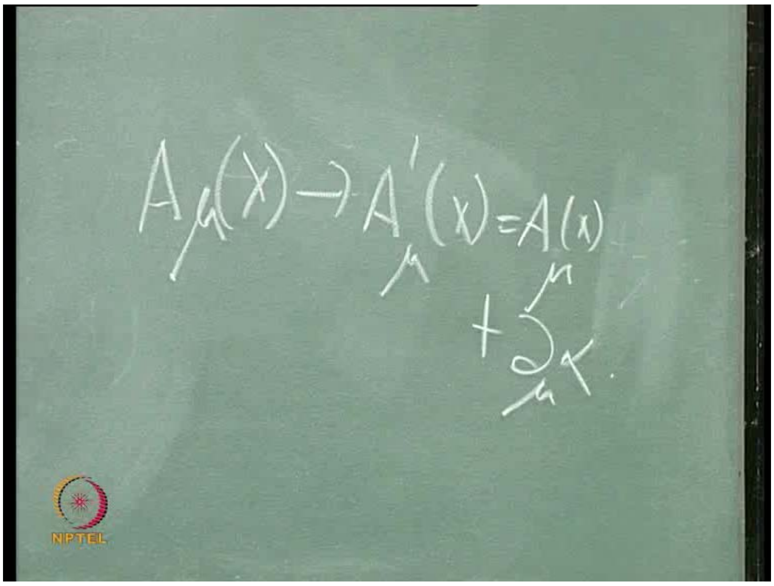

So, x and t, x and t so; this is known as the gauze transformation; in 4 vector notation I can write it as A mu the 4 vector A mu. And under this transformation goes to strictly speaking A mu of (x) f goes to a mu prime of (x) which is A mu of (x) plus del mu alpha plus del mu alpha. So, this happens at the same point in this space time this is an internal symmetry transformation. So, you have Poincare transformation and you also have internal symmetry under this transformation also the equation of motion in electro dimension remains invariant.

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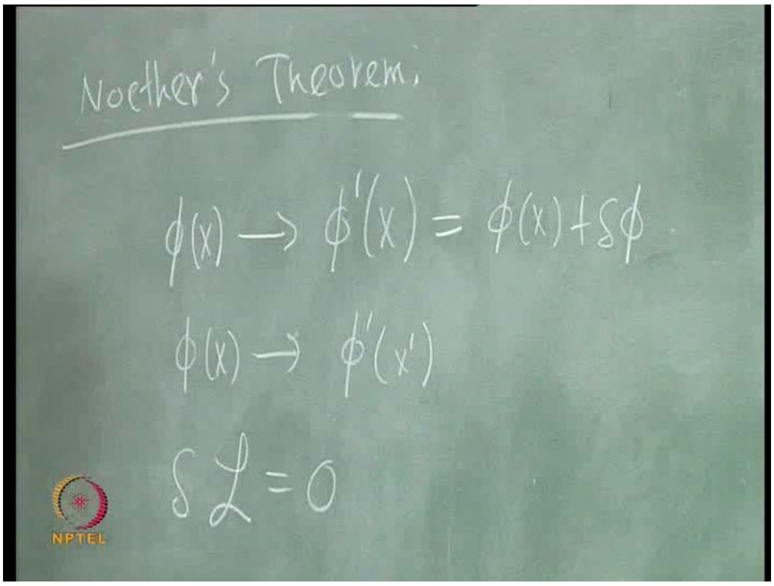

So, there are such transformations; we will discuss what happens under such transformation under internal symmetry transformation phi of (x); the field phi of (x) goes to phi prime of (x) which is equal to phi of (x) plus some delta phi. If I consider infinity transformation then this is what I get more generally the field of (x) will transformed to some phi prime of (x prime) where both the field as well as the arguments says under this transformation. So, this will be combination of some internal symmetry transformation as well as the point transformation; what we will do is that will first consider this second similar case. And then we will see how do we get conservation laws under such a transformation; then we will discuss about the second possibility which is much more general then this one.

So, let us consider the variation here strictly speaking we should require the action to be invariants under such transformation; what we will do is that will restrict our self to the special case where the Lagrangian density remains invariants under such transformations. If the Lagrangian density remains invariant then; obviously, the S in invariant however, the Lagrangian density cans actually saying up total derivative. And if the fields when it sufficiently first infinity then also the action will remain invariant; we will not worry about this second possibility here. So, we will consider the case where the Lagrangian density L remains in variant under such transformation.

