**Quantum Field Theory**

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**Module - 3**
**Free Field quantization Spinor and Vector Fields**
**Lecture - 16**
**Quantization of Electromagnetic Field II**

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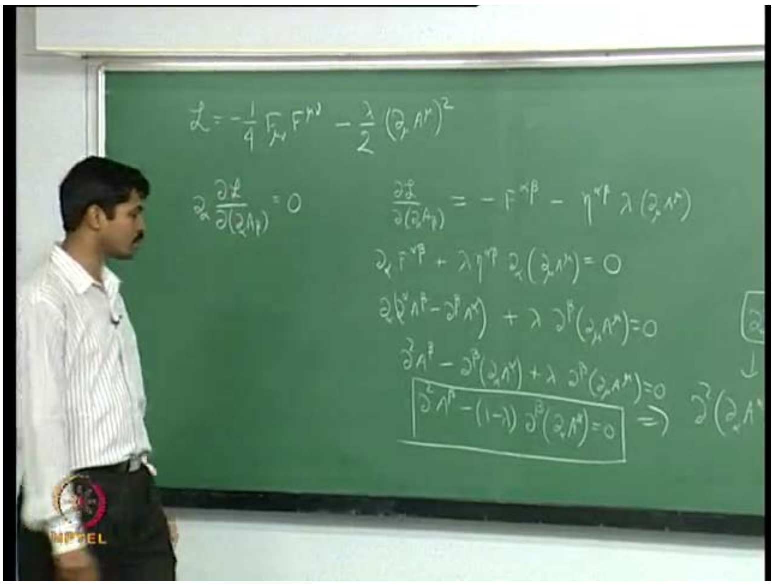

So, we are discussing the covariant quantization, the Lagrangian density is given by f mu nu f mu nu. We have modified this Lagrangian density by A d, which is proportional to del dot a del mu a mu whole square. This is the Lagrangian density we will we consider and then we will quantize this. Of course, this is not as the same as the Maxwell say Lagrangian density, but because of the presence of this additional term, but we will make sure that the additional term does not have any physical effect adding this additional term does not have any physical effect.

So let us so the Feynman gets in this notation correspond to lambda equal to one. Although, you can consider this for any general lambda its we will fix a lambda to be a specific value. We will do quantize you can consider lambda to be any arbitrary number. Then you can quantize it nothing will change. So, yes I could include yesterdays notation it will be lambda equal to half, which is the gauge will we will be working the equation of motion is a given by del alpha A beta del alpha equal to 0.

So, let us compute this quantity del l over del alpha A beta this one will give me A minus f alpha beta. This one will give me A minus eta alpha beta times lambda times del mu a mu. Now, you know why I chosen this vector two here, because I want to do this to cancel, here that is the reason I am choosing to.

So, the equation of motion is just del alpha of f alpha beta minus lambda theta alpha beta del alpha del mu A mu equal to 0 plus. So, let us work out what we get del alpha del alpha a beta minus del beta A alpha plus lambda times, del beta del mu A mu equal to 0. This one is nothing but del square acting on a beta minus del beta acting on del alpha A alpha plus lambda del beta acting on del mu A mu equal to 0. Because, mu is a dummy index here I can change it to alpha.

Then what I see is the equation of motion is basically del square a beta minus 1 minus lambda times del beta del alpha A alpha equal to 0. This is the equation of motion you can see that. This equation here implies del square acting on del alpha A alpha equal to 0. So, del alpha A alpha acts like a free field by suitable boundary condition, by choosing suitable boundary condition.

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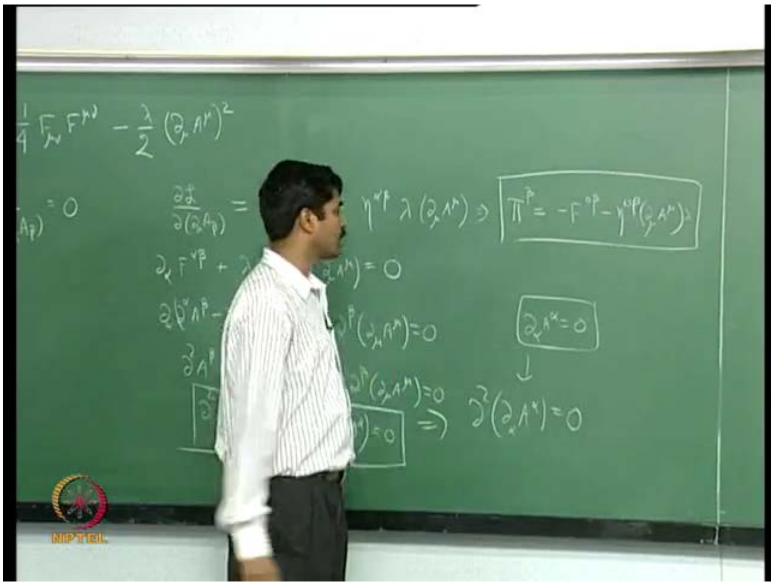

You can set del alpha A alpha to be 0 classically, which is equivalent Lawrence sketch. So, if you choose this Lawrence gauge that del alpha a alpha equal to 0 throughout this space time. Then this is equivalent to saying that you actually restrict yourselves to the

Maxwell Lagrangian, but this is all classically, quantum mechanically it is A not substitute forward.

We will see what the this equation here simply implies that the conjugate momentum to the field A beta, which I will denote by pi beta is given by minus f 0 beta minus eta 0 beta times del mu A mu with a factor of lambda. So, this is the conjugate momentum to the field A beta you see that pi 0 specially is no longer zero. So, as such you they not have any difficulty in considering the equal time commutation relations, which are given by A mu of x t pi nu of Y t is basically i delta mu nu delta x minus y.