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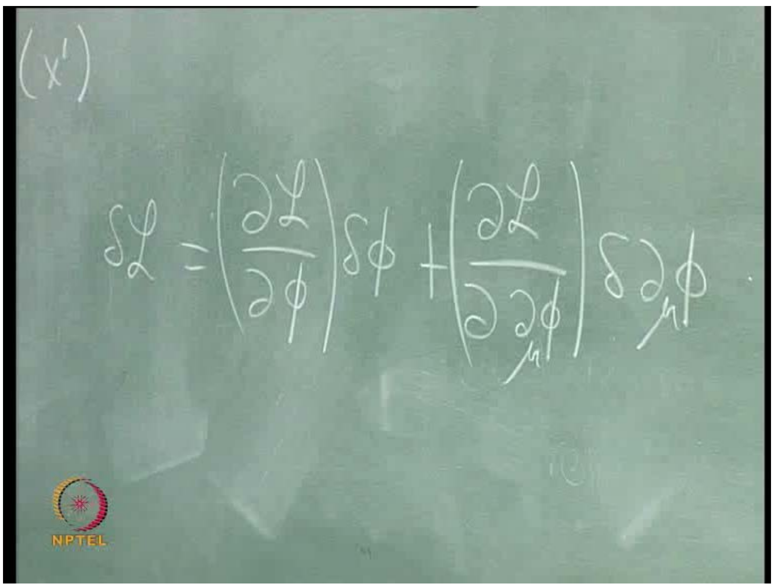

So, delta L is 0 under such transformation what is the delta L? delta L is nothing but del L over del phi times delta phi plus del L over del of del mu phi, delta of del mu phi; we will consider the field configurations which obey the classical field equation, the Euler-Lagrange equations remember here this variation is done at every point in this space time. And we will consider the field consideration with satisfy the Lagrangian equation.

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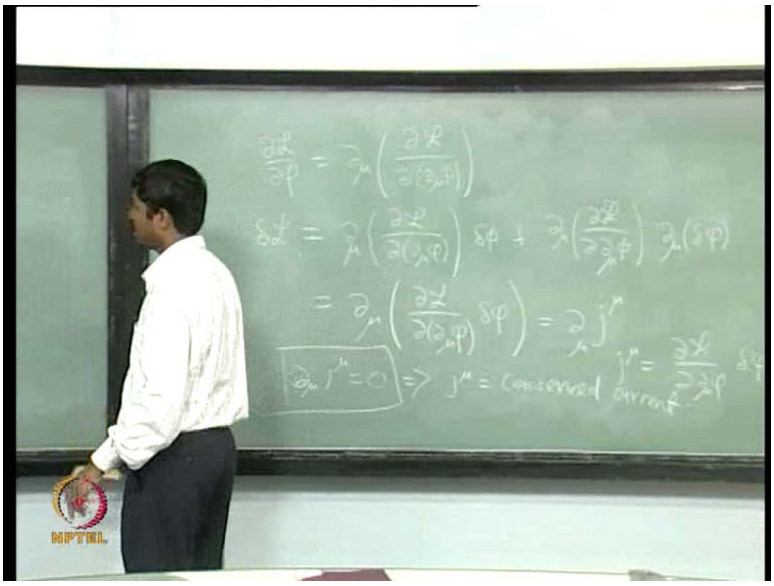

So, we will use the Euler Lagrangian equation to substitute for the first term here; using Euler-Lagrangian equation we find that Del L over Del phi is equal to Del mu of Del L over Del, Del mu phi. So, this will substitute in the expression for delta L to get delta L equal to del mu of del L of del mu phi; delta phi plus del mu of del L over del L mu phi times del mu of delta phi. As you can see again I have exchange delta and Del mu; because this delta transformation here does not depend on x on this space times point x mu. Now, you can see if you combine both these terms you will get a total derivative. So, this quantity is nothing but Del mu of Del L over Del mu phi times delta phi this is of the for Del mu j mu where this 4vector j mu is equal to Del L over Del, Del mu phi times delta phi. So, the invariant of the Lagrangian means that delta L equal to 0.

So, if delta L equal to 0 then Del mu, j mu equal to 0 where j mu is the quantity given by this expression; whenever, this happens you call this quantity j mu to be a conserved current. So, Del mu, j mu equal to 0; implies j mu is a conserve current. So, what we have seen is that if we require the system to be invariant under a continuous internal symmetry transformation. Then we find an equation we get an equation like this provided the field configuration over the equation of motion. And hence we get a quantity, we get a 4 vector which remains conserve; what do you mean by some quantities which remains conserve we will see it in a moment.

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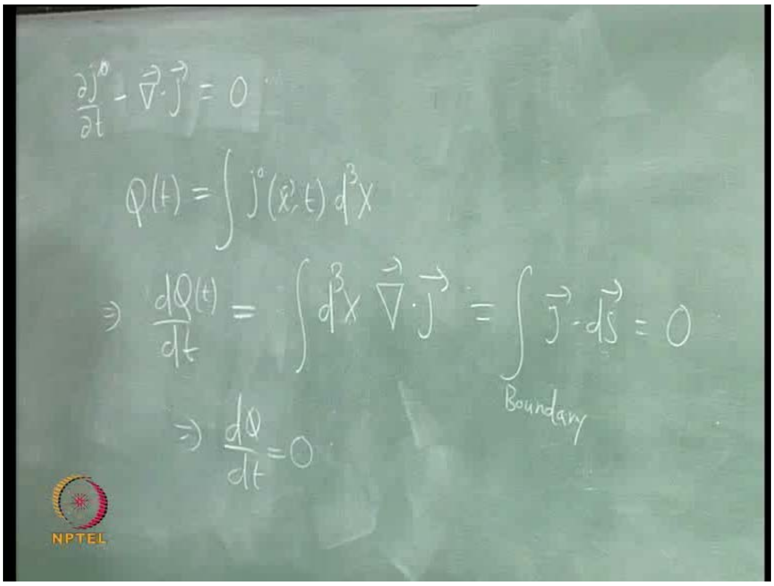

So, let us write this equation in term of components; what we get is Del j mu over Del t plus Del j 0 over Del t plus Del dot j is equal to 0. Now, I will integrate it over the entire space and I define some quaintly which is which I will call as Q of (t); Q of t is define to be integration of j 0 of(x, t) over the entire space d cube x. Then this implies d Q of over d t is equal to minus integration of I think there is the minus here. So, this will be integration of d cube x Del dot j. And if I use gauze diversions theorem; what I will get is integration of j dot d S at the boundary.

And, if the outgoing flux varnishes or if the fields go to 0 sufficiently fast at infinity then this quantity equal to 0; there for the rate of change of this quantity Q of (t) actually is equal to 0.So, this implies d Q over d t equal to 0. So, this quantity here which I define to the Q remains constant in tiles. So, what we get is the quantity which remains conserve. So, what I get is Del 0, j 0 plus Del i, j i equal to 0 right. And I am using the matrix which is eta 0 ,0 equal to 1, eta i j equal to minus delta i j and a del dot j in the usual sense in this just del i, j i right.

So that will give you minus sign when this becomes a covariant j i will require minus sign and here therefore I will have minus sing. So, what we see is that the rate of stage of which I have A is actually 0 provided the field and field when this sufficiently first infinity. So, a continuous symmetry implies a conservation law this is what we have seen; we can see a very simple system and then we can construct the discharge for this system.

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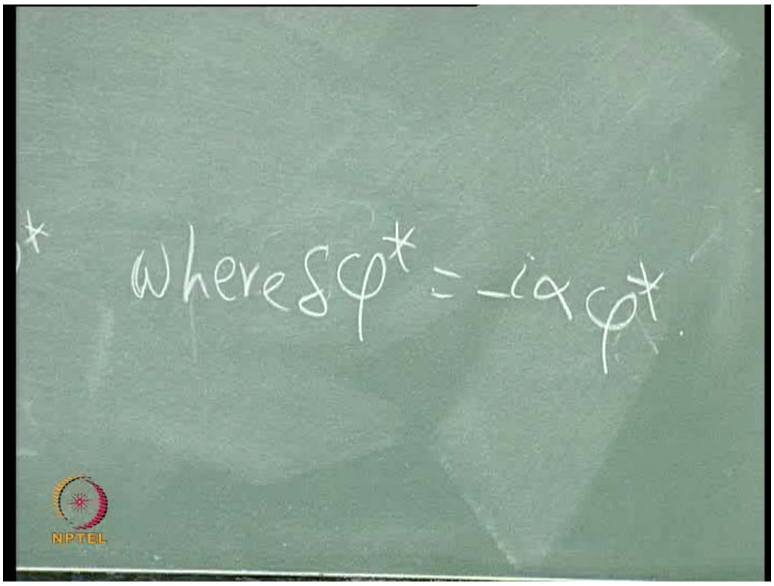