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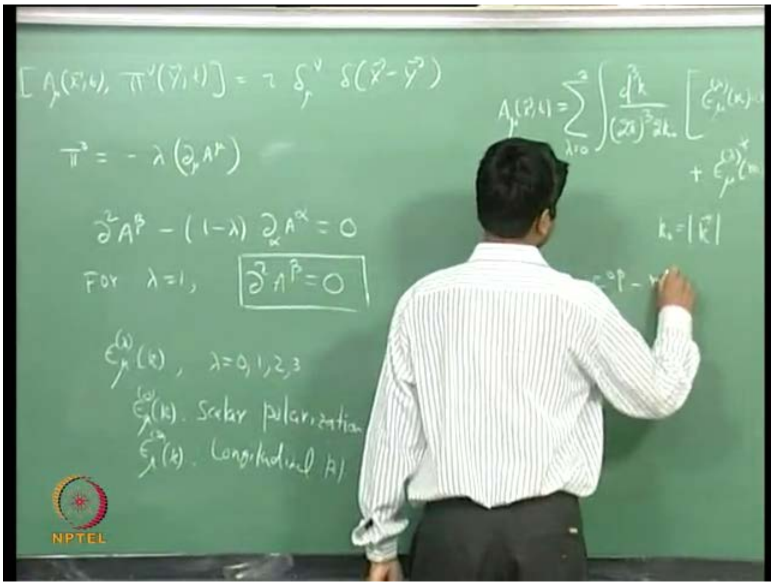


So, as an operator equation you can consider this equal time commutation relation, but you can see that pi 0 from this equation is basically f 0 is 0. So, pi 0 is simply minus lambda times del mu A mu. So, you cannot set del mu A mu equal to 0 as an operator equation quantum mechanically. It will be inconsistent to just set this to be 0, because if you set this to be 0. Then the equal time commutation relation you will have for sum this terms you will have the left hand side 0, but the right hand side is not 0.

So, this is going to pose a problem that we will see how to solve this. Therefore, as an operator equation we cannot impose the condition that del mu A mu equal to 0. So, we will see how actually we will what is the physical condition that we have to impose. So, that we restrict ourselves to the Maxwell's theory. That is what we will discuss in a moment, but before this we will just a carry on with A what we have learnt. So, then we will start quantizing.

So, the equation of motion is del square A beta minus one minus lambda del alpha A alpha equal to 0. As I say this although we can consider this lambda to be any arbitrary parameter. It will simplify much if you if we choose this parameter lambda to be 1, because for when we restrict ourselves to the ourselves to the choice lambda equal to 1. Then the equation of motion becomes del square a beta equal to 0 and we know how to handle this equation. So, we will we will solve this equation.

This restriction of this parameter lambda equal to one is known as the Feynman gauge. So, we work in the Feynman gauge for, which the equation of motion is simply del square acting on a beta equal to 0. So, this almost looks like again the claim Jordan mass less claim Jordan equation for each of these fields A beta. So, we know how to solve this equation exactly, there will be four linearly independent solution. Now, because we are no longer doing any kind of gauge fixing here, therefore here A beta will go like some epsilon beta of k e to the power minus i k dot x.

So, let us lets parameterize this four linearly independent polarization to be epsilon lambda mu of k lambda varies from 0, 1, 2, 3; just like in the previous case. When we have done when we had performed the gauge fixing, we had A 2 linearly independent polarization. We had two transverse polarizations and those we denoted by epsilon r of k with r equal to 1 and 2. Here, there will be four linearly independent polarizations two of them are transverse polarizations. The remaining two are known as the longitudinal polarization, as well as the scalar polarization. So, epsilon 0 mu of k is known as the scalar polarization and epsilon three mu of k is the longitudinal polarization. The most general solution for the field A can be written as A mu of x t is again in terms of the four real components.

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You will have sum over lambda as where is 0 to 3 and integration over d cube k over 2 pi cube to k 0 times epsilon lambda mu of k A lambda k e to the power minus i k dot x plus epsilon lambda star mu of k A lambda dagger of k e to the power i k dot x. Here, k 0 in this integration is given by mode k. Because, this is a mass less field this is the most general solution.

Now, we are considering this a mu to be a quantum field, therefore these operators, now we can look at the canonical commutation relations. Then we can see what to the mean in terms of these operators A and A dagger. So, what in this gauge lambda equal to 1 the conjugate momentum pi beta is given by minus f 0 beta minus eta 0 beta del mu A mu. You can see that this implies A dot mu of x t A nu of y t the commutator of A dot and A is given by i eta mu nu delta of x minus y. Whereas, A dot mu of x t A nu of y t A mu dot of y t is equal to 0, similarly A mu of x t a nu of y t equal to 0.

So, this is the set of equal time commutation relations, we can substitute this here. Then we can see what do we get in for the creation for the operators A and A dagger before to do that we will make a choice for the polarizations. First of all we will assume that the epsilons are real.

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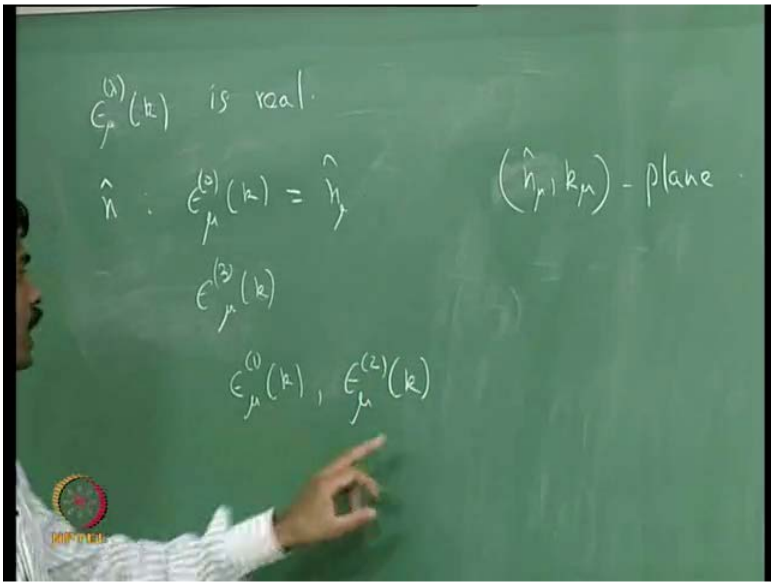


So, epsilon mu lambda of k is real and we will choose these polarizations such that. Suppose, n hat is unit vector along the time direction, then we will make our choice such that epsilon epsilon 0 mu is equal to n hat mu. Then we will choose epsilon three mu of k to be the vector, which lies in the t k x 0 plane or in the n mu hat k mu. This A plane and this is one linearly independent vector in this plane spin by n hat and k mu. You choose epsilon three to be the unit vector, which is orthogonal to n hat mu, which lies in this plane. So, that is the choice for epsilon 3, epsilon 1 and epsilon 2 are two linearly independent vectors, which are orthogonal to each other, which are orthogonal to this plane

I mean let say your time is along x 0 direction and let us assume that k is along x 3 direction. Then these four linearly independent vectors are simply epsilon 0 is equal to 1 0 0 0 epsilon 1 is a 0 1 0 0 epsilon 2 to be 0 0 1 0 and epsilon 3 to be 0 0 0 1. I mean not exactly k mu, but you just look at A. I mean you can this is just one choice that I have given you need not restrict yourselves to four linearly independent.