So, let us consider the system of a complex scalar field phi and it is complex conjugate phi star of (x) are linearly independent. And let us consider the Lagrangian to be Lagrangian density L of phi del mu phi to be half nothing half is there del mu phi star del mu phi minus some potential V of phi star phi; this Lagrangian density is in variant under a continuous global transformation that is if phi goes to e to the power i alpha times phi. And hence phi star goes to e to the power minus i alpha time phi star; under such a transformation phi star phi the remains invariant. Therefore, potential term here remains invariant.

And, in the kinetic term alpha also you get the eta to the minus power here from the first term and e to the power plus i alpha from the second term. Because alpha is a constant it does not depend on space time variables and hence the first term also remains invariants. So, such a system is invariants under a continuous global symmetry we will see; what is the corresponding conserve current for such symmetry; we will consider the infinity for this transformation. So, if I consider infinity smile transformation phi goes to phi plus delta phi over delta phi is i alpha times phi. And phi star here goes to phi star plus delta phi star; which is, where phi star is minus i alpha phi star delta phi star is minus i alpha times phi star.

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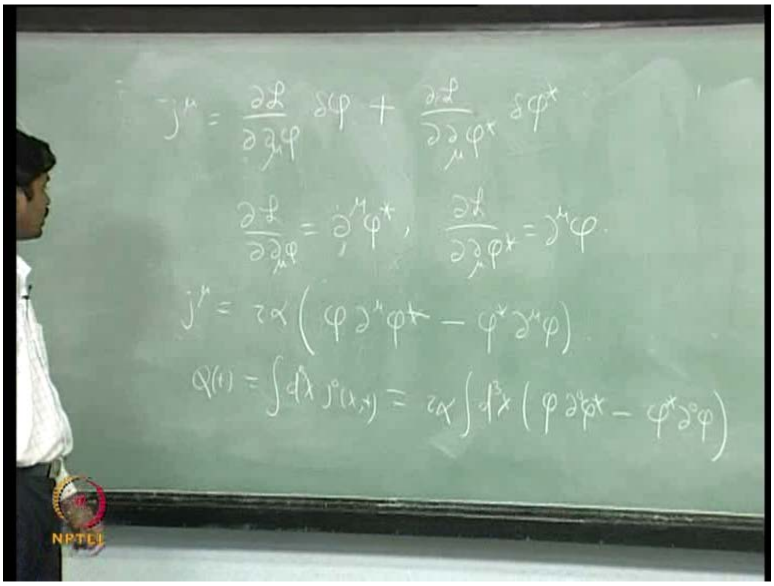

So, you will substitute this in our expression for the conserve current j mu which is given by Del L over Del mu phi time’s delta phi. In this case we have 2 independent field phi and phi there. So, whenever say there are more than 1 field; we have to sum over the field here. So, in this simple system they conserve current is actually this plus Del L over Del, Del mu phi star delta, delta phi star. If there are more than 2 fields than you have to sum over all such fields; from the Lagrangian density you can derive what is del L over del mu phi del L over del mu phi is nothing but del mu phi star and del L over del, del mu phi star is del mu phi.

So, I will substitute this and also the expression for delta phi then what I see is j mu is equal to i alpha times phi Del mu phi star minus phi star Del mu phi. So, for these very simple systems the conserve current have this expression, it is this expression. And the corresponding stars Q of (t) which is integration d Q x j 0 (x, t) is given by i alpha integration d to x phi Del 0 phi star minus phi star Del 0 phi all right. So, this is the quantity which remains concert as a continuous of as a consequence of the continuous local symmetry.

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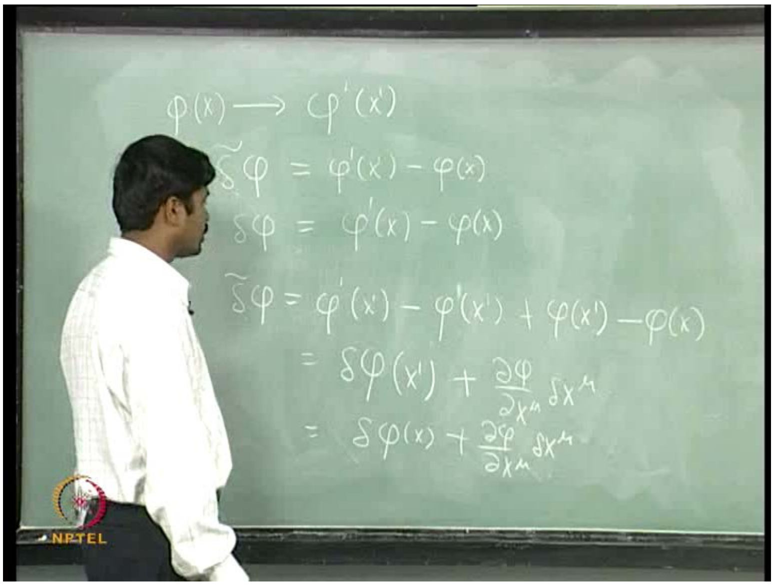

Now, in the remaining part of this lecture; we will consider the more general scenario that is when the field, when the argument of the field. So, phi of (x) goes to phi prime of (x prime) phi prime of we will introduce some variation which I will called as delta till of phi which I will define to be phi prime of x prime minus phi of (x). In order to order to distinguish this from the variation of delta of why I have introduce earlier which is phi prime of (x) minus phi of (x) we need to keep in mind that in this case we have to require a the action to be invariants; it is not sufficient to consider the invariants of the Lagrangian only is the argument of the fields also get is transferred.

Let us rewrite this variation delta till of phi in a way which we will need for later purpose. So, delta till of phi can written as phi prime of (x prime) minus phi prime of minus phi of (x prime) plus phi of (x prime) minus phi of (x). So, what I did is I have added phi of (x prime) and also I have subtracted the same thing this. So, I just I write in rewrite in a tree value A.

However, now you can see that this quantity here is nothing but delta phi of (x prime) the first 2 terms give me delta phi of (x prime). And the last 2 terms gave me for infinity smile transformation this is nothing but Del phi our Del x mu delta x mu right. So, delta till up of i is given by this. But if I keep the transformation to first order in variation then I can rewrite this here H delta phi of (x) plus Del phi our Del x mu delta x mu, because this 2 term here will defer by second order quantities.

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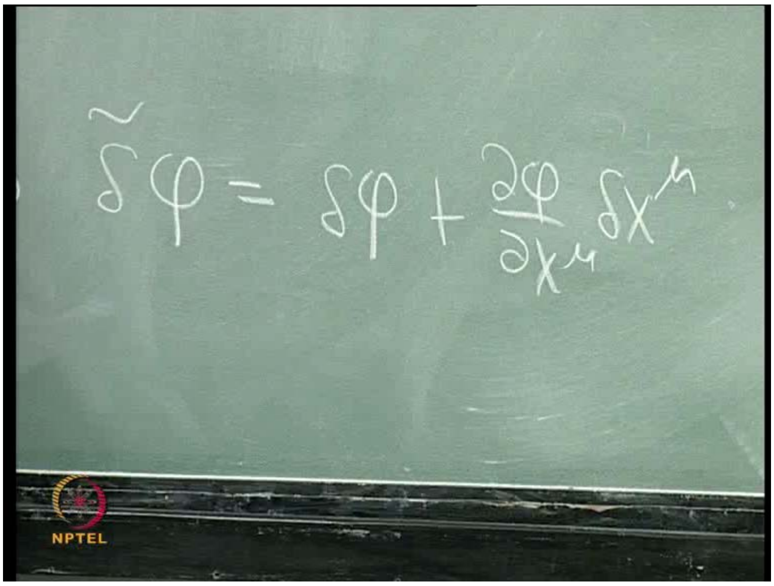

So, what I get here is delta till the phi is equal to delta phi plus Del phi over Del x mu delta x mu. Let we will consider the invariants reaction under this transformation and find the corresponding conserve quantities.

Thank you.