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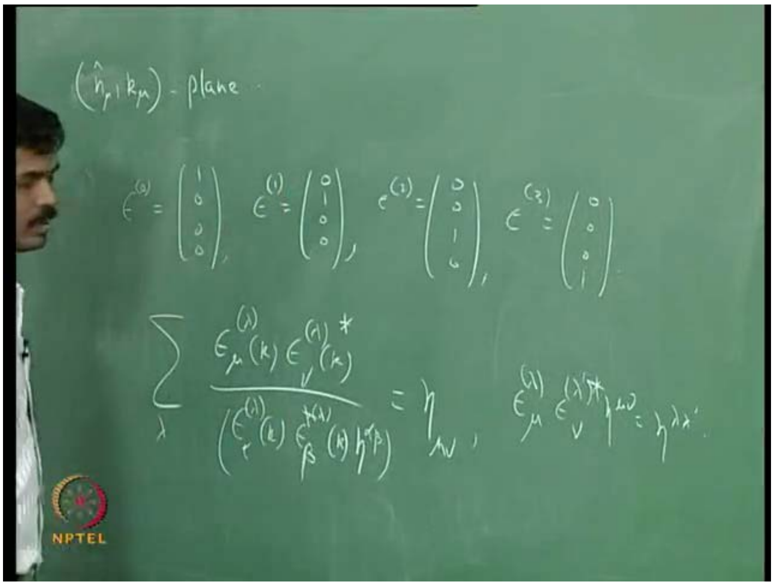


In general they with satisfy this equation that is A given by sum over lambda epsilon lambda mu of k epsilon lambda nu of k star. You do not have to choose them to be real and epsilon alpha lambda k epsilon beta lambda k eta alpha beta. This is eta mu nu and epsilon lambda mu epsilon lambda prime mu eta mu nu is eta lambda lambda prime with A star.

So, with this choice for the polarizations we can look at the equal time commutation relations. You can simply substitute the mode expansions in these equations. Then you can show the I mean this is almost identical to the free scalar field case, except for the presence of these vectors here the polarization vectors. Now, we are making this particular choice for the polarizations. Then you can substitute them and you can show that the operators a and a daggers satisfy the following relation, a lambda of k a lambda prime dagger of k prime is given by minus 2 pi cube 2 k 0 eta lambda lambda prime delta of k minus k prime.

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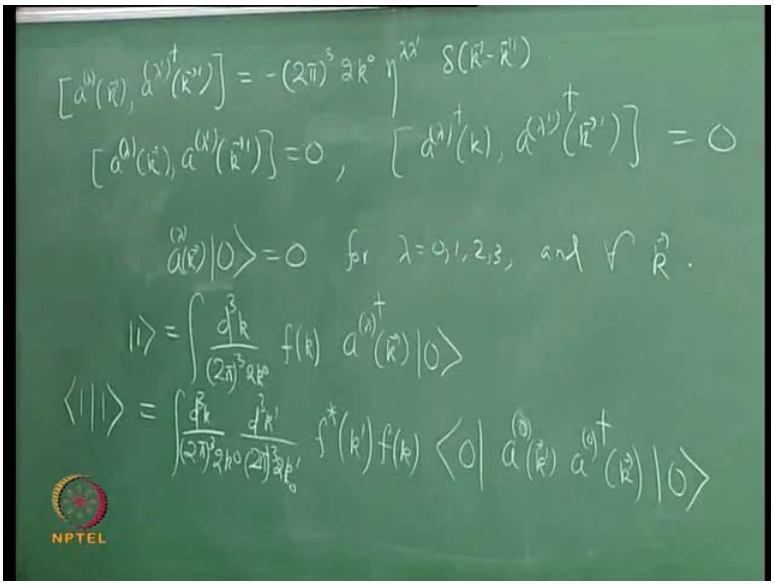


Whereas, a lambda k a lambda prime k prime equal to 0 and show is a lambda dagger of k a lambda prime dagger of k prime. This is what we get for the commutation relations for the a s and a daggers. So, you can again interpret these a s to be the annihilation operators a daggers to be the creation operators. You can define their vacuum to be the one, which is annihilated by a lambda k for lambda equal to 0, 1, 2, 3. For all k then the one particle states two particle states etcetera will be obtained from the vacuum by acting the creation operators submit.

So, you might think that everything is perfectly all right you have now the whole spectrum. Then you can consider the Hamiltonian and then what are the n and j Eigen values and so on, but it is not so simple. The reason is the following you look at the one particle states, which correspond to the let say scalar polarization. So, folks space of one prime states are usually of this form. That you just consider the cube k over 2 pi cube to k 0 sum distribution in the momentum space f of k a lambda dagger of k acting on the vacuum. So, a typical one particle state, which has a finite spread in the momentum space looks like this. You consider such a state for the scalar polarization. So, you just set a lambda equal to.

So, you just consider the case when lambda equal to 0 and then look at the norm for this one particle state. So, what do you get for the norm of this one particle state. So, this will be the d cube k prime over 2 pi cube 2 k 0 prime. Then you have f star of k prime f of k then a lambda I am considering the one particle state corresponding to this scalar polarization. So, a lambda acting on k prime a 0 dagger k this is what is the norm for the one particle state. Now, I can use this commutation relation to flip this a and a dagger here, but when I do that what I get is the following. So, a 0 k prime a 0 dagger of k is basically the commutator of a 0 k prime a 0 dagger of k and plus a 0 dagger of k a 0 of k prime.

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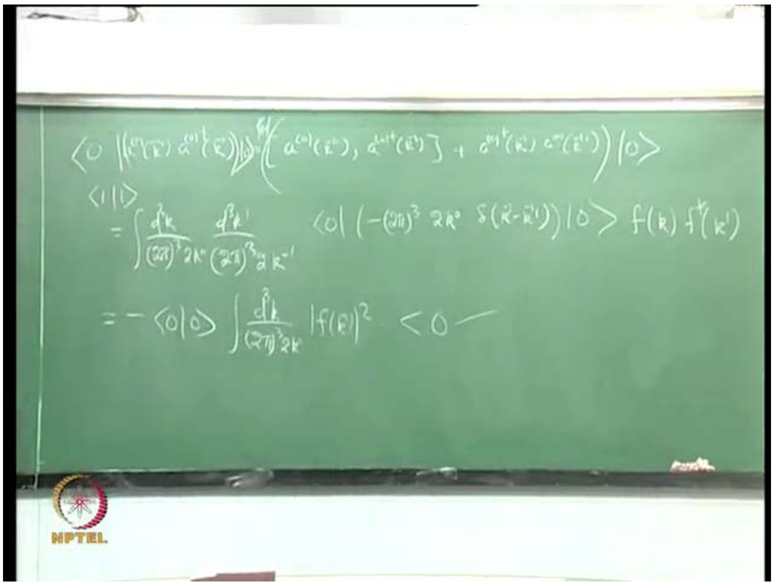


Now, I will substitute this in the expression for the norm of this one particle state corresponding to the scalar polarization. When I substitute that the contribution from this term vanishes, because I am considering this inside this. So, therefore you have a vacuum here and there is a ground state acting on this is an annihilation operator. It will just when acting on the ground state it will give you 0. So, this term vanishes therefore, what we get for the norm here is d cube k over 2 pi cube 2 k 0 d cube k prime over 2 pi cube 2 k 0 prime. This the commutator here, but what is the commutator? If you look at the commutator its minus 2 pi cube 2 k 0 eta lambda lambda prime.

So, for lambda equal to 0 and lambda prime equal to 0, it will give you minus 2 pi cube 2 k 0. Because, of this minus sign here you have a minus 2 pi cube 2 k 0 times delta k minus k prime, this is all you get. It has zero, 0 is one, therefore this is what we have. Now, you can perform the k prime integration and then there is f of k f star of k prime. If you perform the k prime integration what you get is norm of the ground state. There is a minus sign integration d cube k over 2 pi cube 2 k 0 mode f of k square.

So, what you saw here is that the norm of this one particle states are negative, therefore when you quantize when you consider this theory. Then you quantize it you see that there are negative norm states present in the Hilbert space. Therefore, you this is of course, a problem you need to get it off all this negative norms state. So, need to consider how to consider, how to impose a physical condition, which will remove all the negative norms states from the Hilbert space. Remember, we had avoided one of the issues earlier, which was the condition setting the condition del mu a mu equal to 0.

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If we set this as an operator equation, then we run into inconsistency. In the sense that the canonical commutation relations that we have adapted cannot be satisfied, if we consider this as an operator equation left hand side becomes zero, but its right hand side none 0 and so on. So, we need to worry how to impose this condition also in order to recover the Maxwell’s theory. What we will see is that we will see the states in the Hilbert space are physical. If they satisfy this condition that suppose psi is A state for which del mu A mu the expectation value of this operator in this state is 0.

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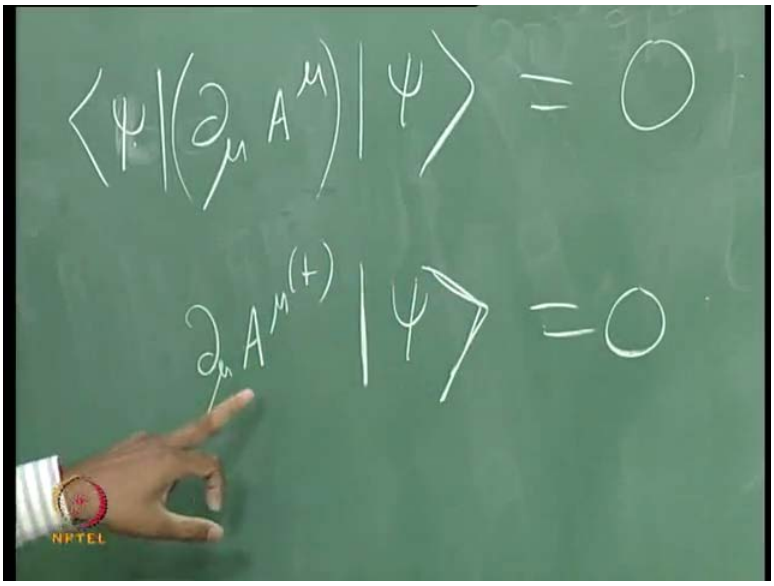

Then this is a physical state. So, this is the physical state condition that that we are going to impose. Although, we cannot set del mu a mu equal to 0 as an operator equation. We can consider the expectation value of this operator in any arbitrary state. Then we will require that this expectation value is always 0. So, we are saying that state is a physical state if it satisfies these conditions that expectation value of del mu A mu equal to 0. Otherwise, it is not a physical state. Then we will see what is the implication of this physical state condition?

We will see that this physical state condition not only will give us the Maxwell's theory. It will also remove all the unwanted negative norms states that appear in this spectrum here. So, the physical Hilbert space will not content of any of these negative norm states, that is what we will see in a moment. So, this condition here is equivalent to imposing this that to del mu A mu plus acting on this state psi equal to 0. You can see that this specially implies this one.

So, let us look at this condition here A mu plus contents the positive frequency part of A the mode expansion. Therefore, A mu plus of x according to our definition is d cube k over 2 pi cube 2, k 0 sum over lambda equal to 0 to 3 epsilon lambda mu of k a lambda, of k e to the power minus i k dot x.

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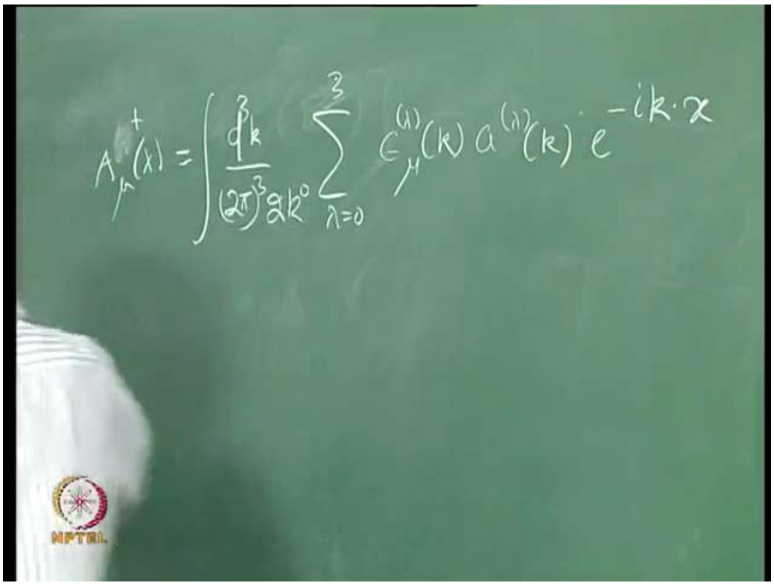


So, what do we get when I impose this condition here? When I impose the condition that this quantity acting on some state gives you 0, so to do that lets consider a basis in the Hilbert space ,which is given by this form a basis of states, if I can choose a basis of states, which are of this form psi t phi.

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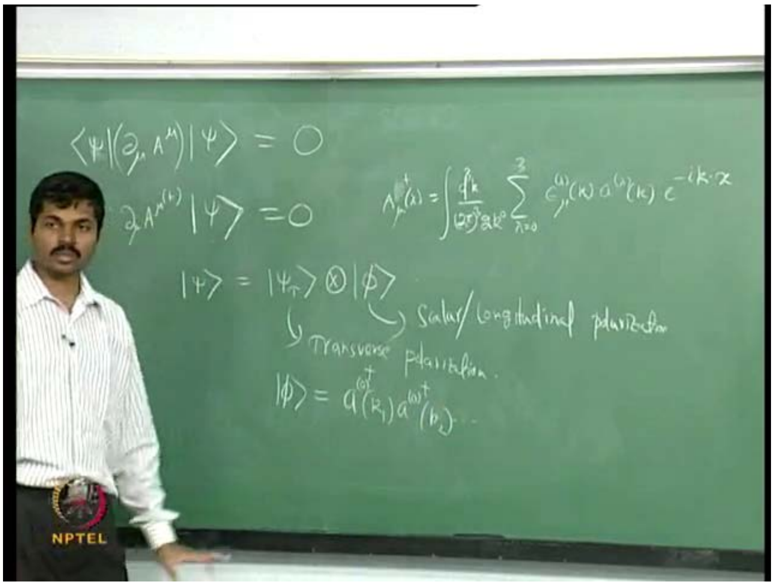


Do you understand what is this psi t correspond to the states, which I raise due to the transverse polarization, whereas this phi here correspond to the longitudinal and scalar polarization. So, this is a scalar or this correspond to transverse polarization. So, this now what we get is we will see what happens, when A mu plus acts on a state like this, because this corresponds to this scalar polarization. Therefore, this state phi here will be of this form it will be obtained by a 0 k 1 a 0 dagger k 1 a 0 dagger k 2. There will be a number of a 0 operators.

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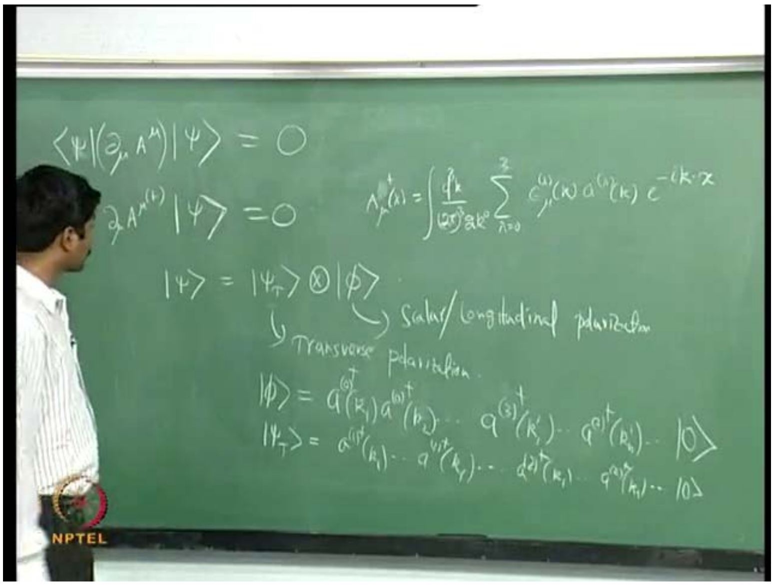


Then there will be a number of a 3 dagger k 1 prime a 3 dagger of k n prime etcetera acting on the ground state here. So, a general n particle state corresponding to this will have n such operators here. Whereas, this psi t here is just given by again this state psi t will be of this form, where you have transverse polarization a 1 dagger of k 1 and a 1 dagger of k r number of such operators. Then you will have also a 2 dagger of k 1 a dagger of k n acting on the translate here. Now, what happens is that, now what we will see what a mu plus of x acting on this state psi transverse and phi equal to 0 gets itself, this is there is this del mu a mu acting on this.

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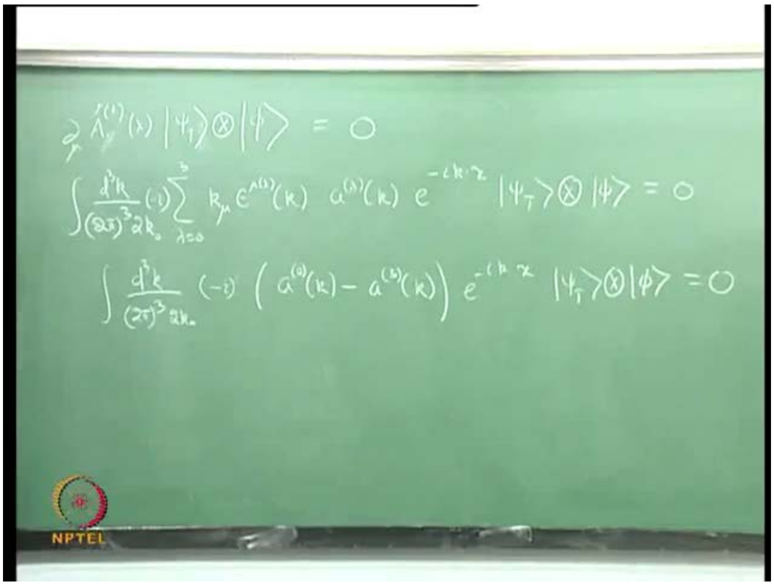

So, what do you get from here as you can see this d cube k over 2 pi cube 2 k 0. This derivative acting on this will give me sum over lambda equal to 0 to 3 k mu epsilon mu lambda of a k with a minus i a lambda k e to the power minus i k dot x phi equal to 0. Now, because of this appearance of this k mu here the transverse polarization will simply go away, because for lambda equal to 1 and 2 this is simply 0. Although, lambda is summed over all the values lambda equal to 1 and 2, correspond to transverse polarization.

So, this simply kills the lambda equal to 1 and 2. Therefore, what is left here is a d cube k over 2 pi cube 2 k 0. There is a minus i and then a 0 of k minus a 3 k times e to the power minus i k dot x, is this clear? This lambda equal to 1 and 2 just goes away and k dot epsilon just gives you this minus sign here, k is just a null vector and k you can choose to be something like 1 0 0 1.

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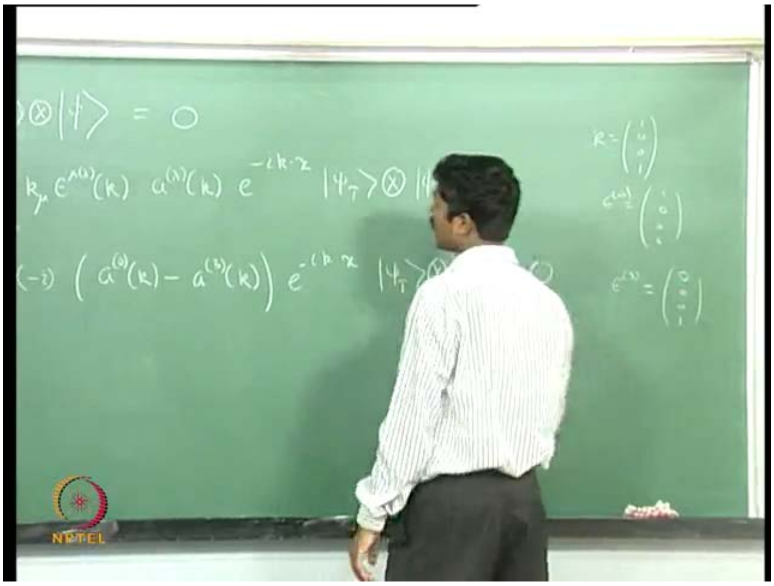


Then this will simply imply if according to our choice epsilon 0 is just a 1 0 0 0. Whereas, epsilon 3 0 0 0 1 and then this factor will simply gave you this, because this involves only the scalar and longitudinal polarization. It’s irrelevant to consider this transverses part here.

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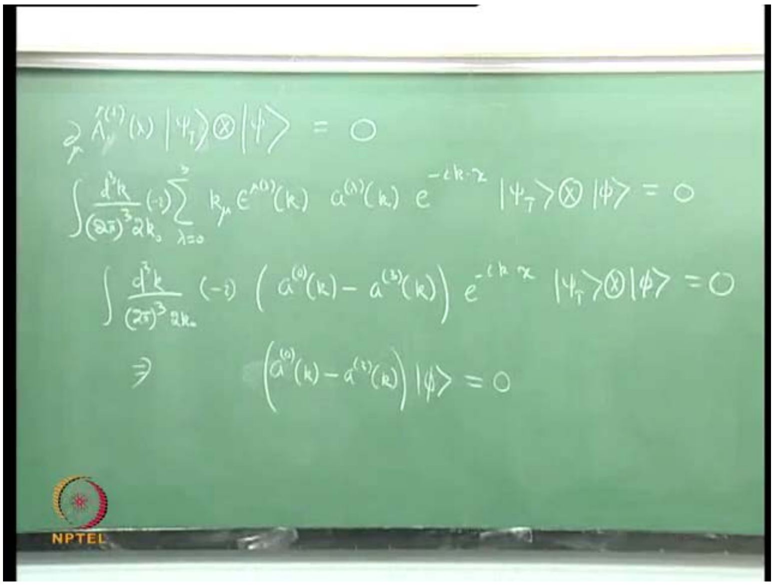


So, these simply imply that a 0 k minus a 3 k acting on some n particles state gives you 0. So, this is what is equivalent to imposing this condition del mu a mu plus on this state psi equal to 0. Let say consider the number operator corresponding to this scalar and longitudinal polarization.

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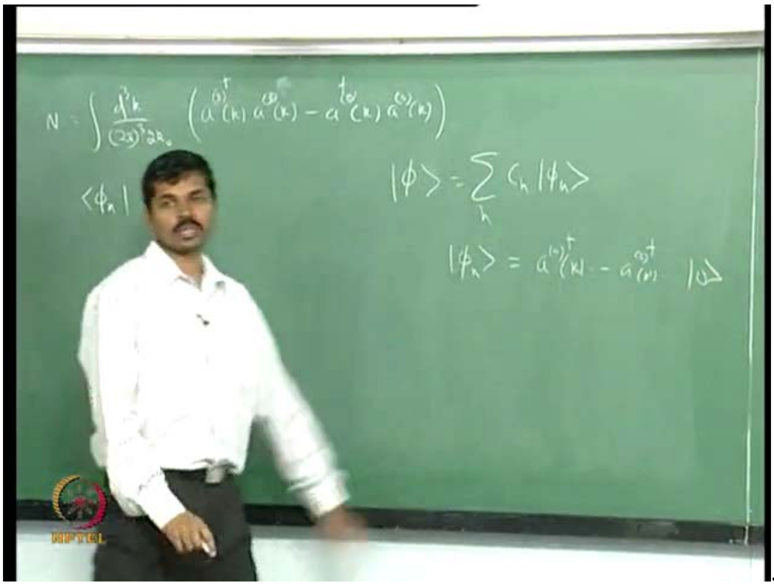

You can work out the expression for the number operator you can show that the number operator is simply d cube k over 2 pi cube 2 k 0 a 0 k a 0 dagger minus a 3. I think there is a 3 here, there is a 3 here minus a 0 k a 0 dagger of k a 0. This is the number operator corresponding to the longitudinal and scalar polarization. You can look at some n particles state phi n any arbitrary state phi here can be express as sum coefficient c n times phi n summed over n, where phi n here is n n particles state corresponding to this scalar and longitudinal polarization. So, there will be a 0 daggers and a 3 daggers and there will be n such objects acting on the vacuum. That is what I am denoting by phi n and a phi in general is summed over all these things.

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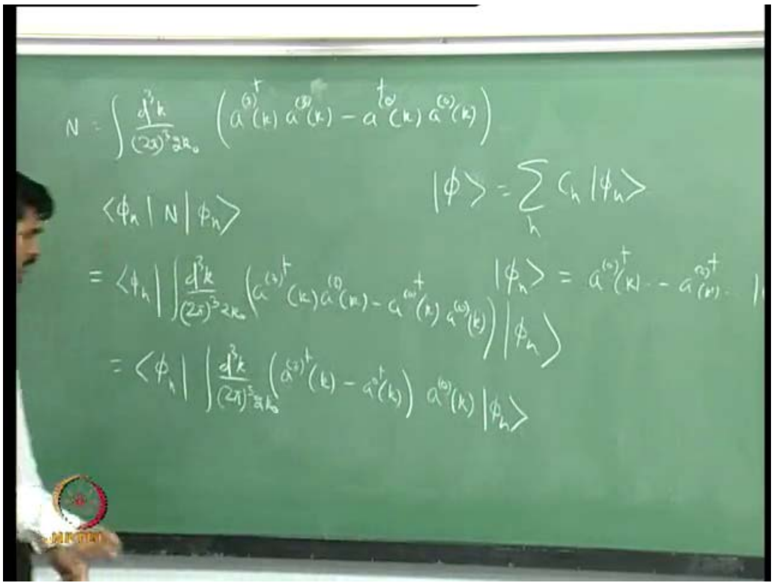

So, now if you look at this quantity here you can show that this 2 k 0 and a 3 dagger k a three of k minus a 0 dagger k a 0 k phi n here, but now this phi n satisfies this condition, that a 0 acting on phi n is equal to a 3 acting on phi n. So, I can used that relation when I used that this will become phi n d cube k over 2 pi cube 2 k 0. This is a 3 dagger k minus a 0 dagger of k a 0 of k acting on phi n. Suppose, phi n is not equal to phi 0 suppose phi n of course, in general it will have c 0 times ground state plus c times one particles state and so on.

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In general phi n will have something, so suppose n is not 0 if n is not 0 then this is not ground state. So, in that case what you can see is that this quantity here is simply equal to 0, because you can consider this acting on this is simply this is always equal to 0, this quantity here is always equal to 0. This is, because if n is ground state then it of course, it annihilates ground state even if n is not equal to ground state you can consider this. Then this when it acts on this one, from the right again you can used the conjugate of this equation here. This conjugate shows that this is equal to 0. So, what does it mean? It is simply means that if n is the number operator.

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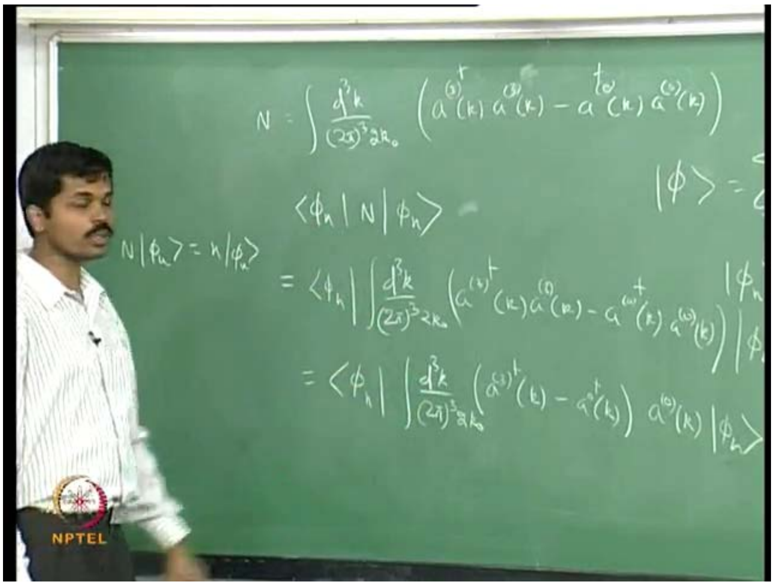


Then n acting on phi n is just n times phi n right, but this calculation shows that this is equal to 0. Therefore, phi n, which is n times phi n phi n, which is equal to 0.

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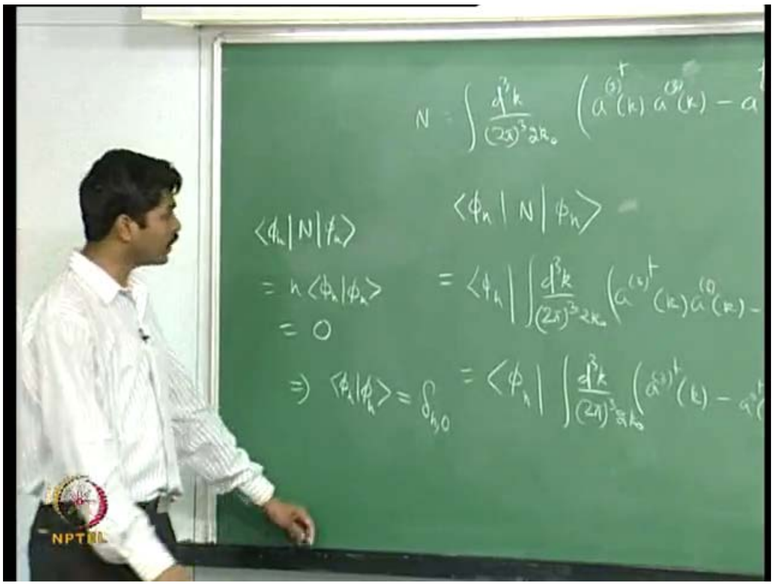

Basically, implies that phi n phi n equal to 0, when n is not equal to 0. In other words if n equal to 0 this is just a ground state, which is unique norm, therefore this quantity is delta m 0. Therefore, if you consider an arbitrary state corresponding to scalar and longitudinal polarization, which satisfy this condition here, then this state can either be ground state or it will have a 0 norm.

So, these are all null states expect for the ground state all the n particle states corresponding to this scalar and longitudinal polarization, which satisfy the physical state condition are all null state. Then you can see that if you consider any physical observable and then if you compute the expectation value. For example, let say you consider the Hamiltonian of the theory.

Then you compute the expectation value of the Hamiltonian in the physical Hilbert space, then these states these null states completely decoupled. They just drop out from the expectation values that is what you can see when you do in an explicit computation. So, although these states are there they do not contribute to any physical measurable quantity. We have said the physical Hilbert space is formed by states of this type psi equal to psi t and phi.

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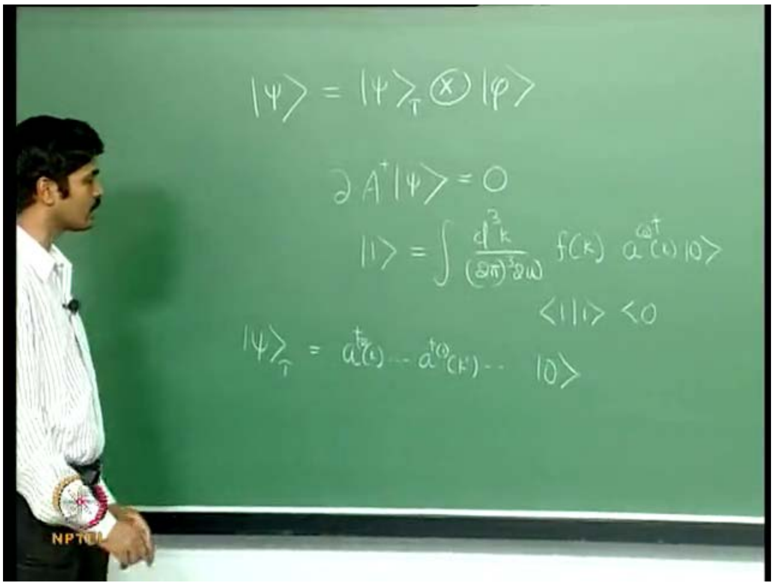


These involve the transverse polarization whereas, these involve the scalar polarization. However, not all these states in the in the full Hilbert space satisfy the physical state condition, which is given by del dot a plus acting on psi is equal to 0. For example, we have considered the state the one particle state, which is given by 1 is equal to integration d cube k over 2 pi cube 2 omega f of k times a 0 dagger k, acting on the vacuum. Then we have seen that this state has negative norm the norm of the state is less than 0.

So, this state for example does not satisfy the physical state condition del dot a plus acting on psi equal to 0 any state of this type, which is psi t. If you consider any state of this type, which is actually given by the creation, which is actually generated by acting creation operators for the transverse polarizations, which is a dagger 2 k, so on a dagger 1 k prime and so on like this.

So, states of this type do satisfy the physical state condition. So, they are they are in the physical Hilbert space there are other states. For example, states like this they do not satisfy the physical state condition. So, they do not belong to the physical Hilbert space. However, in addition to these states there are a bunch of states, which are also generated by this scalar. So, which are of this type a 0 dagger k acting on this or a 3 dagger k acting on this or a bunch of operators, like a 0 dagger k a 3 dagger k, so on acting on the state.

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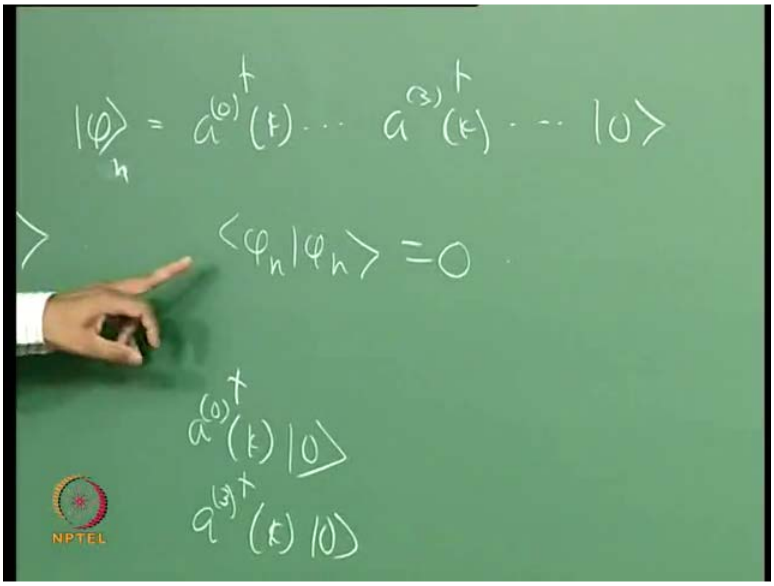


So, phi 0 I will call them phi n these are the n particle states, which have 0 norms. So, phi n phi n norm of these states are 0. There are a bunch of states, which are generated by acting these operators on the vacuum with 0 norm, this will be there in the physical Hilbert space however, because these states have 0 norm. Therefore, you consider any observable quantity expectation values of any operator. These states do not contribute at all. It is only these states, which are generated by acting the transverse on the by acting the creation operators corresponding to the transverse polarizations. They only generate the physical Hilbert space.